

## A NOTE ON UNIVERSALLY GOING-DOWN

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We shall assume throughout that all rings and algebras are commutative with identity and that all homomorphisms are unital. Recall that a ring-homomorphism  $R \rightarrow T$  is said to be universally going-down in case  $S \rightarrow S \otimes_R T$  satisfies going-down (henceforth abbreviated GD) for each change of base  $R \rightarrow S$ . This concept was introduced in [7] and studied extensively in [4]. The most natural examples of universally going-down homomorphisms  $R \rightarrow T$  arise when  $\dim(R) = 0$  or  $T$  is  $R$ -flat [4, Proposition 3.3]. It is interesting that, in some cases, universally going-down reduces to one of these two archetypes. For zero-dimensionality, this is essentially well known and summarized in Proposition 1 below. Our main result, Theorem 3, is that any universally going-down overring extension of an integrally closed domain must be flat. This will follow easily from the main results of [4] and [5], with which we assume familiarity.

**Proposition 1.** *For a ring  $R$ , the following five conditions are equivalent:*

- (1)  $R \rightarrow T$  is universally going-down for each  $R$ -algebra  $T$ ;
- (2) The canonical map  $R \rightarrow R/P$  is universally going-down for each nonminimal  $P \in \text{Spec}(R)$ ;
- (3) The canonical map  $R \rightarrow R/P$  satisfies GD for each non-minimal  $P \in \text{Spec}(R)$ ;
- (4)  $R/P$  is  $R$ -flat for each nonminimal  $P \in \text{Spec}(R)$ ;
- (5)  $\dim(R) = 0$ .

*Proof.* The equivalence of (3), (4), and (5) was observed in [3, Proposition 2.1]. As noted above, [4, Proposition 3.3] yields that (5)  $\Leftrightarrow$  (1); and (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3) trivially, to complete the proof.

The above theme that (universally) GD-behavior often entails flatness was also noted in [3, Remark 2.6 (c)]. It was shown there that if  $R$  is a reduced ring, then the canonical map  $R \rightarrow R/P$  satisfies (universally) GD for each nonmaximal  $P \in \text{Spec}(R)$  (if and) only if each such  $R/P$  is  $R$ -flat. As this result is false without the "reduced" hypothesis [3, Remark 2.6 (b)], we are motivated to consider the "reduced" case of Proposition 1.

**Corollary 2.** *For a ring  $R$ , the following six conditions are equivalent :*

- (1)  *$R$  is reduced and  $R \rightarrow T$  is universally going-down for each  $R$ -algebra  $T$ ;*
- (2) *Each  $R$ -algebra is  $R$ -flat ;*
- (3)  *$R$  is reduced and the canonical map  $R \rightarrow R/P$  is universally going-down for each maximal ideal  $P$  of  $R$  ;*
- (4)  *$R$  is reduced and the canonical map  $R \rightarrow R/P$  satisfies GD for each maximal ideal  $P$  of  $R$  ;*
- (5)  *$R/P$  is  $R$ -flat for each maximal ideal  $P$  of  $R$  ;*
- (6)  *$R$  is von Neumann regular (i.e., absolutely flat).*

*Proof.* As cited in [3], the equivalence (2)  $\Leftrightarrow$  (6) is in well known work of Harada and Auslander. Also, Akiba [1, Corollary 4] (and, much later, [3, Remark 2.6 (e)]) established (6)  $\Leftrightarrow$  (5). Moreover, since (6) is well known to be equivalent to the condition that  $R$  be reduced and zero-dimensional, Proposition 1 yields (1)  $\Leftrightarrow$  (6). Similarly, (4)  $\Leftrightarrow$  (6), as it is easy to see that the GD condition in (4) implies  $\dim(R) = 0$ . Finally, (1)  $\Leftrightarrow$  (3)  $\Leftrightarrow$  (4) trivially, completing the proof.

Before stating our main result, we recall a definition from [4]. A ring-homomorphism  $f: R \rightarrow T$  is said to be quasi-going-up (in short, QGU) if, for each pair of primes  $P_1 \subset P_2$  of  $R$  such that  $f(P_2) \neq T$  and each  $Q_1 \in \text{Spec}(T)$  such that  $f^{-1}(Q_1) = P_1$ , there exists  $Q_2 \in \text{Spec}(T)$  such that  $Q_1 \subset Q_2$  and  $f^{-1}(Q_2) = P_2$ . The key fact used in the next proof is that universally going-down overring extensions of domains satisfy this weak form of going-up, even after change of base.

**Theorem 3.** *Let  $T$  be an overring of a domain  $R$  such that  $R$  is integrally closed in  $T$ . Then the inclusion map  $R \rightarrow T$  is universally going-down (if and) only if  $T$  is  $R$ -flat.*

*Proof.* The parenthetic assertion holds since flat implies universally going-down. Conversely, to show  $T$  is  $R$ -flat, a criterion of Richman (cf. proof of [8, Theorem 2]) reduces us to verifying the following: if  $P \in \text{Spec}(R)$  and  $PT \neq T$ , then  $T_P = R_P$ . (As usual,  $T_P$  denotes the ring of fractions  $T_{R \setminus P}$ .) For any such  $P$ , the hypothesis yields that  $R_P$  is integrally closed in  $T_P$ . Thus, it suffices to verify that, for each  $P \in \text{Spec}(R)$  such that  $PT \neq T$ , one has that  $T_P$  is integral over  $R_P$ . In the terminology of [5], our task is thus to show that the inclusion map  $f: R \rightarrow T$  is quasi-

integral. By the main result of [5], namely [5, Theorem 3.2], this is equivalent to showing that  $f$  is universally QGU, in the sense that  $S \rightarrow S \otimes_R T$  is QGU for each change of base  $R \rightarrow S$ . However, the main result of [4], namely [4, Theorem 3.17], assures that each universally going-down overring extension of a domain is universally QGU. The proof is complete.

**Remark 4.** (a) A result of Papick (cf. [6, (3.14)]) asserts that if  $T$  is an overring of a coherent domain  $R$  such that  $R$  is integrally closed in  $T$ , then  $R \subset T$  satisfies GD (if and) only if  $T$  is  $R$ -flat. One may regard the assertion of Theorem 3 in the same vein, where the finiteness hypothesis of coherence has been eliminated, at the expense of enhancing the GD hypothesis to universally going-down.

(b) One way to motivate the "integrally closed in" hypothesis in Theorem 3 is via Corollary 2, for any von Neumann regular ring is trivially integrally closed. Another way is to note that the "dual" situation, that of an *integral* overring extension of domains that is universally going-down, has been extensively characterized (cf. [4, Corollaries 3.19 and 3.20]).

(c) The above "flat" impact of universally going-down should be contrasted with the effect of another type of "enhanced GD" condition considered in some of our recent work. Let  $R$  be a domain such that  $A \subset B$  satisfies GD for all pairs  $A \subset B$  of subrings of  $R$ . Then by [2, Theorem 2.1 and Proposition 2.5],  $\dim(A) \leq 1$  and  $\dim(B) \leq 1$  for all subrings  $A \subset B$  of  $R$ , but it need *not* follow that  $B$  is  $A$ -flat.

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