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AN EXTENSION OF A THEOREM OF D. GROMOLL AND J. A. WOLF

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0. A complete Riemannian manifold M is said to be *without focal points* if any geodesic has no focal point as a 1-dimensional submanifold in M. Many properties and results for manifolds of nonpositive curvature were extended to those manifolds. In case we extend a theorem of Gromoll and Wolf (cf. [2], Theorem 9.1 and [3]) by same argument as theirs, only one lemma corresponding to Lemma 9.6 in [2] has not been established yet. In the present note we prove the lemma. Combined with some properties corresponding to their lemmas, we have the following theorem by using the terminology in the book of J. Cheeger and D. G. Ebin ([2], Theorem 9.1).

Theorem. *Let M be a closed totally convex subset of a simply connected Riemannian manifold without focal points and let Γ be a properly discontinuous group of semi-simple isometries of M. Let Σ be a solvable subgroup of Γ. Then M contains a flat totally geodesic submanifold E, which is isometric to $\mathbf{R}^k$ and invariant under Σ such that:*

(1) *Σ acts with finite kernel Φ on Σ,*

(2) *E/Σ is compact, in particular, Σ is finitely generated and Σ/Φ is a crystallographic group of rank k.*

In particular, (1) *and* (2) *hold if M is the universal covering space of a compact manifold and Γ the group of covering translations.*

The very similar statement was given by H. B. Lawson and S. —T. Yau ([5]) for manifolds of nonpositive curvature, and was extended to manifolds without focal points by J. O'Sullivan ([6], [7]). Nevertheless, we write this note to emphasize that the lemma is true and Theorem can be proved by the same argument as in the case of nonpositive curvature.

1. We state the lemma and prove it.

Lemma 1. *Let M be a totally convex subset in a simply connected complete Riemannian manifold without focal points. Let φ be a semi-simple isometry of M and let C be a nonempty closed totally convex subset in M which is invariant under the action of φ. Then, $C \cap C_\varphi \neq \phi$, where C_φ is* *the minimum set of the displacement function* d_φ of φ.

Proof. We say that a point $q \in C$ is a foot of a point p on C if $d(p, C) = d(p, q)$. Since C is totally convex, there exists the unique foot on C of each point $p \in M$. We denote by $f(p)$ the foot of p on C. Obviously, $\varphi f = f\varphi$. First we treat the case where φ is elliptic, i.e., $min\ d_\varphi = 0$. If $p \in C_\varphi$, then $f(p) \in C \cap C_\varphi$, since

$$d(\varphi(f(p)), p) = d(\varphi(f(p)), \varphi(p)) = d(f(p), p),$$

and, hence, $\varphi(f(p)) = f(p)$. Next assume that φ is axial, i.e., $min\ d_\varphi > 0$. Let $\alpha: (-\infty, \infty) \to M$ be an axis of φ, namely $\varphi\alpha(t) = \alpha(t+a)$ for some constant a and for any $t \in (-\infty, \infty)$. Put $\beta(t) = f(\alpha(t))$ for each $t \in (-\infty, \infty)$. Then, by Proposition 2.8 in [4], $\beta: (-\infty, \infty) \to M$ be a geodesic and biasymptotic to α. Hence, $\beta(t) \in C \cap C_\varphi$, since

$$\varphi\beta(t) = \varphi(f(\alpha(t))) = f(\varphi(\alpha(t))) = f(\alpha(t+a)) = \beta(t+a)$$

for any $t \in (-\infty, \infty)$, namely β is an axis of φ, and, therefore, $\beta(\mathbf{R}) \subset C_\varphi$ because of Theorem 32.3 in [1]. Lemma 1 is proved.

2. The following fixed point theorem was known already. However, we give an elementary proof here. The idea of the proof is due to K. Shiohama.

Lemma 2. *Let C be a totally convex subset in a complete simply connected Riemannian manifold N without focal points. Let φ be an isometry of C. If there is a compact subset A in C which is invariant under the action of φ, then φ has a fixed point in C.*

Proof. Let r be the minimum of the radii of all geodesic balls in N which contain A. Suppose a geodesic ball $B(p, r)$ contains A. We want to prove that $p \in C$ and p is a fixed point of φ. Suppose $p \notin C$. Let q be the foot of p on C. Since C is totally convex and the distance function to each $x \in C$ is convex in N, we see that $d(x, q) < d(x, p)$ for any $x \in C$. From this we have a contradiction to the choice of r. Suppose $p \neq \varphi p$ for indirect proof. $\varphi B(p, r) \cap C = B(\varphi p, r) \cap C$ contains A also, since $\varphi A = A$. Let q be the midpoint of p and φp. Then,

$$d(x, q) < \max\{d(x, p), d(x, \varphi p)\} \leq r$$

for all $x \in A$, since all geodesic balls are strictly convex in N. If $r_0 =$

max $\{d(x, q);\ x \in A\}$, then $r_0 < r$. This contradicts the choice of r, since $B(q, r_0)$ contains A. Lemma 2 is proved.

3. The proof of Theorem 9.1 in [2] needs Lemma 9.2, 9.3, 9.5, 9.6 in [2] and nothing else. Here, we give the list of correspondence. Since Lemma 9.2 is free from the assumption of nonpositive curvature, it holds in our case also. The author does not know whether Lemma 9.3 is true under the nonfocality properties. However, the proof of Theorem 9.1 needs only the total convexity of the minimum set C_φ of the displacement function d_φ, φ being a semi-simple isometry. This fact is proved as follows: If $min\ d_\varphi = 0$, then C_φ is the set of all fixed points of φ, and therefore C_φ is totally convex because of the existence of the unique minimizing geodesic joining two points. If $min\ d_\varphi > 0$, then C_φ is the set of all axes of φ, i.e. a geodesic $\alpha: (-\infty, \infty) \to M$ such that $\varphi\alpha(t) = \alpha(t+a)$ for any $t \in (-\infty, \infty)$, $a = min\ d_\varphi$, and therefore $d(\alpha(t), \beta(t))$, $t \in (-\infty, \infty)$, is bounded for any axes α and β of φ. In this case the flat strip theorem (cf. Theorem 1.13, [4]) implies the total convexity of C_φ. Lemma 9.5 is proved by using Lemma 2 and the flat strip theorem (cf. [4]). Lemma 9.6 corresponds to Lemma 1. Now, we can prove Theorem by the parallel argument to Gromoll and Wolf's.

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