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ON THE KO-COHOMOLOGIES OF THE STUNTED LENS SPACES

Dedicated to Professor Hirosi Toda on his 60th birthday

AKIE TAMAMURA and SUSUMU KÔNO

1. Introduction. In this note, we consider the *KO*-cohomologies of the stunted lens spaces. In order to state our theorem we recall some notation in [4].

For the $(2n+1)$-dimensional standard lens space $L^n(q) = S^{2n+1}/Z_q$ mod q, we set

$$(1.1)\qquad \begin{aligned} L_q^{2n+1} &= L^n(q), \\ L_q^{2n} &= \{[z_0, \cdots, z_n] \in L^n(q) \mid z_n \text{ is real} \geqq 0\}. \end{aligned}$$

We denote by ord G the order of a finite group G.

The following theorem is immediately obtained (cf. [9]).

Theorem 1. *Let $q \geqq 3$ be an odd integer, then*

(1) $\widetilde{KO}^{-2j-1}(L_q^{2m}/L_q^{2n}) \cong 0$.

(2) i) $\widetilde{KO}^{-2j}(L_q^{2m}/L_q^{2n}) \cong \tilde{K}^{-2j}(L_q^{2m}/L_q^{2n})/(1-t)\tilde{K}^{-2j}(L_q^{2m}/L_q^{2n})$,
where t: $K \to K$ is the complex conjugation.

ii) ord $\widetilde{KO}^{-2j}(L_q^{2m}/L_q^{2n}) = q^{[(m+j)/2]-[(n+j)/2]}$.

(3) $\widetilde{KO}^{-2j-1}(L_q^{2m}/L_q^{2n+1}) \cong \widetilde{KO}(S^{2n+2j+3})$.

(4) $\widetilde{KO}^{-2j}(L_q^{2m}/L_q^{2n+1}) \cong \widetilde{KO}^{-2j}(L_q^{2m}/L_q^{2n+2}) \oplus \widetilde{KO}(S^{2n+2j+2})$.

(5) $\widetilde{KO}^{-2j-1}(L_q^{2m+1}/L_q^{2n}) \cong \widetilde{KO}(S^{2m+2j+2})$.

(6) $\widetilde{KO}^{-2j}(L_q^{2m+1}/L_q^{2n}) \cong \widetilde{KO}^{-2j}(L_q^{2m}/L_q^{2n}) \oplus \widetilde{KO}(S^{2m+2j+1})$.

(7) $\widetilde{KO}^{-2j-1}(L_q^{2m+1}/L_q^{2n+1}) \cong \widetilde{KO}(S^{2m+2j+2}) \oplus \widetilde{KO}(S^{2n+2j+3})$.

(8) $\widetilde{KO}^{-2j}(L_q^{2m+1}/L_q^{2n+1})$
$\cong \widetilde{KO}^{-2j}(L_q^{2m}/L_q^{2n+2}) \oplus \widetilde{KO}(S^{2n+2j+2}) \oplus \widetilde{KO}(S^{2m+2j+1})$.

As defined in [1], we denote by $\phi(n_1, n_2)$ the number of integers s with

$n_2 < s \leqq n_1$ and $s \equiv 0, 1, 2$ or $4 \pmod 8$, so that $\phi(n_1, n_2) = \phi(n_1, 0) - \phi(n_2, 0)$ with

$$\phi(n, 0) = \begin{cases} [n/2] & (n \equiv 0, 6 \text{ or } 7 \pmod 8) \\ [n/2]+1 & (\text{otherwise}). \end{cases}$$

Let $h\colon Z \to Z$ be the function defined by

$$h(s) = \begin{cases} 2 & (s \equiv 1 \pmod 8) \\ 1 & (s \equiv 0 \text{ or } 2 \pmod 8) \\ 0 & (\text{otherwise}). \end{cases} \tag{1.2}$$

Main result is the following theorem.

Theorem 2. (1) *If $j \equiv 0 \pmod 4$ and $n \not\equiv 3 \pmod 4$, then we have*

$$\operatorname{ord} \widetilde{KO}^{-j}(L^m_{2q}/L^n_{2q}) = 2^{\phi(m+j, n+j)} q^{[m/4]-[n/4]}.$$

(2) *If $j \equiv 0 \pmod 4$ and $n \equiv 3 \pmod 4$, then we have*

$$\widetilde{KO}^{-j}(L^m_{2q}/L^n_{2q}) \cong \widetilde{KO}^{-j}(L^m_{2q}/L^{n+1}_{2q}) \oplus Z.$$

(3) *If $j \equiv 1 \pmod 4$ and $m \geqq n+2$, then the groups $\widetilde{KO}^{-j}(L^m_{2q}/L^n_{2q})$ are tabled as follows.*

$n+j \pmod 8$ \ $m+j \pmod 4$	0	1	2	3
0	$Z \oplus Z_2 \oplus Z_2$	$Z_2 \oplus Z_2$	$Z_2 \oplus Z_2$	$Z_2 \oplus Z_2$
1	$Z \oplus Z_2$	Z_2	Z_2	Z_2
2	Z	0	0	0
3	Z	0	0	0
4	Z	0	0	0
5	Z	0	0	0
6	Z	0	0	0
7	$Z \oplus Z_2$	Z_2	Z_2	Z_2

(4) *If $j \equiv 2 \pmod 4$, $n \not\equiv 1 \pmod 4$ and $m \geqq n+3$, then we have*

$$\operatorname{ord} \widetilde{KO}^{-j}(L^m_{2q}/L^n_{2q}) = 2^{h(m+j)+h(n+j)} q^{[(m+2)/4]-[(n+2)/4]}.$$

(5) *If $j \equiv 2 \pmod 4$, $n \equiv 1 \pmod 4$ and $m \geqq n+3$, then we have*

$$\widetilde{KO}^{-j}(L^m_{2q}/L^n_{2q}) \cong \begin{cases} \widetilde{KO}^{-j}(L^m_{2q}/L^{n+1}_{2q}) \oplus Z & (n+j \equiv 3 \pmod 8) \\ (\widetilde{KO}^{-j}(L^m_{2q}/L^{n+1}_{2q})/G) \oplus Z & (n+j \equiv 7 \pmod 8)\,, \end{cases}$$

where G denotes the kernel of the homomorphism

$$(p_{m,n})^!: \widetilde{KO}^{-j}(L^m_{2q}/L^{n+1}_{2q}) \to \widetilde{KO}^{-j}(L^m_{2q}/L^n_{2q})$$

induced by the projection $p_{m,n}$: $L^m_{2q}/L^n_{2q} \to L^m_{2q}/L^{n+1}_{2q}$, and ord G *is equal to* 2.

(6) *If $j \equiv 3 \pmod 4$ and $m \geqq n+2$, then we have*

$$\widetilde{KO}^{-j}(L^m_{2q}/L^n_{2q}) \cong \begin{cases} Z & (m \equiv 1 \pmod 4)) \\ Z_2 \oplus Z_2 & (m+j \equiv 2 \pmod 8)) \\ Z_2 & (m+j \equiv 1 \text{ or } 3 \pmod 8)) \\ 0 & (\text{otherwise}). \end{cases}$$

Remark 1. If $m = n+1$, then we have a homeomorphism

$$S^j(L^{n+1}_{2q}/L^n_{2q}) \approx S^{j+n+1}\,,$$

and the group $\widetilde{KO}(S^{j+n+1})$ is well known.

Remark 2. If $m = n+2$, then we have homotopy equivalences

$$S^j(L^m_{2q}/L^n_{2q}) \simeq \begin{cases} S^{j-n+2} \vee S^{j+n+1} & (n\text{: odd}) \\ S^{j+n}(L^2_{2q}) & (n\text{: even})\,, \end{cases}$$

and parts (4) and (5) are true except the following cases:

i) If $j \equiv 2 \pmod 4$ and $m+j = n+j+2 \equiv 3 \pmod 8$, then we have

$$\widetilde{KO}^{-j}(L^m_{2q}/L^n_{2q}) \cong Z_2\,.$$

ii) If $j \equiv 2 \pmod 4$ and $m+j = n+j+2 \equiv 1 \pmod 8$, then we have

$$\widetilde{KO}^{-j}(L^m_{2q}/L^n_{2q}) \cong Z \oplus Z_2\,.$$

Remark 3. The partial results for the case $n = 0$ of Theorem 2 have been obtained in [11] (see also (2.3)).

Remark 4. The corresponding results for the case $q = 1$ of Theorem 2 are included in [6]. Our results imply

$$\widetilde{KO}^{-2j-1}(L^m_{2q}/L^n_{2q}) \cong \widetilde{KO}^{-2j-1}(RP(m)/RP(n))$$

for any q.

Consider the homomorphism

$$(p_n{}^m)^!:\ \widetilde{KO}^{-j}(L^m_{2q}/L^n_{2q}) \to \widetilde{KO}^{-j}(L^m_{2q})$$

induced by the projection

$$p_n{}^m:\ L^m_{2q} \to L^m_{2q}/L^n_{2q}.$$

Then, we have following result.

Corollary 3. *If* $j \equiv 0 \pmod 4$ *and* $n \not\equiv 3 \pmod 4$, *then the homomorphism* $(p_n{}^m)^!$ *is a monomorphism.*

We prove Theorem 2 in section 3 after recalling some known facts in section 2.

The authors would like to thank for the referee for pointing out the Additional Remark in the end of section 3.

2. Preliminaries. In this section we fix integers m and n with $0 \leqq n < m$. Set $e^k = L^k_q - L^{k-1}_q$ for $0 \leqq k \leqq m$. Then the space L^m_q has the cell decomposition

$$L^m_q = e^0 \cup e^1 \cup \cdots \cup e^m,$$
$$\partial(e^{2k+1}) = 0,\ \partial(e^{2k}) = qe^{2k-1}.$$

Thus we have

(2.1) *The cohomology groups of the pair* (L^m_{2q}, L^n_{2q}) *are given by*

$$H^j(L^m_{2q}, L^n_{2q};\ Z) \cong \begin{cases} Z & (j = m \equiv 1 \pmod 2) \text{ or } j = n+1 \equiv 0 \pmod 2) \\ Z_{2q} & (j \equiv 0 \pmod 2) \text{ and } n+1 < j \leqq m) \\ 0 & (\text{otherwise}), \end{cases}$$

$$H^j(L^m_{2q}, L^n_{2q};\ Z_2) \cong \begin{cases} Z_2 & (n < j \leqq m) \\ 0 & (\text{otherwise}). \end{cases}$$

We recall some known facts which are useful for the proof of Theorem 2.

(2.2) [11, (1.2) and (1.3)].

$$Sq^2\tilde{H}^j(L^m_{2q}/L^n_{2q};\ Z_2) \cong \begin{cases} Z_2 & (j \equiv 2 \text{ or } 3 \pmod 4) \text{ and } n < j \leqq m-2) \\ 0 & (\text{otherwise}). \end{cases}$$

(2.3) [11, Theorems (0.1) and (0.2)].

(1) *If* $j \equiv 0 \pmod 4$, *then*

$$\operatorname{ord} \widetilde{KO}^{-j}(L_{2q}^m) = 2^{\varphi(m+j,j)} q^{[m/4]}.$$

(2) *If* $j \equiv 1 \pmod 4$, *then the groups* $\widetilde{KO}^{-j}(L_{2q}^m)$ *are tabled as follows.*

$j \pmod 8$ \ $m+j \pmod 4$	0	1	2	3
1	$Z \oplus Z_2$	Z_2	Z_2	Z_2
5	Z	0	0	0

(3) *If* $j \equiv 2 \pmod 4$ *and* $m \geqq 2$, *then we have*

$$\operatorname{ord} \widetilde{KO}^{-j}(L_{2q}^m) = 2^{h(m+j)+h(j)} q^{[(m+2)/4]},$$

where $h\colon Z \to Z$ *is the function defined by* (1.2).

(4) *If* $j \equiv 3 \pmod 4$, *then*

$$\widetilde{KO}^{-j}(L_{2q}^m) \cong \begin{cases} Z & (m \equiv 1 \pmod 4)) \\ Z_2 \oplus Z_2 & (m+j \equiv 2 \pmod 8)) \\ Z_2 & (m+j \equiv 1 \text{ or } 3 \pmod 8)) \\ 0 & (\textit{otherwise}). \end{cases}$$

Consider the homomorphism

$$(i_n^m)^!\colon \widetilde{KO}^{-j}(L_{2q}^m) \to \widetilde{KO}^{-j}(L_{2q}^n)$$

induced by the inclusion

$$i_n^m\colon L_{2q}^n \to L_{2q}^m.$$

Then we have the following.

Lemma 2.4. (1) *If* $j \equiv 0 \pmod 4$, *then the homomorphism* $(i_n^m)^!$ *is an epimorphism.*

(2) *If* $j \equiv 3 \pmod 4$ *and* $m \geqq n+3$, *then the homomorphism* $(i_n^m)^!$ *is a zero-map.*

Proof. (1) is obtained immediately from [7, Lemma 2.9]. From (4) of (2.3), (2) is clear except the case $m = n+3$ and $n+j \equiv 1 \pmod 8$. Assume $n+j \equiv 1 \pmod 8$, then $\widetilde{KO}^{-j}(L_{2q}^{n+2}) \cong Z_2$ and $\widetilde{KO}^{-j}(L_{2q}^{n+2}/L_{2q}^n) \cong$

$\widetilde{KO}^{-j}(S^n(L_{2q}^2)) \cong Z_2$. In the exact sequence

$$\to \widetilde{KO}^{-j-1}(L_{2q}^{n+2}) \xrightarrow{(i_n^{n+2})^!} \widetilde{KO}^{-j-1}(L_{2q}^n) \to \widetilde{KO}^{-j}(L_{2q}^{n+2}/L_{2q}^n)$$
$$\xrightarrow{(p_n^{n+2})^!} \widetilde{KO}^{-j}(L_{2q}^{n+2}) \xrightarrow{(i_n^{n+2})^!} \widetilde{KO}^{-j}(L_{2q}^n) \to$$

associated to the cofibration

$$L_{2q}^n \xrightarrow{i_n^{n+2}} L_{2q}^{n+2} \xrightarrow{p_n^{n+2}} L_{2q}^{n+2}/L_{2q}^n,$$

the homomorphism $(i_n^{n+2})^!$: $\widetilde{KO}^{-j-1}(L_{2q}^{n+2}) \to \widetilde{KO}^{-j-1}(L_{2q}^n)$ is an epimorphism by (1). Hence, $(p_n^{n+2})^!$ is an isomorphism. So the homomorphism $(i_n^{n+2})^!$: $\widetilde{KO}^{-j}(L_{2q}^{n+2}) \to \widetilde{KO}^{-j}(L_{2q}^n)$ is a zero-map and the homomorphism $(i_n^{n+3})^!$: $\widetilde{KO}^{-j}(L_{2q}^{n+3}) \to \widetilde{KO}^{-j}(L_{2q}^n)$ is a zero-map. q.e.d.

Remark. By the above proof, we see that (2) of (2.4) is also true for the cases $j \equiv 3 \pmod 4$, $m = n+2$ and $n+j \not\equiv 2 \pmod 8$.

Now consider the Atiyah-Hirzebruch spectral sequences for $\widetilde{KO}^*(L_{2q}^m/L_{2q}^n)$:

$$\widetilde{E}_2^{*,*} \cong \widetilde{H}^*(L_{2q}^m/L_{2q}^n;\ KO^*(pt)) \Rightarrow \widetilde{KO}^*(L_{2q}^m/L_{2q}^n).$$

Then, as mentioned in [11], we have (cf. [5] and [10])

(2.5) *The differentials*

$$d_2^{i,8k}:\ E_2^{i,8k} \to E_2^{i+2,8k-1},$$
$$d_2^{i,8k-1}:\ E_2^{i,8k-1} \to E_2^{i+2,8k-2}$$

and

$$d_3^{i,8k-2}:\ E_3^{i,8k-2} \to E_3^{i+3,8k-4}$$

are induced by $Sq^2\rho_2$, Sq^2 *and* $\beta_2 Sq^2$ *respectively, where* ρ_2 *is the reduction* mod 2, Sq^2 *is the Steenrod operation and* β_2 *is the Bockstein operation associated to the exact sequence*

$$0 \to Z \to Z \to Z_2 \to 0.$$

Using (2.1), (2.2) and (2.5), we obtain the following results, where $\alpha(m, n)$ and $\beta(m, n)$ denote non-negative integers with

$$\alpha(m, n) \leqq \begin{cases} 1 & (m \equiv 0, 1 \text{ or } 2 \pmod 4) \\ 2 & (m \equiv 3 \pmod 4)), \end{cases}$$

$$\beta(m, n) \leqq \begin{cases} 1 & (n \equiv 0 \text{ or } 2 \pmod 4) \\ 2 & (n \equiv 3 \pmod 4). \end{cases}$$

(2.6) *Assume* $m \geqq n+3$, *then we have*

(1) *If* $j \equiv 0 \pmod 4$, *then we have*

$$\operatorname{ord} \widetilde{KO}^{-j}(L_{2q}^m / L_{2q}^n) = \begin{cases} 2^{\varphi(m+j, n+j)} q^{[m/4]-[n/4]} \\ \quad (m+j \equiv 0, 1, 6 \text{ or } 7 \pmod 8) \text{ and } n+j \equiv 1, 2, 4 \text{ or } 5 \pmod 8)) \\ 2^{\varphi(m+j, n+j)-\alpha(m,n)} q^{[m/4]-[n/4]} \\ \quad (m+j \equiv 2, 3, 4 \text{ or } 5 \pmod 8) \text{ and } n+j \equiv 1, 2, 4 \text{ or } 5 \pmod 8)) \\ 2^{\varphi(m+j, n+j)-\beta(m,n)} q^{[m/4]-[n/4]} \\ \quad (m+j \equiv 0, 1, 6 \text{ or } 7 \pmod 8) \text{ and } n+j \equiv 0 \text{ or } 6 \pmod 8)) \\ 2^{\varphi(m+j, n+j)-\alpha(m,n)-\beta(m,n)} q^{[m/4]-[n/4]} \\ \quad (m+j \equiv 2, 3, 4 \text{ or } 5 \pmod 8) \text{ and } n+j \equiv 0 \text{ or } 6 \pmod 8)). \end{cases}$$

(2) *If* $j \equiv 1 \pmod 4$, *then we have*

$$\widetilde{KO}^{-j}(L_{2q}^m / L_{2q}^n) \cong \begin{cases} Z & (m \equiv 3 \pmod 4) \text{ and } n+j \equiv 3, 4 \text{ or } 5 \pmod 8)) \\ 0 & (m \not\equiv 3 \pmod 4) \text{ and } n+j \equiv 2, 3, 4, 5 \text{ or } 6 \pmod 8)), \end{cases}$$

and

$$\operatorname{ord} \widetilde{KO}^{-j}(L_{2q}^m / L_{2q}^n) = \begin{cases} 2^{2-\beta(m,n)} & (m \not\equiv 3 \pmod 4) \text{ and } n+j \equiv 0 \pmod 8)) \\ 2^{1-\beta(m,n)} & (m \not\equiv 3 \pmod 4) \text{ and } n+j \equiv 1 \text{ or } 7 \pmod 8)). \end{cases}$$

(3) *If* $j \equiv 2 \pmod 4$ *and* $n \not\equiv 1 \pmod 4$, *then we have*

$$\operatorname{ord} \widetilde{KO}^{-j}(L_{2q}^m / L_{2q}^n) = 2^{h(m+j)+h(n+j)} q^{[(m+2)/4]-[(n+2)/4]},$$

where $h\colon Z \to Z$ *is the function defined by* (1.2).

(4) *If* $j \equiv 3 \pmod 4$, *then we have*

$$\widetilde{KO}^{-j}(L_{2q}^m / L_{2q}^n) \cong \begin{cases} Z & (m+j \equiv 0 \pmod 8) \text{ and } n \not\equiv 3 \pmod 4)) \\ 0 & (m+j \equiv 5, 6 \text{ or } 7 \pmod 8)), \end{cases}$$

and

$$\operatorname{ord} \widetilde{KO}^{-j}(L_{2q}^m / L_{2q}^n) = \begin{cases} 2^{2-\alpha(m,n)} & (m+j \equiv 2 \pmod 8)) \\ 2^{1-\alpha(m,n)} & (m+j \equiv 1 \text{ or } 3 \pmod 4)). \end{cases}$$

3. Proof of Theorem 2. In this section we complete the proof of Theorem 2. Consider the exact sequence

$$(3.1) \qquad \to \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \xrightarrow{(p_n{}^m)^!} \widetilde{KO}^{-j}(L_{2q}^m) \xrightarrow{(i_n{}^m)^!} \widetilde{KO}^{-j}(L_{2q}^n) \to$$

associated to the cofibration

$$L_{2q}^n \xrightarrow{i_n{}^m} L_{2q}^m \xrightarrow{p_n{}^m} L_{2q}^m/L_{2q}^n,$$

where $i_n{}^m$ and $p_n{}^m$ are maps given in sections 2 and 1 respectively. It follows from the exactness of (3.1) by making use of (2.3), (2.6) and Lemma 2.4 that

$$(3.2) \qquad \alpha(m, n) = \beta(m, n) = 0 \ \textit{for} \ m \geqq n+3.$$

(3.3) *The homomorphism* $(p_n{}^m)^!$ *is an isomorphism for* $j \equiv 3 \pmod 4$ *and* $m \geqq n+3$.

Thus, using Remarks 1 and 2, we see

(3.4) (1) *If* $j \equiv 0 \pmod 4$ *and* $n \not\equiv 3 \pmod 4$, *then we have*

$$\operatorname{ord} \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) = 2^{\varphi(m+j,n+j)} q^{[m/4]-[n/4]}.$$

(2) *If* $j \equiv 1 \pmod 4$ *and* $m \geqq n+2$, *then we have*

$$\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \cong Z_2$$
$$(m \not\equiv 3 \pmod 4) \ \textit{and} \ n+j \equiv 1 \ \textit{or} \ 7 \pmod 8),$$

and

$$\operatorname{ord} \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) = 4$$
$$(m \not\equiv 3 \pmod 4) \ \textit{and} \ n+j \equiv 0 \pmod 8)).$$

(3) *If* $j \equiv 3 \pmod 4$ *and* $m \geqq n+2$, *then we have*

$$\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \cong \begin{cases} Z & (m \equiv 1 \pmod 4)) \\ Z_2 \oplus Z_2 & (m+j \equiv 2 \pmod 8)) \\ Z_2 & (m+j \equiv 1 \ \textit{or} \ 3 \pmod 8)) \\ 0 & (\textit{otherwise}). \end{cases}$$

Remark. (1) (3.4) implies Corollary 3.

(2) (3.3) is also true for the cases $j \equiv 3 \pmod 4$, $m = n+2$ and $n+j \not\equiv 2 \pmod 8$.

Now, consider the exact sequence

$$\widetilde{KO}^{-j}(L_{2q}^{m+1}/L_{2q}^{n}) \to \widetilde{KO}^{-j}(L_{2q}^{m}/L_{2q}^{n}) \to \widetilde{KO}^{-j+1}(S^{m+1})$$

associated to the cofibration

$$L_{2q}^{m}/L_{2q}^{n} \to L_{2q}^{m+1}/L_{2q}^{n} \to S^{m+1}.$$

Then from (3.4), we obtain

(3.5) *If* $j \equiv 1 \pmod 4$, $m \equiv 3 \pmod 4$ *and* $m \geqq n+2$, *then we have*

$$\widetilde{KO}^{-j}(L_{2q}^{m}/L_{2q}^{n}) \cong \begin{cases} Z \oplus Z_2 & (n+j \equiv 1 \text{ or } 7 \pmod 8) \\ Z & (n+j \equiv 2 \text{ or } 6 \pmod 8). \end{cases}$$

Consider the commutative diagram

$$\begin{array}{ccccc} \widetilde{KO}^{-j+1}(S^{m+1}) & = & \widetilde{KO}^{-j+1}(S^{m+1}) & & \\ \uparrow{\scriptstyle \delta_1} & & \uparrow{\scriptstyle \delta} & & \\ \widetilde{KO}^{-j}(L_{2q}^{m}/L_{2q}^{n+2}) & \xrightarrow{g} & \widetilde{KO}^{-j}(L_{2q}^{m}/L_{2q}^{n}) & \xrightarrow{f} & \widetilde{KO}^{-j}(L_{2q}^{n+2}/L_{2q}^{n}) \\ \uparrow & & \uparrow{\scriptstyle k} & & \| \\ \widetilde{KO}^{-j}(L_{2q}^{m+1}/L_{2q}^{n+2}) & \xrightarrow{g_1} & \widetilde{KO}^{-j}(L_{2q}^{m+1}/L_{2q}^{n}) & \xrightarrow{f_1} & \widetilde{KO}^{-j}(L_{2q}^{n+2}/L_{2q}^{n}) \end{array}$$

of the exact sequences, where the upper row is associated to the cofibration

$$L_{2q}^{n+2}/L_{2q}^{n} \to L_{2q}^{m}/L_{2q}^{n} \to L_{2q}^{m}/L_{2q}^{n+2}.$$

If $j \equiv 1 \pmod 4$, $m \equiv 3 \pmod 4$ and $n+j \equiv 0 \pmod 8$, then $L_{2q}^{n+2}/L_{2q}^{n} \simeq S^{n+2} \vee S^{n+1}$, $\widetilde{KO}^{-j}(L_{2q}^{m+1}/L_{2q}^{n+2}) \cong 0$ and ord $\widetilde{KO}^{-j}(L_{2q}^{m+1}/L_{2q}^{n})$ is equal to 4. This implies that f_1 is an isomorphism and the short exact sequence

$$0 \to \widetilde{KO}^{-j}(L_{2q}^{m}/L_{2q}^{n+2}) \xrightarrow{g} \widetilde{KO}^{-j}(L_{2q}^{m}/L_{2q}^{n}) \xrightarrow{f} \widetilde{KO}^{-j}(S^{n+2} \vee S^{n+1}) \to 0$$

splits. Thus we obtain

(3.6) *If* $j \equiv 1 \pmod 4$, $n+j \equiv 0 \pmod 8$ *and* $m \geqq n+2$, *then we have*

$$\widetilde{KO}^{-j}(L_{2q}^{m}/L_{2q}^{n}) \cong \begin{cases} Z \oplus Z_2 \oplus Z_2 & (m \equiv 3 \pmod 4) \\ Z_2 \oplus Z_2 & (m \not\equiv 3 \pmod 4). \end{cases}$$

Now we turn to the case $j \equiv 0 \pmod 2$ and $n+j \equiv 3 \pmod 4$. Consider the exact sequence

$$(3.7)\qquad \widetilde{KO}^{-j-1}(L_{2q}^m/L_{2q}^{n+1}) \xrightarrow{(p_{m,n})^!} \widetilde{KO}^{-j-1}(L_{2q}^m/L_{2q}^n) \xrightarrow{(i_{m,n})^!} \widetilde{KO}^{-j-1}(S^{n+1})$$
$$\xrightarrow{\delta} \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^{n+1}) \xrightarrow{(p_{m,n})^!} \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \xrightarrow{(i_{m,n})^!} \widetilde{KO}^{-j}(S^{n+1})$$
$$\xrightarrow{\delta} \widetilde{KO}^{-j+1}(L_{2q}^m/L_{2q}^{n+1}) \xrightarrow{(p_{m,n})^!} \widetilde{KO}^{-j+1}(L_{2q}^m/L_{2q}^n) \xrightarrow{(i_{m,n})^!} \widetilde{KO}^{-j+1}(S^{n+1})$$

associated to the cofibration

$$S^{n+1} \xrightarrow{i_{m,n}} L_{2q}^m/L_{2q}^n \xrightarrow{p_{m,n}} L_{2q}^m/L_{2q}^{n+1}.$$

In this case $\widetilde{KO}^{-j+1}(S^{n+1}) \cong 0$, and

$$\operatorname{rk} \widetilde{KO}^{-j+1}(L_{2q}^m/L_{2q}^{n+1}) = \operatorname{rk} \widetilde{KO}^{-j+1}(L_{2q}^m/L_{2q}^n)$$

by (2.6), (3.4), (3.5) and (3.6). Hence we have

(3.8) *If $j \equiv 0 \pmod 2$ and $n+j \equiv 3 \pmod 4$, then the image of the homomorphism*

$$\delta\colon\ \widetilde{KO}^{-j}(S^{n+1}) \to \widetilde{KO}^{-j+1}(L_{2q}^m/L_{2q}^{n+1})$$

has a finite order.

If $n+j \equiv 3 \pmod 8$, then we have

$$\widetilde{KO}^{-j-1}(S^{n+1}) \cong 0.$$

It follows from (3.8) that we obtain a short exact sequence

$$0 \to \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^{n+1}) \to \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \to Z \to 0.$$

Thus we have

(3.9) *If $j \equiv 0 \pmod 2$ and $n+j \equiv 3 \pmod 8$, then we have*

$$\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \cong \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^{n+1}) \oplus Z.$$

Assume $j \equiv 0 \pmod 4$, $n+j \equiv 7 \pmod 8$, $m \not\equiv 3 \pmod 4$ and $m \geqq n+2$. Then we have

$$\widetilde{KO}^{-j-1}(L_{2q}^m/L_{2q}^{n+1}) \cong Z_2,$$
$$\widetilde{KO}^{-j-1}(L_{2q}^m/L_{2q}^n) \cong Z_2 \oplus Z_2,$$
$$\widetilde{KO}^{-j-1}(S^{n+1}) \cong Z_2$$

by (3.4) and (3.6). This implies that, in the exact sequence (3.7), the homomorphism

$$(i_{m,n})^!: \widetilde{KO}^{-j-1}(L^m_{2q}/L^n_{2q}) \to \widetilde{KO}^{-j-1}(S^{n+1})$$

is an epimorphism, and the homomorphism

$$\delta: \widetilde{KO}^{-j-1}(S^{n+1}) \to \widetilde{KO}^{-j}(L^m_{2q}/L^{n+1}_{2q})$$

is a zero-map. Note that the homomorphism δ is also a zero-map for the cases $m=n+1$ or $m \equiv 3 \pmod 4$. It follows from (3.8) that we obtain a short exact sequence

$$0 \to \widetilde{KO}^{-j}(L^m_{2q}/L^{n+1}_{2q}) \to \widetilde{KO}^{-j}(L^m_{2q}/L^n_{2q}) \to Z \to 0.$$

Thus we obtain

(3.10) *If $j \equiv 0 \pmod 4$ and $n+j \equiv 7 \pmod 8$, then we have*

$$\widetilde{KO}^{-j}(L^m_{2q}/L^n_{2q}) \cong \widetilde{KO}^{-j}(L^m_{2q}/L^{n+1}_{2q}) \oplus Z.$$

Finally, assume $j \equiv 2 \pmod 4$, $n+j \equiv 7 \pmod 8$ and $m \geqq n+3$. Then, in the exact sequence (3.7), the homomorphism

$$(p_{m,n})^!: \widetilde{KO}^{-j-1}(L^m_{2q}/L^{n+1}_{2q}) \to \widetilde{KO}^{-j-1}(L^m_{2q}/L^n_{2q})$$

is an isomorphism by (3.3). Hence we obtain an exact sequence

$$0 \to Z_2 \to \widetilde{KO}^{-j}(L^m_{2q}/L^{n+1}_{2q}) \xrightarrow{(p_{m,n})^!} \widetilde{KO}^{-j}(L^m_{2q}/L^n_{2q}) \to Z \to 0$$

by (3.8). Thus we have

(3.11) *If $j \equiv 2 \pmod 4$, $n+j \equiv 7 \pmod 8$ and $m \geqq n+3$, then we have*

$$\widetilde{KO}^{-j}(L^m_{2q}/L^n_{2q}) \cong (\widetilde{KO}^{-j}(L^m_{2q}/L^{n+1}_{2q})/G) \oplus Z,$$

where G denotes the kernel of the homomorphism $(p_{m,n})^!$.

Now, summarizing (2.6), (3.4), (3.5), (3.6), (3.9), (3.10) and (3.11), we obtain Theorem 2. The proof is thus completed.

Additional Remark. In view of the fact that the stunted lens spaces are Thom complexes, one may think of using the *KO*-theory Thom isomorphism in the computation of $\widetilde{KO}^{-j}(L^m_{2q}/L^n_{2q})$. It turns out that there is an isomorphism

$$\widetilde{KO}^{-j}(L^m_{2q}/L^n_{2q}) \cong KO^{-j-n-1}(L^{m-n-1}_{2q}) \text{ for } n \equiv 3 \pmod 4,$$

which is a generalized form of Proposition (2.1) of [6]. Combining this isomorphism with the results in Theorem 2, one can now get further information about $\widetilde{KO}^{-j}(L^m_{2q}/L^n_{2q})$.

References

[1] J. F. ADAMS: Vector fields on spheres, Ann. of Math. 75 (1962), 603–632.

[2] D. W. ANDERSON: A new cohomology theory, Thesis, Univ. of California, Berkeley, 1964.

[3] R. BOTT: Lectures on $K(X)$, Benjamin, 1969.

[4] K. FUJII, T. Kobayashi and M. Sugawara: Stable homotopy types of stunted lens spaces, Mem. Fac. Sci. Kochi Univ. (Math.) 3 (1982), 21–27.

[5] M. FUJII: K_0-groups of projective spaces, Osaka J. Math. 4 (1967), 141–149.

[6] M. FUJII and T. YASUI: K_0-groups of the stunted real projective spaces, Math. J. Okayama Univ. 16 (1973), 47–54.

[7] T. KOBAYASHI: KO-cohomology of the lens space mod 4, Mem. Fac. Sci. Kochi Univ. (Math.) 6 (1985), 45–63.

[8] T. KOBAYASHI: KO-cohomology of the lens space mod 8, Mem. Fac. Sci. Kochi Univ. (Math.) 7 (1986) 33–57.

[9] S. KÔNO and A. TAMAMURA: J-groups of the suspensions of the stunted lens spaces mod p, Osaka J. Math. 24 (1987), 481–498.

[10] C. R. F. MAUNDER: The spectral sequence of an extraordinary cohomology theory, Proc. Cambridge Philos. Soc. 59 (1963), 567–574.

[11] M. YASUO: On the KO-cohomology of the lens space $L^n(q)$ for q even, Mem. Fac. Sci. Kyushu Univ. Ser. A 32 (1978), 153–163.

DEPARTMENT OF APPLIED MATHEMATICS
OKAYAMA UNIVERSITY OF SCIENCE
DEPARTMENT OF MATHEMATICS
OSAKA UNIVERSITY

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