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# THE TAYLOR COEFFICIENTS OF $\zeta(s)$, $(s-1)\zeta(s)$ AND $(z/(1-z))\zeta(1/(1-z))$

MASUMI NAKAJIMA

**1. Introduction.** The coefficients $\Gamma_k$ in the Laurent expansion of the Riemann zeta-function at $s=1$

$$\zeta(s)=\frac{1}{s-1}+\sum_{k=0}^{\infty}\Gamma_k(s-1)^k \tag{1}$$

have been considered by many mathematicians. (Stieltjes [4], Briggs and Chowla [1], Berndt [2], Israilov [3], recently Matsuoka [5] and other authors. See references in [2, 3] and Ivić [4], p. 49.) Among them Briggs and Chowla, Berndt, Israilov and Matsuoka have made the interesting contributions to the estimation of $|\Gamma_k|$. In this paper, we consider the coefficients $D_k'$, $D_k$ and $c_k$ in the Taylor expansions of $\zeta(s)$, $(s-1)\zeta(s)$ and $(z/(1-z))\zeta(1/(1-z))$ at any point $s_0\neq 1$ and at $z=0$ respectively:

$$\zeta(s)=\sum_{k=0}^{\infty}D_k'(s-s_0)^k,\ (s_0\neq 1) \tag{2}$$

$$(s-1)\zeta(s)=\sum_{k=0}^{\infty}D_k(s-s_0)^k,\ (s_0\neq 1) \tag{3}$$

$$(z/(1-z))\zeta(1/(1-z))=\sum_{k=0}^{\infty}c_k z^k,\ (|z|<1) \tag{4}$$

(4) is the transformation of $(s-1)\zeta(s)$ from the complex $s$-plane to the complex $z$-plane by the Möbius transformation $s=1/(1-z)$ which transforms the right half $s$-plane $\{s:\ \mathrm{Re}\ s>1/2\}$ into the $z$-disk $\{z:\ |z|<1\}$. Note that the Riemann hypothesis is true if and only if (4) has no zero in the unit disk $|z|<1$. ([9])

In § 2, we shall give the explicit expressions of $D_k'$, $D_k$ and $c_k$ which are our generalizations of the Stieltjes formula [4]. The Stieltjes formula has seemed to be thought as the special case that such a formula exists because the point $s=1$ is the only pole of $\zeta(s)$. In [2], Berndt states the estimate:

$$|\Gamma_k|\leqq\{3+(-1)^k\}/k\pi^k\leqq 4/k\pi^k \tag{5}$$

but it can be easily seen that his argument leads to the following result:

$$|\Gamma_k| \leqq 4(2\pi)^{-N} N^{-k-1} \prod_{j=1}^{N} (N+j) \tag{6}$$

for any integer $N$ such that $1 \leqq N \leqq k$. In particular, if we take $N = [k/2]$ where [ ] is the Gauss symbol in (6), then

$$|\Gamma_k| \leqq 4\pi^{-[k/2]}[k/2]^{-[k/2]} \prod_{j=1}^{[k/2]} \{([k/2]+j)(2[k/2])^{-1}\} \tag{7}$$

$$\leqq 4[k/2]^{-1}\{\pi[k/2]\}^{-[k/2]} \tag{8}$$

In § 3, we shall also give the estimation for the value $|D'_k|$, $|D_k|$ and $|c_k|$ by the argument similar to that of Berndt [2]. As for the estimation of Taylor's coefficients of $\zeta(s)$, Mitrovič [6, 7] showed the following result:

$$|D'_k| \leqq (\sigma_0-1)^{-k-1}, \ (k \geqq 0) \tag{9}$$

where $\sigma_0 = \mathrm{Re}\ s_0$ and $\sigma_0 > 1$. Our result in case $\sigma_0 > 1$ is sharper than that of Mitrovič.

We use hereafter the following notations:

The letter $a$ always takes the value 1 or 0.

$\binom{n}{r}$ denotes the binomial coefficient, $f_k(u) = (k!)^{-1}u^k e^u = \sum_{r=k}^{\infty} \binom{r}{k}(r!)^{-1}u^r$, $f^{(n)}_k(u) = (d^n/du^n)f_k(u)$, $s_0 = 1+s_1$, $g_{(a),k}(u) = u^{-1}f^{(a)}_k(-s_1 \log u)$, $g^{(n)}_{(a),k}(u) = (d^n/du^n)g_{(a),k}(u)$, $o(\ )$ denotes Landau's small $o$ symbol, $O(\ )$ denotes Landau's large $O$ symbol, $c_\nu(j)$ denotes the Stirling number of the first kind which is defined by the relation:

$$(z-1)(z-2)\cdots(z-\nu) = \sum_{j=0}^{\nu} c_\nu(j) z^j,$$

$$c_0(0) = 1, \ c_\nu(-1) = c_\nu(\nu+1) = 0.$$

Our $c_\nu(j)$ is equal to $a_{j+1}^{(\nu+1)}$ in Berndt [2] and to $(-1)^{\nu+j}\nu!\ b_{j,\nu}$ in Israilov [3]. $B_n(u)$ denotes the Bernoulli polynomial which is defined by

$$ze^{zu}/(e^z-1) = \sum_{n=0}^{\infty} (n!)^{-1}B_n(u)z^n, \ (|z| < 2\pi)$$

$B_n$ denotes the Bernoulli number which is equal to $B_n(0)$. $P_n(u) = (n!)^{-1} \cdot B_n(\{u\})$ where $\{u\} = u-[u]$, $[u]$ is the Gauss symbol which denotes the greatest integer not exceeding $u$. $h_r(u) = u^{-1}(\log u)^r$, $h^{(m)}_r(u) = (d^m/du^m) \cdot h_r(u)$,

$$B_{(k)}(n) = \sum_{r=0}^{\min(n,k)} \binom{n}{r} f_{k-r}(-s_1 \log u)$$

$L_k(u)$ denotes the Laguerre polynomial which is defined by

$$L_k(u) = \sum_{r=0}^{k} \binom{k}{r} (-1)^r (r!)^{-1} u^r.$$

$\alpha = \mathrm{Re}\, s_0 - 1$. $s_0 = 1 + s_1$. $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{C}$ and $\mathbf{R}$ denote the set of natural numbers, integers, complex numbers and real numbers respectively.

**2. The explicit expressions of $D'_k$, $D_k$ and $c_k$.** $\zeta(s)$ can be expanded at any point $s = s_0$, $(s_0 \neq 1)$ through (1) as follows:

$$\zeta(s) = \sum_{k=0}^{\infty} \left\{ -(1-s_0)^{-k-1} + \sum_{n=k}^{\infty} \binom{n}{k} \Gamma_n (s_0-1)^{n-k} \right\} (s-s_0)^k, \qquad (10)$$
$$(|s-s_0| < |1-s_0|)$$

so that $D'_k$ in (2) is

$$D'_k = -(1-s_0)^{-k-1} + D^{(0)}{}_k \qquad (11)$$

where

$$D^{(0)}{}_k = \begin{cases} \sum_{n=k}^{\infty} \binom{n}{k} \Gamma_n s_1{}^{n-k}, \ (k \geqq 1) \\ \sum_{n=0}^{\infty} \Gamma_n s_1{}^n = \zeta(s_0), \ (k=0) \end{cases}$$
$$s_0 = 1 + s_1$$

And similarly we have

$$D_k = D^{(1)}{}_k = \begin{cases} \sum_{n=k}^{\infty} \binom{n}{k} \Gamma_{n-1} s_1{}^{n-k}, \ (k \geqq 1) \\ 1 + \sum_{n=0}^{k} \Gamma_n s_1{}^{n+1} = (s_0-1)\zeta(s_0), \ (k=0) \end{cases} \qquad (12)$$

and

$$c_k = \begin{cases} \sum_{n=0}^{k-1} \binom{k-1}{n} \Gamma_{k-1-n} = \sum_{n=0}^{k-1} \binom{k-1}{n} \Gamma_n, \ (k \geqq 1) \\ 1, \ (k=0) \end{cases} \qquad (13)$$

Next we consider the following sum:

$$\sum_{n=1}^{x} g_{(\alpha),k}(n), \ (k \geqq 1) \qquad (14)$$

Before the calculation of (14), we need the following two lemmas. As for the Stieltjes formula [4]:

$$\gamma_r = (-1)^r r!\,\Gamma_r = \lim_{x\to\infty}\left\{\sum_{n=1}^{x} n^{-1}(\log n)^r - \int_1^x u^{-1}(\log u)^r du\right\} \quad (15\text{-a})$$

or

$$\sum_{n=1}^{x} n^{-1}(\log n)^r - \int_1^x u^{-1}(\log u)^r du = \gamma_r + E(r, x), \quad (15\text{-b})$$

$$E(r, x) = o(1)$$

we have

**Lemma 1.** *For the above $E(r, x)$, we have*

$$E(r, x) = (1/2)x^{-1}(\log x)^r + \sum_{m=2}^{N}(-1)^m(B_m/m!)h^{(m-1)}{}_r(x)$$
$$+(-1)^N\int_x^\infty P_N(u)h^{(N)}{}_r(u)du, \ (r \geqq 0,\ N \geqq 1) \quad (16)$$

*where*

$$h^{(m)}{}_r(u) = u^{-m-1}\sum_{j=0}^{\min(m,r)} c_m(j)r(r-1)$$
$$\cdots(r-j+1)(\log u)^{r-j}, \ (m \geqq 0,\ r \geqq 0) \quad (17)$$

*Proof.* An easy calculation gives (17) (see Israirov [3]). And the Euler-Maclaurin summation formula yields

$$\sum_{n=1}^{x} h_r(n) - \int_1^x h_r(u)du = (1/2)h_r(x) + \sum_{m=2}^{N}(-1)^m(B_m/m!)h^{(m-1)}{}_r(x)$$
$$-\sum_{m=r+1}^{N}(-1)^m(B_m/m!)h^{(m-1)}{}_r(1)$$
$$+(-1)^{N+1}\int_1^x P_N(u)h^{(N)}{}_r(u)du \quad (18)$$

which shows

$$\gamma_r = -\sum_{m=r+1}^{N}(-1)^m(B_m/m!)h^{(m-1)}{}_r(1)$$
$$+(-1)^{N+1}\int_1^\infty P_N(u)h^{(N)}{}_r(u)du \quad (19)$$

We substitute (19) and (18) for $\gamma_r$ in (15-b) and the left-hand side of (15-b) respectively, then we have the lemma.

**Lemma 2.**

$$g^{(m)}_{(a),k}(u) = u^{-m-1}\sum_{r=0}^{m} c_m(r)B_{(k)}(r+a)(-s_1)^r,\ (m \geqq 0,\ k \geqq 1) \quad (20)$$

*Proof.* This lemma is proved by induction on $m$ by using the properties that $(d/du)f_k(u) = f_k(u)+f_{k-1}(u)$, $(k \geqq 1)$, $(d/du)B_{(k)}(n) = -s_1u^{-1}B_{(k)}(n+1)$, $(k \geqq 1)$ and $c_{m+1}(j) = c_m(j-1)+(-m-1)c_m(j)$.

By an easy calculation and applying (15) to (14), we have

$$\begin{aligned}\sum_{n=1}^{x} g_{(a),k}(n) &= \sum_{n=1}^{x} n^{-1}\left\{\sum_{r=k}^{\infty}\binom{r}{k}(-1)^{r-a}\{(r-a)!\}^{-1}(s_1\log n)^{r-a}\right\}\\ &= \sum_{r=k}^{\infty}\binom{r}{k}s_1^{\,r-a}(-1)^{r-a}\{(r-a)!\}^{-1}\sum_{n=1}^{x} n^{-1}(\log n)^{r-a}\\ &= \int_1^x g_{(a),k}(u)du+s_1^{\,k-a}D^{(a)}_k+S(a,k) \end{aligned}\quad (21)$$

where $S(a,k) = \sum_{r=k}^{\infty}\binom{r}{k}s_1^{\,r-a}(-1)^{r-a}\{(r-a)!\}^{-1}E(r-a,x)$.

As for the above $S(a,k)$, we have

**Lemma 3.**

$$S(a,k) = \begin{cases} o(1),\ (\mathrm{Re}\ s_0 > 0)\\ (2x)^{-1}\sum_{v=0}^{a}\binom{a}{v}f_{k-v}(-s_1\log x)+o(1),\\ \qquad (-1 < \mathrm{Re}\ s_0 \leqq 0)\\ (2x)^{-1}\sum_{v=0}^{a}\binom{a}{v}f_{k-v}(-s_1\log x)\\ \quad +\sum_{m=2}^{M}(-1)^m(B_m/m!)g^{(m-1)}_{(a),k}(x)+o(1),\\ \qquad (-M < \mathrm{Re}\ s_0 \leqq -(M-1))\ \textit{where}\ M \in \mathbb{N} \end{cases}$$

*Proof.* From Lemma 1 and Lemma 2,

$$\begin{aligned} S(a,k) &= (2x)^{-1}f^{(a)}_k(-s_1\log x)\\ &\quad +\sum_{m=2}^{N}(-1)^m(B_m/m!)g^{(m-1)}_{(a),k}(x)\end{aligned}$$

$$+(-1)^{N}\int_{x}^{\infty}P_{N}(u)g^{(N)}_{(a,k)}(u)du$$

$$=(2x)^{-1}\sum_{v=0}^{a}\binom{a}{v}f_{k-v}(-s_1\log x)+\sum_{m=2}^{N}(-1)^{m}(B_m/m!)$$

$$x^{-m}\sum_{r=0}^{m-1}c_{m-1}(r)\sum_{v=0}^{\min(k,r+a)}\binom{r+a}{v}f_{k-v}(-s_1\log x)(-s_1)^{r}$$

$$+(-1)^{N}\int_{x}^{\infty}P_{N}(u)(d^{N}/du^{N})g_{(a),k}(u)du$$

Noting that $f_k(-s_1\log x) = O(x^{-\sigma_1+\varepsilon})$, ($\varepsilon$ is an arbitrary small positive number, $\sigma_1 = \mathrm{Re}\ s_1$) and

$$\left|\int_{1}^{\infty}P_{N}(u)g^{(N)}_{(a),k}(u)du\right| \leqq \int_{1}^{\infty}|g^{(N)}_{(a),k}(u)|du < +\infty,$$

we have

$$S(a,k) = O(x^{-\sigma_0+\varepsilon})+\sum_{m=2}^{N}O(x^{-(m-1)-\sigma_0+\varepsilon})+o(1),\ (\sigma_0 = \mathrm{Re}\ s_0)$$

and have the desired result.

From (21) and Lemma 3, we obtain the following

**Theorem 1.**(*A generalization of the Stieltjes formula*). *If* Re $s_0 > 0$, *then*

$$D'_k = -(-s_1)^{-k-1}+D^{(0)}_k$$

$$= -(-s_1)^{-k-1}+s_1^{-k}\lim_{x\to\infty}\left\{\sum_{n=1}^{x}n^{-1}f_k(-s_1\log n)\right.$$

$$\left.-\int_{1}^{x}u^{-1}f_k(-s_1\log u)du\right\}, \tag{22-a}$$

*and*

$$D_k = D^{(1)}_k$$

$$= s_1^{-k+1}\lim_{x\to\infty}\left\{\sum_{n=1}^{x}n^{-1}f^{(1)}_k(-s_1\log n)\right.$$

$$\left.-\int_{1}^{x}u^{-1}f^{(1)}_k(-s_1\log u)du\right\}, \tag{22-b}$$

*If* $-1 <$ Re $s_0 \leqq 0$, *then*

$$D'_k = -(-s_1)^{-k-1}+D^{(0)}_k$$

$$= -(-s_1)^{-k-1} + s_1^{-k} \lim_{x\to\infty} \left\{ \sum_{n=1}^{x} n^{-1} f_k(-s_1 \log n) \right.$$

$$\left. - \int_1^x u^{-1} f_k(-s_1 \log u) du - (2x)^{-1} f_k(-s_1 \log x) \right\} \qquad (23\text{-a})$$

*and*

$$D_k = D^{(1)}{}_k$$

$$= s_1^{-k-1} \lim_{x\to\infty} \left\{ \sum_{n=1}^{x} n^{-1} f^{(1)}{}_k(-s_1 \log n) \right.$$

$$\left. - \int_1^x u^{-1} f^{(1)}{}_k(-s_1 \log u) du - (2x)^{-1} f^{(1)}{}_k(-s_1 \log x) \right\} \qquad (23\text{-b})$$

*And if* $-M < \mathrm{Re}\ s_0 \leqq -(M-1)$, $M \in \mathbf{N}$, $M \geqq 2$, *then*

$$D_k = -(-s_1)^{-k-1} + D^{(0)}{}_k$$

$$= -(-s_1)^{-k-1} + s_1^{-k} \lim_{x\to\infty} \left\{ \sum_{n=1}^{x} n^{-1} f_k(-s_1 \log n) \right.$$

$$- \int_1^x u^{-1} f_k(-s_1 \log u) du - (2x)^{-1} f_k(-s_1 \log x)$$

$$\left. - \sum_{m=2}^{M} (-1)^m (B_m/m!) g^{(m-1)}{}_{(0),k}(x) \right\} \qquad (24\text{-a})$$

*and*

$$D_k = D^{(1)}{}_k$$

$$= s_1^{-k+1} \lim_{x\to\infty} \left\{ \sum_{n=1}^{x} n^{-1} f^{(1)}{}_k(-s_1 \log n) \right.$$

$$- \int_1^x u^{-1} f^{(1)}{}_k(-s_1 \log u) du - (2x)^{-1} f^{(1)}{}_k(-s_1 \log x)$$

$$\left. - \sum_{m=2}^{M} (-1)^m (B_m/m!) g^{(m-1)}{}_{(1),k}(x) \right\} \qquad (24\text{-b})$$

Note the following

**Remark 1.** When $\mathrm{Re}\ s_0 > 0$,

$$D^{(0)}{}_k = (-1)^k \{k!\}^{-1} \lim_{x\to\infty} \left\{ \sum_{n=1}^{x} n^{-s_0} (\log n)^k \right.$$

$$\left. - \int_1^x u^{-s_0} (\log u)^k du \right\}, \qquad (25\text{-a})$$

and

$$D^{(1)}{}_k = s_1 D^{(0)}{}_k + D^{(0)}{}_{k-1}.$$

When $-1 < \operatorname{Re} s_0 \leqq 0$,

$$D^{(0)}{}_k = (-1)^k \{k!\}^{-1} \lim_{x\to\infty} \left\{ \sum_{n=1}^{x} n^{-s_0} (\log n)^k - \int_1^x u^{-s_0} (\log u)^k du - (2x)^{-1} (\log x)^k \right\}, \tag{25-b}$$

and

$$D^{(1)}{}_k = s_1 D^{(0)}{}_k + D^{(0)}{}_{k-1}$$

We have the similar result for $\operatorname{Re} s_0 \leqq -1$, but in this case, the expression of $D^{(0)}{}_k$ is more complicated.

A similar calculation gives the following

**Theorem 2.**(*Another generalization of the Stieltjes formula*). *For* $c_{k+1}$ *in* (4),

$$c_{k+1} = \lim_{x\to\infty} \left\{ \sum_{n=1}^{x} n^{-1} L_k(\log n) - \int_1^x u^{-1} L_k(\log u) du \right\} \tag{26}$$

*where* $L_k(\ )$ *denotes the Laguerre polynomial.*

*Proof.* Instead of (14), we start from the following sum:

$$\sum_{n=1}^{x} n^{-1} L_k(\log n)$$

and we omit the calculation which is the same as that of Theorem 1. (In fact the calculation is simpler than that of Theorem 1, for $L_k$ is the finite sum.) The expressions (22), (23), (24), (26) are the generalizations of the Stieltjes formula (15).

**3. The estimation of $|D^{(a)}{}_k|$ and $|c_k|$.** To estimate $|D^{(a)}{}_k|$, we need the following

**Remark 2.**

$$g^{(m)}{}_{(a),k}(x) = O(x^{-m-\sigma_0+\varepsilon}),\ \sigma_0 = \operatorname{Re} s_0,\ (m \geqq 0,\ k \geqq 1)$$
$$g^{(m)}{}_{(a),k}(1) = 0,\ (0 \leqq m \leqq k-1-a)$$

which can be seen from Lemma 2.

**Theorem 3.** *For* $s_0 \neq 1$, $-M < \operatorname{Re} s_0$, $1 \leqq M \leqq N \leqq k-a$, $k \geqq 1$, $M \in \mathbf{N}$

$$D^{(a)}_k = (-1)^{N+1} s_1^{a-k} \int_1^{\infty} P_N(u) g^{(N)}_{(a),k}(u)\,du \tag{27}$$

*Proof.* We apply the Euler-Maclaurin summation formula to the sum (14) and use Remark 2 and Theorem 1 to obtain (27). Say, for $\operatorname{Re} s_0 > 0$,

$$\begin{aligned} D^{(a)}_k &= s_1^{a-k} \lim_{x\to\infty} \left\{ \sum_{n=1}^{x} g_{(a),k}(n) - \int_1^x g_{(a),k}(u)\,du \right\} \\ &= s_1^{a-k} \lim_{x\to\infty} \Big\{ (1/2)(g_{(a),k}(1) + g_{(a),k}(x)) \\ &\quad + \sum_{m=2}^{N} (-1)^m (B_m/m!) g^{(m-1)}_{(a),k}(u) \Big|_{u=1}^{u=x} \\ &\quad + (-1)^{N+1} \int_1^x P_N(u) g^{(N)}_{(a),k}(u)\,du \Big\} \end{aligned}$$

For $-M < \operatorname{Re} s_0 \leqq -(M-1)$, $(M \geqq 1)$

$$\begin{aligned} D^{(a)}_k &= s_1^{a-k} \lim_{x\to\infty} \Big\{ \sum_{n=1}^{x} g_{(a),k}(n) \\ &\quad - \int_1^x g_{(a),k}(u)\,du - (1/2) g_{(a),k}(x) \\ &\quad - \sum_{m=2}^{M} (-1)^m (B_m/m!) g^{(m-1)}_{(a),k}(x) \Big\} \\ &= s_1^{a-k} \lim_{x\to\infty} \Big\{ (1/2) g_{(a),k}(1) \\ &\quad + \sum_{m=2}^{N} (-1)^m (B_m/m!) g^{(m-1)}_{(a),k}(u) \Big|_{u=1}^{u=x} \\ &\quad - \sum_{m=2}^{M} (-1)^m (B_m/m!) g^{(m-1)}_{(a),k}(x) \\ &\quad + (-1)^{N+1} \int_1^x P_N(u) g^{(N)}_{(a),k}(u)\,du \Big\} \end{aligned}$$

Hence, when $x$ tends to infinity, the desired result follows.

From Theorem 3, we can estimate $|D^{(a)}_k|$ as follows.

**Theorem 4.** *For every fixed* $s_0 \neq 1$, $-M < \operatorname{Re} s_0$, $1 \leqq M \leqq N \leqq k-a$, $M \in \mathbf{N}$,

$$|D^{(a)}{}_k| \leqq \{|s_1|+N+\alpha|^a 4(2\pi)^{-N}(N+\alpha)^{-k-1}\prod_{r=1}^{N}(|s_1|+N+\alpha+r), \quad (28)$$

*where* $\alpha=$ Re $s_0-1$. *In particular, if we choose* $N=[k/2]$, *then*

$$|D^{(0)}{}_k| \leqq 4\pi^{-[k/2]}([k/2]+\alpha)^{-k-1+[k/2]}\prod_{r=1}^{[k/2]}\{(|s_1|+[k/2]+\alpha+r)$$
$$/(2[k/2]+2\alpha)\}$$
$$=4(\pi/b)^{-[k/2]}([k/2]+\alpha)^{-k-1+[k/2]}\prod_{r=1}^{[k/2]}\{(|s_1|+[k/2]+\alpha+r)$$
$$/(2b[k/2]+2\alpha b)\}, \quad (29\text{-a})$$
$$|D_k| \leqq 4\pi^{-[k/2]}([k/2]+\alpha)^{-k-1+[k/2]}\prod_{r=0}^{[k/2]}\{(|s_1|+[k/2]+\alpha+r)$$
$$/(2[k/2]+2\alpha)\}$$
$$=4(\pi/b)^{-[k/2]}([k/2]+\alpha)^{-k-1+[k/2]}\prod_{r=0}^{[k/2]}\{(|s_1|+[k/2]+\alpha+r)$$
$$/(2b[k/2]+2\alpha b)\}, \quad (29\text{-b})$$

*where* $b$ *is an arbitrary positive number. And if* $k \geqq |s_1|/(b-1)+2$, $1<b<\pi$, *then*

$$|D^{(0)}{}_k| \leqq 4(\pi/b)^{-[k/2]}\{[k/2]+\alpha\}^{-k-1+[k/2]}, \quad (30\text{-a})$$

$$|D_k| = |D^{(1)}{}_k| \leqq 4(\pi/b)^{-[k/2]}\{[k/2]+\alpha\}^{-k+[k/2]}, \quad (30\text{-b})$$

*Proof.* From (27) and Remark 2,

$$D^{(a)}{}_k=(-1)^{N+1}s_1{}^{a-k}\sum_{r=0}^{N}\sum_{j}^{a+r}\binom{a+r}{j}(-s_1)^r c_N(r)\cdot$$
$$\int_1^\infty \{(k-j)!\}^{-1}P_N(u)(-s_1\log u)^{k-j}u^{-N-s_0}du$$

The above equality leads to the following estimation:

$$|D^{(a)}{}_k| \leqq |s_1|^{a-k}\sum_{r=0}^{N}\sum_{j=0}^{a-r}\binom{a+r}{j}|s_1|^{k+r-j}|c_N(r)|4(2\pi)^{-N}\cdot$$
$$\int_1^\infty \{(k-j)!\}^{-1}u^{-N-1-\alpha}(\log u)^{k-j}du,$$

because $|P_N(u)| \leqq 4(2\pi)^{-N}$. Since the integral in the right-hand side of the above inequallity is equal to $(N+\alpha)^{-k+j-1}$,

$$|D^{(a)}{}_k| \leqq |s_1|^a 4(2\pi)^{-N}\sum_{r=0}^{N}|c_N(r)||s_1|^r\cdot$$

$$\sum_{j=0}^{a+r}\binom{a+r}{j}|s_1|^{-j}(N+\alpha)^j(N+\alpha)^{-k-1}$$
$$= |s_1|^a 4(2\pi)^{-N}\sum_{r=0}^{N}|c_N(r)|\,|s_1|^r\cdot$$
$$|1+(N+\alpha)/|s_1||^{a+r}(N+\alpha)^{-k-1}$$
$$= (|s_1|+N+\alpha)^a 4(2\pi)^{-N}(N+\alpha)^{-k-1}\prod_{r=1}^{N}||s_1|+N+\alpha+r|$$

by using the definition of $c_N(r)$. This completes the proof.

**Remark 3.** If we use the expression in Remark 1, we can have the better estimation of $|D^{(0)}{}_k|$ applying the Euler-Maclaurin summation formula to $\sum_{n=1}^{x} n^{-s_0}(\log n)^k$, say

$$|D^{(0)}{}_k| \leqq 4(2\pi)^{-N}(N+\alpha)^{-k-1}\prod_{j=0}^{N-1}\{|N+\alpha|+|s_0+j|\},\ (N\leqq k) \quad (31)$$

We also have some expressions of $c_k$ (Theorem 5). For the proof of the theorem, we use the following two lemmas.

**Lemma 4.**

$$(d^m/du^m)\{u^{-1}L_k(\log u)\}$$
$$= u^{-m-1}\sum_{j=0}^{\min(m,k)} c_m(j)\sum_{r=j}^{k}\binom{k}{r}\{(r-j)!\}^{-1}(\log u)^{r-j}(-1)^r,\ (m\geqq 1)$$

*Proof.* This lemma is also proved by induction on $m$.

**Lemma 5.**

$$(d^m/dx^m)\{x^{-1}L_k(\log x)\} = o(1),\ (m\geqq 0)$$

*Proof.* Obvious.
From Lemma 5, the following theorem is to be proved.

**Theorem 5.**

$$c_{k+1} = \sum_{n=1}^{h-1} n^{-1}L_k(\log n)$$
$$-\int_0^{\log h} L_k(t)dt+(1/2)h^{-1}L_k(\log h)$$

$$+\sum_{m=2}^{N}(-1)^m h^{-m}(B_m/m!)\sum_{j=0}^{\min(m-1,k)} c_{m-1}(j)\cdot$$
$$\sum_{r=j}^{k}\binom{k}{r}\{(r-j)!\}^{-1}(\log h)^{r-j}(-1)^r$$
$$+(-1)^{N+1}\int_h^\infty P_N(u)u^{-N-1}\sum_{j=0}^{\min(N,k)} c_N(j)\cdot$$
$$\sum_{r=j}^{k}\binom{k}{r}\{(r-j)!\}^{-1}(\log u)^{r-j}(-1)^r du,\ (N\geqq 1,\ h\geqq 1) \quad (32)$$

*Especially, for* $h=1$,

$$c_{k+1}=(1/2)+\sum_{m=2}^{N}(-1)^m(B_m/m!)\sum_{j=0}^{\min(m-1,k)} c_{m-1}(j)\binom{k}{j}(-1)^j$$
$$+(-1)^{N+1}\int_1^\infty P_N(u)u^{-N-1}\sum_{j=0}^{\min(N,k)} c_N(j)\cdot$$
$$\sum_{r=j}^{k}\binom{k}{r}\{(r-j)!\}^{-1}(\log u)^{r-j}(-1)^r du,\ (N\geqq 1) \quad (33)$$

*In particular,*

$$c_{k+1}=(1/2)+\int_1^\infty \{u\}u^{-2}L'_{k+1}(\log u)du$$
$$=(1/2)+\int_0^\infty \{e^t\}e^{-t}L'_{k+1}(t)dt, \quad (34)$$

*where* $\{u\}=u-[u]$, $[\ ]$ *is Gauss's symbol and* $L'_n(x)=(d/dx)L_n(x)$.

*Proof.* The proof is the same as that of Theorem 3. (Note that $(d/dx)\cdot\{L_n(x)-L_{n+1}(x)\}=L_n(x)$)

As for the estimation of $|c_k|$, the result is not so good as that of $|D^{(a)}_k|$. (From (13), the trivial result that $|c_k|\leqq 2^k$ is derived.) so we conclude with the following two

**Conjecture 1.**

$$c_k=O(1)$$

**Conjecture 2.**

$$c_{k+1}-c_k>0 \text{ for all integers } k\geqq 1$$

*in other words,*

$$\int_1^\infty \{u\}u^{-2}L_k(\log u)du<0 \text{ for all integers } k\geqq 1.$$

The similar results for some other Dirichlet series and sharper results depending on $s_0$, as for the upper estimates of the Taylor coefficients of $\zeta(s)$ by using the complex integral and the saddle point method can be obtained. These results will be treated in the next paper [8].

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DEPARTMENT OF MATHEMATICS
RIKKYO UNIVERSITY
IKEBUKURO, TOKYO 171, JAPAN

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