

ON AN EXPONENTIAL SAM INVOLVING THE ARITHMETIC FUNCTION $\sigma_a(n)$

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1. Introduction. Let $-1 < a \leq 0$ and $\sigma_a(n) = \sum_{d|n} d^a$ be the sum of the a -th powers of positive divisors of the positive integer n , so that $\sigma_0(n) = d(n)$. Jutila [4] has investigated the exponential sum

$$D(X; h/k) = \sum'_{n \leq X} d(n) \cdot e(hn/k)$$

for large X , where h and k are co-prime integers with $1 \leq k$, $e(t) = e^{2\pi it}$, and the symbol $\sum'_{n \leq X}$ denotes that if X is an integer, then the term corresponding to X is to be halved. Let $s = \sigma + it$ be a complex variable, and

$$E(s; h/k) = \sum_{n=1}^{\infty} d(n) \cdot e(hn/k) \cdot n^{-s}, \quad \text{Re}(s) > 1.$$

The function $E(s; h/k)$ can be analytically continued to a meromorphic function in the whole complex plane (Estermann [2]). We put

$$(1.1) \quad \begin{aligned} \Delta(X; h/k) &= D(X; h/k) - k^{-1} \cdot (\log X + 2\gamma - 1 - 2 \cdot \log k) \cdot X - E(0; h/k), \end{aligned}$$

where γ is Euler's constant. Using the truncated Voronoi summation formula, Jutila [4] proved that if $1 \leq k \leq X$, and N is a positive integer such that $1 \leq N \ll X$, then

$$(1.2) \quad \begin{aligned} \Delta(X; h/k) &= (\pi\sqrt{2})^{-1} \cdot k^{\frac{1}{2}} \cdot X^{\frac{1}{2}} \cdot \sum'_{n \leq N} d(n) \cdot e(-\bar{h}n/k) \cdot n^{-\frac{3}{2}} \cdot \cos\left(4\pi k^{-1}n^{\frac{1}{2}}X^{\frac{1}{2}} - \frac{1}{4}\pi\right) \\ &\quad + O(k \cdot N^{-\frac{1}{2}} \cdot X^{\frac{1}{2} + \epsilon}), \end{aligned}$$

where the class

$$(1.3) \quad \bar{h} \pmod{k}$$

is defined by $h\bar{h} \equiv 1 \pmod{k}$.

The first purpose of this paper is to derive a formula of the Voronoi type (Theorem 1) for the exponential sum

$$(1.4) \quad D_a(X; h/k) = \sum'_{n \leq X} \sigma_a(n) \cdot e(hn/k).$$

Theorem 2 gives an asymptotic formula for the mean square of the error term in the asymptotic formula for $D_a(X; h/k)$. Jutila's proof of the above formula (1.2) can be modified so as to give an analogous expression for $D_a(X; h/k)$, and such a modification is the basis of our proof.

We may appeal to an analogy between the function $E(s; h/k)$ and

$$(1.5) \quad E_a(s; h/k) = \sum_{n=1}^{\infty} \sigma_a(n) \cdot e(hn/k) \cdot n^{-s}, \quad \text{Re}(s) > 1.$$

The basic properties of the function $E_a(s; h/k)$ are given in Lemma 1. In what follows, ε is taken an arbitrarily small positive number, and not necessarily the same in each occurrence. The constants implied by the symbols \ll and $O(\)$ always depend at most on ε and a . $\zeta(s)$ is the Riemann zeta-function.

To formulate the analogue of (1.1) for $D_a(X; h/k)$, we define that if $a = 0$, then

$$(1.6) \quad \Delta_0(X; h/k) = \Delta(X; h/k),$$

and if $-1 < a < 0$, then

$$(1.7) \quad \begin{aligned} \Delta_a(X; h/k) &= D_a(X; h/k) - k^{a-1} \cdot \zeta(1-a) \cdot X - k^{-1-a} \cdot (1+a)^{-1} \cdot \zeta(1+a) \cdot X^{1-a} \\ &\quad - E_a(0; h/k), \end{aligned}$$

which are the "right" analogy of (1.1) for $D_a(X; h/k)$. Then we have the following

Theorem 1. For $-1 < a \leq 0$, $k \leq X$, and $1 \leq N \ll X$ we have

$$(1.8) \quad \begin{aligned} \Delta_a(X; h/k) &= (\pi\sqrt{2})^{-1} \cdot k^{\frac{1}{2}} \cdot X^{\frac{1}{2}+a} \cdot \sum_{n \leq N} \sigma_a(n) \cdot e(-\bar{h}n/k) \cdot n^{-\frac{1}{2}-a} \cdot \cos\left(4\pi k^{-1} n^{\frac{1}{2}} X^{\frac{1}{2}} - \frac{1}{4}\pi\right) \\ &\quad + O(k \cdot N^{-\frac{1}{2}} \cdot X^{\frac{1}{2}+\varepsilon}). \end{aligned}$$

Remark 1. As an immediate corollary of (1.8) with $N = k^{2/13-2a} \cdot X^{1-2a/13-2a}$ we have

$$(1.9) \quad \Delta_a(X; h/k) \ll k^{2-2a/13-2a+\varepsilon} \cdot X^{1/13-2a+\varepsilon}$$

for $k \leq X$, and $-1 < a \leq 0$.

Remark 2. In case $k = 1$, other formulas of the Voronoi type were

studied first by Wigert [9], next by Cramér [1] and later by Oppenheim [5].

A plausible conjecture as to $\Delta_a(X; h/k)$ may be inferred from the following mean value theorem.

Theorem 2. For $1 \leq k \leq X$, and $-\frac{1}{2} < a \leq 0$ we have

$$(1.10) \quad \int_1^X \left| \Delta_a(t; h/k) \right|^2 \cdot dt \\ = k \cdot [(6+4a)\pi^2]^{-1} \cdot \zeta(3/2-a) \cdot \zeta(3/2+a) \cdot \zeta^2(3/2) \cdot [\zeta(3)]^{-1} \cdot X^{\frac{1}{2}+a} \\ + O(k^2 \cdot X^{1+\varepsilon}) + O(k^{\frac{1}{2}} \cdot X^{\frac{1}{2}+a+\varepsilon}).$$

This argument is essentially similar to that of Cramér (see e.g. [3]). Theorem 2 suggests the following

Conjecture. For $-\frac{1}{2} < a \leq 0$, and $k^{2/(1+2a)} \leq X$,

$$(1.11) \quad \Delta_a(X; h/k) \ll k^{\frac{1}{2}} \cdot X^{\frac{1}{2}+a+\varepsilon}.$$

2. Some lemmas.

Lemma 1. The function $E_a(s; h/k)$ can be analytically continued to a meromorphic function, which is regular in the whole complex plane up to two simple poles at $s = 1, 1+a$ ($-1 < a < 0$), and is regular in the whole complex plane up to a double pole at $s = 1$ ($a = 0$). The function $E_a(s; h/k)$ satisfies the functional equation

$$(2.1) \quad E_a(s; h/k) \\ = \pi^{-1} \cdot [k/(2\pi)]^{1+a-2s} \cdot \Gamma(1-s) \cdot \Gamma(1+a-s) \cdot \left[\cos\left(\frac{1}{2}\pi a\right) \right. \\ \left. \cdot E_a(1+a-s; \bar{h}/k) - \cos\left(\pi s - \frac{1}{2}\pi a\right) \cdot E_a(1+a-s; -\bar{h}/k) \right],$$

and has for $-1 < a < 0$ the Laurent expansions

$$(2.2) \quad E_a(s; h/k) = \begin{cases} k^{-1+a} \cdot \zeta(1-a) \cdot (s-1)^{-1} + \dots, & \text{at } s = 1, \\ k^{-1-a} \cdot \zeta(1+a) \cdot (s-a-1)^{-1} + \dots, & \text{at } s = 1+a, \end{cases}$$

and has for $a = 0$ the Laurent expansion

$$(2.3) \quad E_a(s; h/k) = k^{-1} \cdot (s-1)^{-2} + k^{-1} \cdot (2\gamma - 2 \cdot \log k) \cdot (s-1)^{-1} + \dots, \\ \text{at } s = 1.$$

Proof. We start from the following identity, valid for $\operatorname{Re}(s) > 1$:

$$(2.4) \quad \begin{aligned} E_a(s; h/k) &= \sum_{n=1}^{\infty} \sum_{l|n} t^a \cdot e(hn/k) \cdot n^{-s} \\ &= \lim_{N \rightarrow \infty} \left[\sum_{t \leq N} t^{a-s} \sum_{r \leq N/t} e(hrt/k) \cdot r^{-s} \right] \\ &= \sum_{1 \leq b \leq k} \zeta(s-a; b, k) \cdot \zeta(s; e(hb/k)), \end{aligned}$$

where

$$(2.5) \quad \zeta(s; a, k) = \sum_{n \equiv a \pmod{k}} n^{-s},$$

and

$$(2.6) \quad \zeta(s; e(a/k)) = \sum_{n=1}^{\infty} e(an/k) \cdot n^{-s}.$$

The functional equations (2.5) and (2.6) have been investigated by Estermann [2]. We quote from [2] that

$$(2.7) \quad \begin{aligned} \zeta(s; a, k) &= G(s) \cdot k^{-s} \cdot \left[e\left(\frac{1}{4}s\right) \cdot \zeta(1-s; e(a/k)) \right. \\ &\quad \left. - e\left(-\frac{1}{4}s\right) \cdot \zeta(1-s; e(-a/k)) \right], \end{aligned}$$

and

$$(2.8) \quad \begin{aligned} \zeta(s; e(a/k)) &= G(s) \cdot k^{1-s} \cdot \left[e\left(\frac{1}{4}s\right) \cdot \zeta(1-s; -a, k) \right. \\ &\quad \left. - e\left(-\frac{1}{4}s\right) \cdot \zeta(1-s; a, k) \right], \end{aligned}$$

where

$$G(s) = -i(2\pi)^{s-1} \cdot \Gamma(1-s).$$

Applying (2.7) and (2.8) to (2.4), we obtain

$$(2.9) \quad \begin{aligned} E_a(s; h/k) &= 2G(s) \cdot G(s-a) \cdot k^{1+a-2s} \cdot \left[\cos\left(\pi s - \frac{1}{2}\pi a\right) \sum_{1 \leq b \leq k} \zeta(1+a-s; e(b/k)) \right. \\ &\quad \cdot \zeta(1-s; -hb, k) - \cos\left(\frac{1}{2}\pi a\right) \sum_{1 \leq b \leq k} \zeta(1+a-s; e(b/k)) \\ &\quad \left. \cdot \zeta(1-s; hb, k) \right]. \end{aligned}$$

Hence, by (1.3), we have proved the functional equation $E_a(s; h/k)$.

It easily follows from (2.4) that $E_a(s; h/k)$ is regular everywhere except at $s = 1, 1+a$ ($-1 < a < 0$) or $s = 1$ ($a = 0$). We shall investigate the Laurent expansion in the neighbourhood of these points.

In case of the simple pole $s = 1$, by the property of (2.6) and $(h, k) = 1$, we have

$$\begin{aligned} E_a(s; h/k) - \zeta(s) \cdot \zeta(s-a; k, k) \\ = \sum_{1 \leq b \leq k-1} \zeta(s; e(hb/k)) \cdot \zeta(s-a; b, k). \end{aligned}$$

Since $\zeta(s; e(hb/k))$ is regular at $s = 1$ for $1 \leq b \leq k-1$, the above identity means that $E_a(s; h/k)$ has the same meromorphic part as $\zeta(s) \cdot \zeta(s-a; k, k)$ in a neighbourhood of $s = 1$. And in case of the simple pole $s = 1+a$, by (2.5), we have

$$\begin{aligned} E_a(s; h/k) - \zeta(s-a; k, k) \sum_{1 \leq b \leq k} \zeta(s; e(hb/k)) \\ = \sum_{1 \leq b \leq k} [\zeta(s-a; b, k) - \zeta(s-a; k, k)] \cdot \zeta(s; e(hb/k)). \end{aligned}$$

The above identity means that $E_a(s; h/k)$ has the same meromorphic part as

$$k^{1+a-2s} \cdot \zeta(s) \cdot \zeta(s-a)$$

in a neighbourhood of $s = 1+a$. Thus we have proved the Laurent expansions (2.2). Lastly in case of the double pole $s = 1$, the Laurent expansion (2.3) has been investigated in detail by Estermann [1]. Therefore we have proved Lemma 1.

Lemma 2. *Let $F(t)$ and $G(t)$ be real functions, $G(t)/F'(t)$ monotonic, $|F'(t)/G(t)| \geq m > 0$, throughout the interval $[a, b]$. Then*

$$\left| \int_a^b G(t) \cdot \exp(iF(t)) \cdot dt \right| < 4/m.$$

The proof of this lemma depends on Titchmarsh [7].

3. Proof of Theorem 1. Let T be a parameter given by

$$(3.1) \quad k^2 T^2 \cdot (4\pi^2 X)^{-1} = N + \frac{1}{2}.$$

By Perron's formula (see e.g. [7]), and (3.1), we have

$$\begin{aligned} & \sum'_{n \leq X} \sigma_a(n) \cdot e(hn/k) \\ &= (2\pi i)^{-1} \cdot \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} E_a(s; h/k) \cdot X^s \cdot s^{-1} \cdot ds + O(k \cdot N^{-\frac{1}{2}} \cdot X^{\frac{1}{2}+\varepsilon}). \end{aligned}$$

The above integral is evaluated by the theorem of residues using the rectangular contour with vertices at $1 + \varepsilon \pm iT$, $a - \varepsilon \pm iT$. By the equation (2.1), and the Phragmén-Lindelöf principle, we have

$$(3.2) \quad E_a(s; h/k) \ll (k|t|)^{1+\varepsilon-\sigma}, \text{ for } a - \varepsilon \leq \sigma \leq 1 + \varepsilon, |t| \geq 1.$$

By (3.1) and (3.2), the integrals over the horizontal sides are

$$\ll k \cdot N^{-\frac{1}{2}} \cdot X^{\frac{1}{2}+\varepsilon}.$$

If $-1 < a < 0$, by (2.2), the residues of the integrand at $s = 1, 1 + a$ yield the second and third terms on the right-hand side of (1.7), and the residue at $s = 0$ gives $E_a(0; h/k)$. If $a = 0$, by (2.3), the residue of the integrand at $s = 1$ yields the second term on the right-hand side of (1.6), the residue at $s = 0$ gives $E_0(0; h/k) = E(0; h/k)$. Hence we have, for $-1 < a \leq 0$,

$$\begin{aligned} & \Delta_a(X; h/k) \\ &= (2\pi i)^{-1} \cdot \int_{a-\varepsilon-iT}^{a-\varepsilon+iT} E_a(s; h/k) \cdot X^s \cdot s^{-1} \cdot ds + O(k \cdot N^{-\frac{1}{2}} \cdot X^{\frac{1}{2}+\varepsilon}). \end{aligned}$$

After substituting the expression (2.1) in the right-hand side, this becomes

$$\begin{aligned} (3.3) \quad & \Delta_a(X; h/k) \\ &= (\pi i)^{-1} \cdot (2\pi)^{-a-2} \cdot k^{a+1} \cdot \cos\left(\frac{1}{2}\pi a\right) \cdot \sum_{n=1}^{\infty} \sigma_a(n) \cdot e(-\bar{h}n/k) \cdot n^{-a-1} \cdot I_1 \\ & \quad - (\pi i)^{-1} \cdot (2\pi)^{-a-2} \cdot k^{a+1} \cdot \sin\left(\frac{1}{2}\pi a\right) \cdot \sum_{n=1}^{\infty} \sigma_a(n) \cdot e(-\bar{h}n/k) \cdot n^{-a-1} \cdot I_2 \\ & \quad + \pi^{-2} \cdot (2\pi)^{-a-1} \cdot k^{a+1} \cdot \cos\left(\frac{1}{2}\pi a\right) \cdot \sum_{n=1}^{\infty} \sigma_a(n) \cdot \sin(2\pi\bar{h}n/k) \cdot n^{-a-1} \cdot I_3 \\ & \quad + O(k \cdot N^{-\frac{1}{2}} \cdot X^{\frac{1}{2}+\varepsilon}), \end{aligned}$$

where

$$\begin{aligned}
 I_1 &= \int_{a-\varepsilon-iT}^{a-\varepsilon+iT} \Gamma(1-s) \cdot \Gamma(1+a-s) \cdot [1 - \cos(\pi s)] \cdot (4\pi^2 k^{-2} nX)^s \cdot s^{-1} \cdot ds, \\
 I_2 &= \int_{a-\varepsilon-iT}^{a-\varepsilon+iT} \Gamma(1-s) \cdot \Gamma(1+a-s) \cdot \sin(\pi s) \cdot (4\pi^2 k^{-2} nX)^s \cdot s^{-1} \cdot ds, \\
 I_3 &= \int_{a-\varepsilon-iT}^{a-\varepsilon+iT} \Gamma(1-s) \cdot \Gamma(1+a-s) \cdot (4\pi^2 k^{-2} nX)^s \cdot s^{-1} \cdot ds,
 \end{aligned}$$

The integral I_3 is estimated by the Stirling formula as

$$(3.4) \qquad \ll k^{-2a} \cdot N^\varepsilon \cdot X^{a+\varepsilon} \cdot n^{a-\varepsilon}.$$

And, it is easily shown that the contribution of (3.4) to (3.3) is

$$\ll k^{-a+1} \cdot N^\varepsilon \cdot X^{a+\varepsilon}.$$

Hence we have

$$\begin{aligned}
 (3.5) \quad \Delta_a(X; h/k) &= (\pi i)^{-1} \cdot (2\pi)^{-a-2} \cdot k^{a+1} \cdot \cos\left(\frac{1}{2}\pi a\right) \cdot \sum_{n=1}^{\infty} \sigma_a(n) \cdot e(-\bar{h}n/k) \cdot n^{-a-1} \cdot I_1 \\
 &\quad - (\pi i)^{-1} \cdot (2\pi)^{-a-2} \cdot k^{a+1} \cdot \sin\left(\frac{1}{2}\pi a\right) \cdot \sum_{n=1}^{\infty} \sigma_a(n) \cdot e(-\bar{h}n/k) \cdot n^{-a-1} \cdot I_2 \\
 &\quad + O(k \cdot N^{-\frac{1}{2}} \cdot X^{\frac{1}{2}+\varepsilon}).
 \end{aligned}$$

Now, we consider the contribution of the terms with $n > N$ in the first two terms of the right-hand side of (3.5). Each of the integrals I_1 and I_2 can be divided into the three parts :

$$\begin{aligned}
 (3.6) \quad &\left[\int_{a-\varepsilon+i}^{a-\varepsilon+iT} + \int_{a-\varepsilon-i}^{a-\varepsilon+i} + \int_{a-\varepsilon-iT}^{a-\varepsilon-i} \right] \cdot \Gamma(1-s) \cdot \Gamma(1+a-s) \cdot [1 - \cos(\pi s)] \\
 &\cdot (4\pi^2 k^{-2} nX)^s \cdot s^{-1} \cdot ds = I_{1,1} + I_{1,2} + I_{1,3},
 \end{aligned}$$

and

$$\begin{aligned}
 (3.7) \quad &\left[\int_{a-\varepsilon+i}^{a-\varepsilon+iT} + \int_{a-\varepsilon-i}^{a-\varepsilon+i} + \int_{a-\varepsilon-iT}^{a-\varepsilon-i} \right] \cdot \Gamma(1-s) \cdot \Gamma(1+a-s) \cdot \sin(\pi s) \\
 &\cdot (4\pi^2 k^{-2} nX)^s \cdot s^{-1} \cdot ds = I_{2,1} + I_{2,2} + I_{2,3}, \text{ say.}
 \end{aligned}$$

The integral $I_{1,1}$ is estimated by Lemma 2 and the Stirling formula as

$$\ll k^{-a} \cdot N^{-\frac{1}{2}a+\varepsilon} \cdot X^{\frac{1}{2}a} \cdot n^{a-\varepsilon}.$$

By similar estimations, The integrals $I_{1,3}$, $I_{2,1}$, and $I_{2,3}$ are

$$\ll k^{-a} \cdot N^{-\frac{1}{2}a+\varepsilon} \cdot X^{\frac{1}{2}a} \cdot n^{a-\varepsilon}.$$

And we easily estimate that the integrals $I_{1,2}$ and $I_{2,2}$ are

$$\ll k^{-2a} \cdot N^\varepsilon \cdot X^{a+\varepsilon} \cdot n^{a-\varepsilon}.$$

Hence, it is easily shown that the contribution of (3.6) and (3.7) to (3.5) is

$$\ll k \cdot N^{-\frac{1}{2}a+\varepsilon} \cdot X^{\frac{1}{2}a}.$$

Hence we have

$$\begin{aligned} (3.8) \quad \Delta_a(X; h/k) &= (\pi i)^{-1} \cdot (2\pi)^{-a-2} \cdot k^{a+1} \cdot \cos\left(\frac{1}{2}\pi a\right) \cdot \sum_{n \leq N} \sigma_a(n) \cdot e(-\bar{h}n/k) \cdot n^{-a-1} \cdot I_1 \\ &\quad - (\pi i)^{-1} \cdot (2\pi)^{-a-2} \cdot k^{a+1} \cdot \sin\left(\frac{1}{2}\pi a\right) \cdot \sum_{n \leq N} \sigma_a(n) \cdot e(-\bar{h}n/k) \cdot n^{-a-1} \cdot I_2 \\ &\quad + O(k \cdot N^{-\frac{1}{2}} \cdot X^{\frac{1}{2}+\varepsilon}). \end{aligned}$$

Next, each of the integrals I_1 and I_2 can be divided into the five parts;

$$\begin{aligned} (3.9) \quad &\left[\int_{-i\infty}^{i\infty} - \left(\int_{iT}^{i\infty} + \int_{-i\infty}^{-iT} + \int_{-iT}^{a-\varepsilon-iT} + \int_{a-\varepsilon+iT}^{iT} \right) \right] \Gamma(1-s) \Gamma(1+a-s) \\ &\cdot [1 - \cos(\pi s)] (4\pi^2 k^{-2} nX)^s s^{-1} ds \\ &= I_{1,4} - (I_{1,5} + I_{1,6} + I_{1,7} + I_{1,8}), \end{aligned}$$

and

$$\begin{aligned} (3.10) \quad &\left[\int_{-i\infty}^{i\infty} - \left(\int_{iT}^{i\infty} + \int_{-i\infty}^{-iT} + \int_{-iT}^{a-\varepsilon-iT} + \int_{a-\varepsilon+iT}^{iT} \right) \right] \cdot \Gamma(1-s) \cdot \Gamma(1+a-s) \\ &\cdot \sin(\pi s) \cdot (4\pi^2 k^{-2} nX)^s s^{-1} ds \\ &= I_{2,4} - (I_{2,5} + I_{2,6} + I_{2,7} + I_{2,8}), \text{ say.} \end{aligned}$$

Firstly, we calculate the integrals $I_{1,4}$ and $I_{2,4}$.

Applying Mellin's inversion formula (see e.g. [6]), we obtain that if $-1 < a < \frac{1}{2}$, then

$$\begin{aligned} I_{1,4} &= -2^{a-2} \cdot (4\pi^2 k^{-2} nX)^{1+a} \cdot \int_{\frac{2+2a-i\infty}{2+2a-i\infty}}^{\frac{2+2a+i\infty}{2+2a+i\infty}} 2^{s-a-3} \cdot \Gamma\left(\frac{1}{2}s\right) \cdot \Gamma\left(\frac{1}{2}s-a-1\right) \\ &\quad \cdot (4\pi k^{-1} n^{\frac{1}{2}} X^{\frac{1}{2}})^{-s} \cdot ds + 2^{a+2} \pi (4\pi^2 k^{-2} nX)^{1+a} \cdot \int_{\frac{2+2a-i\infty}{2+2a-i\infty}}^{\frac{2+2a+i\infty}{2+2a+i\infty}} 2^{s-a-2} \\ &\quad \cdot \Gamma\left(\frac{1}{2}s\right) \cdot \Gamma\left(\frac{1}{2}s-a-1\right) \cdot \pi^{-1} \cdot \cos\left(\frac{1}{2}\pi s - \pi a - \pi\right) \\ &\quad \cdot (4\pi k^{-1} n^{\frac{1}{2}} X^{\frac{1}{2}})^{-s} \cdot ds \\ &= -2i(2\pi)^{a+2} \cdot k^{-a-1} \cdot (nX)^{\frac{1}{2}+a+\frac{1}{2}} \\ &\quad \cdot \left[K_{a+1}(4\pi k^{-1} n^{\frac{1}{2}} X^{\frac{1}{2}}) + \frac{1}{2} \pi Y_{a+1}(4\pi k^{-1} n^{\frac{1}{2}} X^{\frac{1}{2}}) \right], \end{aligned}$$

and

$$\begin{aligned} I_{2,4} &= 2^{a+1} \pi (4\pi^2 k^{-2} nX)^{1+a} \int_{\frac{2+2a-i\infty}{2+2a-i\infty}}^{\frac{2+2a+i\infty}{2+2a+i\infty}} 2^{s-a-2} \Gamma\left(\frac{1}{2}s\right) \left[\Gamma\left(2+a-\frac{1}{2}s\right) \right]^{-1} \\ &\quad \cdot (4\pi k^{-1} n^{\frac{1}{2}} X^{\frac{1}{2}})^{-s} ds \\ &= \pi i (2\pi)^{2+a} \cdot k^{-a-1} \cdot (nX)^{\frac{1}{2}+a+\frac{1}{2}} \cdot J_{a+1}(4\pi k^{-1} n^{\frac{1}{2}} X^{\frac{1}{2}}), \end{aligned}$$

where K_{a+1} , Y_{a+1} , and J_{a+1} are Bessel functions (see e.g. [8]). Next, we estimate $I_{1,5}$, $I_{1,6}$, $I_{2,5}$, and $I_{2,6}$. We must be divide into the following two cases with a view to satisfying the monotone condition in Lemma 2. If $n \leq \left(N + \frac{1}{2}\right) \cdot \exp(-2/a)$, we have, by Lemma 2 and the Stirling formula,

$$I_{1,5} \ll T^a \cdot \left[\log \left(n \cdot \left(N + \frac{1}{2} \right)^{-1} \right) \right]^{-1}.$$

By similar estimations. The integrals $I_{1,6}$, $I_{2,5}$, and $I_{2,6}$ are

$$\ll T^a \cdot \left[\log \left(n \cdot \left(N + \frac{1}{2} \right)^{-1} \right) \right]^{-1}.$$

If $n > \left[N + \frac{1}{2} \right] \cdot \exp(-2/a)$, we have, by Lemma 2 and the Stirling formula,

$$\begin{aligned} I_{1,5} &= c \left[\int_{\tau}^{\tau_0} + \int_{\tau_0}^{\infty} \right] \cdot G(t) \cdot \exp(iF(t)) \cdot [1 + O(t^{-1})] \cdot dt \\ &\ll T_0^a \cdot [\log(4\pi^2 k^{-2} T_0^{-2} nX)]^{-1} \\ &\ll T^a \cdot \left[\log \left(n \cdot \left(N + \frac{1}{2} \right)^{-1} \right) \right]^{-1}, \end{aligned}$$

where c is a constant,

$$F(t) = 2t(-\log kt + \log 2\pi + 1) + t \cdot \log nX,$$

$$G(t) = t^a,$$

and

$$T_0 = 2\pi k^{-1} n^{\frac{1}{2}} X^{\frac{1}{2}} \cdot \exp(1/a).$$

By similar estimations, the integrals $I_{1,6}$, $I_{2,5}$, and $I_{2,6}$ are

$$\ll T^a \cdot \left[\log \left(n \cdot \left(N + \frac{1}{2} \right)^{-1} \right) \right]^{-1}.$$

Lastly, by easy estimations, we estimate that the integrals $I_{1,7}$, $I_{1,8}$, $I_{2,7}$, and $I_{2,8}$ are

$$\ll k^{-a} \cdot N^{-\frac{1}{2}a+\epsilon} \cdot X^{\frac{1}{2}a} \cdot n^{a-\epsilon}.$$

Hence, by the results (3.9) and (3.10), we obtain

$$\begin{aligned} &\Delta_a(X; h/k) \\ &= -2\pi^{-1} \cos\left(\frac{1}{2}\pi a\right) \cdot X^{\frac{1}{2}+\frac{1}{2}a} \cdot \sum_{n \leq N} \sigma_a(n) \cdot e(-\bar{h}n/k) \cdot n^{-\frac{1}{2}-\frac{1}{2}a} \\ &\quad \cdot \left[K_{a+1}(4\pi k^{-1} n^{\frac{1}{2}} X^{\frac{1}{2}}) + \frac{1}{2}\pi Y_{a+1}(4\pi k^{-1} n^{\frac{1}{2}} X^{\frac{1}{2}}) \right] - \sin\left(\frac{1}{2}\pi a\right) \\ &\quad \cdot X^{\frac{1}{2}+\frac{1}{2}a} \cdot \sum_{n \leq N} \sigma_a(n) \cdot e(-\bar{h}n/k) \cdot n^{-\frac{1}{2}-\frac{1}{2}a} \cdot J_{a+1}(4\pi k^{-1} n^{\frac{1}{2}} X^{\frac{1}{2}}) \\ &\quad + O(k \cdot N^{-\frac{1}{2}} \cdot X^{\frac{1}{2}+\epsilon}). \end{aligned}$$

By the usual asymptotic formulas for Bessel functions [8], for $1 \leq N \ll X$,

we have

$$\begin{aligned} \Delta_a(X; h/k) &= (\pi\sqrt{2})^{-1} \cdot k^{\frac{1}{2}} \cdot X^{\frac{1}{2} + \frac{1}{2}a} \cdot \sum_{n \leq N} \sigma_a(n) \cdot e(-\bar{h}n/k) \cdot n^{-\frac{3}{2} - \frac{1}{2}a} \\ &\quad \cdot \cos\left(4\pi k^{-1} n^{\frac{1}{2}} X^{\frac{1}{2}} - \frac{1}{4}\pi\right) + O(k \cdot N^{-\frac{1}{2}} \cdot X^{\frac{1}{2} + \epsilon}). \end{aligned}$$

We have proved Theorem 1.

4. Proof of Theorem 2. It will be sufficient to prove the corresponding formula for the integral over $\left[\frac{1}{2}X, X\right]$ and then to replace $\frac{1}{2}X$ by $\frac{1}{4}X$, $X/8$, and so on, and to add up all the results. We start from the result of Theorem 1 with $N = X$. By integrating term by term and using the first mean-value theorem for integrals, we obtain

$$\begin{aligned} (3.1) \quad &\int_{X/2}^X \left| \Delta_a(t; h/k) \right|^2 \cdot dt \\ &= (2\pi^2)^{-1} \cdot k \cdot \sum_{m, n \leq X} \sigma_a(m) \cdot \sigma_a(n) \cdot (mn)^{-\frac{3}{2} - \frac{1}{2}a} \cdot \int_{X/2}^X t^{\frac{1}{2} + a} \cdot e(-\bar{h}m/k) \\ &\quad \cdot e(\bar{h}n/k) \cdot \cos\left(4\pi k^{-1} m^{\frac{1}{2}} t^{\frac{1}{2}} - \frac{1}{4}\pi\right) \cdot \cos\left(4\pi k^{-1} n^{\frac{1}{2}} t^{\frac{1}{2}} - \frac{1}{4}\pi\right) \cdot dt \\ &\quad + O\left(k^{\frac{1}{2}} \cdot X^{\frac{1}{2} + \frac{1}{2}a + \epsilon} \cdot \int_{X/2}^X \left| \sum_{n \leq X} \sigma_a(n) \cdot e(-\bar{h}n/k) \cdot n^{-\frac{3}{2} - \frac{1}{2}a} \right. \right. \\ &\quad \left. \left. \cdot \cos\left(4\pi k^{-1} n^{\frac{1}{2}} t^{\frac{1}{2}} - \frac{1}{4}\pi\right) \right| dt\right) + O(k^2 \cdot X^{1 + \epsilon}). \end{aligned}$$

In the first sum in the right-hand side of (3.1) we distinguish the cases $m = n$ and $m \neq n$ contribute

$$\begin{aligned} &(2\pi^2)^{-1} \cdot k \cdot \sum_{n \leq X} \sigma_a^2(n) \cdot n^{-\frac{3}{2} - a} \cdot \int_{X/2}^X t^{\frac{1}{2} + a} \cdot \cos^2\left(4\pi k^{-1} n^{\frac{1}{2}} t^{\frac{1}{2}} - \frac{1}{4}\pi\right) \cdot dt \\ &= [(6 + 4a)\pi^2]^{-1} k \cdot \left[X^{\frac{1}{2} + a} - \left(\frac{1}{2}X\right)^{\frac{1}{2} + a}\right] \cdot \sum_{n=1}^{\infty} \sigma_a^2(n) \cdot n^{-\frac{3}{2} - a} \\ &\quad + O(k \cdot X^{1 + \epsilon}) + O(k^2 \cdot X^{1 + a - \epsilon}). \end{aligned}$$

It is seen that the terms in (3.1) for which $m \neq n$ are a multiple of

$$\begin{aligned} & k \sum_{m+n \leq X} \sigma_a(m) \sigma_a(n) e(-\bar{h}m/k) e(\bar{h}n/k) (mn)^{-\frac{3}{4}-\frac{1}{2}a} \\ & \int_{X/2}^X t^{\frac{1}{2}+a} \cos(4\pi k^{-1} m^{\frac{1}{2}} t^{\frac{1}{2}} - 4\pi k^{-1} n^{\frac{1}{2}} t^{\frac{1}{2}}) dt \\ & + k \sum_{m+n \leq X} \sigma_a(m) \sigma_a(n) e(-\bar{h}m/k) e(\bar{h}n/k) (mn)^{-\frac{3}{4}-\frac{1}{2}a} \\ & \int_{X/2}^X t^{\frac{1}{2}+a} \sin(4\pi k^{-1} m^{\frac{1}{2}} t^{\frac{1}{2}} + 4\pi k^{-1} n^{\frac{1}{2}} t^{\frac{1}{2}}) dt \\ & = S_1 + S_2, \text{ say.} \end{aligned}$$

Estimating the integral in S_2 by Lemma 2 we have

$$\begin{aligned} S_2 & \ll k^2 \cdot X^{1+a} \cdot \sum_{m < n \leq X} \sigma_a(m) \cdot \sigma_a(n) \cdot (mn)^{-\frac{3}{4}-\frac{1}{2}a} \cdot (m^{\frac{1}{2}} + n^{\frac{1}{2}})^{-1} \\ & \ll k^2 \cdot X^{1+a}. \end{aligned}$$

Analogously, we obtain

$$\begin{aligned} S_1 & \ll k^2 \cdot X^{1+a} \cdot \left[\sum_{\substack{n \leq m/2 \\ m \leq X}} + \sum_{\substack{m/2 < n \\ m \leq X}} \right] \cdot \sigma_a(m) \cdot \sigma_a(n) \cdot (mn)^{-\frac{3}{4}-\frac{1}{2}a} \cdot (m^{\frac{1}{2}} - n^{\frac{1}{2}})^{-1} \\ & = k^2 \cdot X^{1+a} \cdot [S_{1,1} + S_{1,2}], \text{ say.} \end{aligned}$$

By partial summation formula, we have

$$\begin{aligned} S_{1,1} & \ll \sum_{m \leq X} \sigma_a(m) \cdot m^{-\frac{3}{4}-\frac{1}{2}a} \cdot \sum_{n \leq m/2} \sigma_a(n) \cdot n^{-\frac{3}{4}-\frac{1}{2}a} \ll X^{-a+\varepsilon}, \\ S_{1,2} & \ll \sum_{m \leq X} \sigma_a(m) \cdot m^{-1-a} \cdot \sum_{m/2 < n < m} \sigma_a(n) \cdot (m-n)^{-1} \ll X^{-a+\varepsilon}. \end{aligned}$$

Therefore the first sum in (3.1) is equal to

$$\begin{aligned} & k \cdot [(6+4a)\pi^2]^{-1} \cdot \left[X^{\frac{3}{4}+a} - \left(\frac{1}{2}X\right)^{\frac{3}{4}+a} \right] \cdot \sum_{n=1}^{\infty} \sigma_a^2(n) \cdot n^{-\frac{3}{4}-a} \\ & + O(k^2 \cdot X^{1+a+\varepsilon}) + O(k \cdot X^{1+\varepsilon}). \end{aligned}$$

The first O -term in (3.1) is estimated by the Cauchy-Schwarz inequality as

$$\ll k^{\frac{3}{2}} X^{\frac{3}{4}+\frac{1}{2}a+\varepsilon}$$

when we square out the modulus under the integral sign and treat the terms $m = n$ and $m \neq n$ similarly as before. We have proved Theorem 2.

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