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SOME COMMUTATIVITY CONDITIONS

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Throughout, R will represent a ring with center C. Let A be a non-empty subset of R, $n > 1$ an integer, and α an endomorphism of $\{R, +\}$. We consider the following conditions:

$(\alpha - A)_n$ $x^n - \alpha(x) \in A$ for every $x \in R$.

$(\text{II} - A)_n$ *If* $x, y \in R$ *and* $x - y \in A$, *then either* $x^n = y^n$ *or both* x *and* y *are in the centralizer of* A *in* R.

$(* - A)_n$ *For all positive integers* k, *every element of* R *can be written in the form* $x^{nk} + a$ *with some* $x \in R$ *and* $a \in A$.

$P_7(n)$ $[x, y^n] = [x^n, y]$ *for all* $x, y \in R$.

In this brief note, we shall prove the following theorem which includes the recent result of Laffey [2].

Theorem 1. *The following statements are equivalent:*

1) *R is commutative.*

2) *There exists a commutative non-empty subset A of R, an integer $n > 1$, and a surjective endomorphism α of $\{R, +\}$ for which $(\alpha - A)_n$ and $(\text{II} - A)_n$ are satisfied.*

3) *There exists a commutative non-empty subset A of R and an integer $n > 1$ for which $(* - A)_n$, $P_7(n)$ and $(\text{II} - A)_n$ are satisfied.*

4) *There exists an integer $n > 1$ and a surjective endomorphism α of $\{R, +\}$ for which $(\alpha - C)_n$ is satisfied.*

5) *There exists an integer $n > 1$ for which $(* - C)_n$ and $P_7(n)$ are satisfied.*

Proof. Obviously, 1) implies 2) and 3). Also, one can easily see that 4) implies 5) (cf. [2, Lemmas 1 and 2]).

2) ⇨ 4). By [3, Lemma 1 (3)], $[r^n, a] = 0$ for every $r \in R$ and $a \in A$. Hence $[\alpha(r), a] = [r^n, a] - [r^n - \alpha(r), a] = 0$. Since α is surjective, we see that a is central.

3) ⇨ 5). Again by [3, Lemma 1 (3)], $[r^n, a] = 0$ for every $r \in R$ and $a \in A$. Let y be an arbitrary element of R. Then $y = x^n + b$ with some $x \in R$ and $b \in A$. Hence, for every $a \in A$, $[y, a] = [x^n, a] + [b, a] = 0$, and therefore a is central.

5) ⇨ 1). By [1, Proposition 3 (ii)], there exists a positive integer h such that $[x^{n^h}, y] = 0$ for all $x, y \in R$. This together with $(* - C)_n$ implies

that R is commutative.

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