Math. J. Okayama Univ. 29 (1987), 185—189

RINGS SATISFYING THE IDENTITY $(X-X^n)(Y-Y^n) = 0$

Dedicated to Professor Hisao Tominaga on his 60th birthday

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A well known theorem of Jacobson asserts that if R is a ring with the property that for every x in R there exists an integer $n > 1$ such that $x^n = x$, then R is commutative. With this as motivation, we consider the structure of a ring R which satisfies the identity
(*) $(x-x^n)(y-y^n) = 0$ for all $x, y \in R$, $n > 1$ fixed.
That (*) does not necessarily imply that R is commutative is seen by considering the ring

$$R = \left\{\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\1 & 0\end{pmatrix} \Big| \, 0, 1 \in \mathrm{GF}(2)\right\}.$$

It is readily verified that (*) holds in R for all positive integers n but, of course, R is not commutative. Additional conditions are thus needed in order to imply the commutativity of R. In this paper, we prove three theorems involving such additional conditions, namely the following:

Theorem 1. *Let R be an s-unital ring with center C, and $n > 1$ a fixed integer. Suppose* (*) $(x-x^n)(y-y^n) = 0$ *and* (i) $(xy)^n - x^n y^n \in C$ *for all* $x, y \in R$. *If, for all $x \in R$ and $a \in N$, $(n-1)[x, a] = 0$ implies $[x, a] = 0$, then R is commutative, where N is the set of nilpotents in R.*

Theorem 2. *Let R be an s-unital ring with center C, and $n > 1$ a fixed integer. Suppose* (*) $(x-x^n)(y-y^n) = 0$ *and* (i)′ $(xy)^{n+1} - x^{n+1}y^{n+1} \in C$ *for all $x, y \in R$. Then R is commutative.*

Theorem 3. *Let R be a ring with center C, and $n > 1$ a fixed integer. Suppose* (*) $(x-x^n)(y-y^n) = 0$ *for all $x, y \in R$. Then R is commutative under any of the following hypotheses:* (ii) $(xy)^n - y^n x^n \in C$, (iii) $(xy)^n - (yx)^n \in C$, (iv) $x^n y^n - y^n x^n \in C$ *for all $x, y \in R$.*

In preparation for the proofs of these theorems, we state first the following known results. As usual, $[x, y]$ denotes the commutator $xy - yx$.

Lemma 1. *Let x, y be any elements in R such that $[x, [x, y]] = 0$. Then, for all positive integers k, $[x^k, y] = kx^{k-1}[x, y]$.*

Lemma 2 [1]. *Suppose that* 1) *for every $x \in R$, $x-x^n \in N$ with some integer $n > 1$, where N is the set of nilpotents in R,* 2) *N is commutative, and* 3) *for all $x \in R$ and $a \in N$, $[x, [x, a]] = 0$. Then R is commutative.*

Lemma 3 [3]. *Let R be a ring with identity, and suppose that $[x^h, y^h] = 0$ and $[x^k, y^k] = 0$ for all $x, y \in R$, where h and k are fixed relatively prime positive integers. Then R is commutative.*

Next, we prove

Lemma 4. *Suppose that R satisfies (*). Then N forms an ideal of R with $N^2 = 0$ and $[a, x^{k+n}] = [a, x^{k+1}]$ for all $a \in N$, $x \in R$ and $k \geqq 0$. In particular, if R has 1 and x is invertible then $[a, x^{n-1}] = 0$.*

Proof. By (*), $(a-a^n)^2 = 0$, namely $a^2 = a^2(2a^{n-1}-a^{2(n-1)})$. Since $n-1 > 0$, we can easily see that $a^2 = 0$. Now, let $b \in N$. Then $0 = (a-a^n)(b-b^n) = ab$. Hence N forms an ideal by [4, Lemma 1 (1)]. Finally, noting that $x-x^n \in N$ by (*), we get $[a, x^{k+1}]-[a, x^{k+n}] = a(x-x^n)x^k-x^k(x-x^n)a = 0$.

Lemma 5. (1) *Let R be a ring with 1. If R satisfies* (i)$'$, *then R is normal, that is, every idempotent in R is central.*

(2) *If R satisfies any of* (ii)—(iv), *then R is normal.*

Proof. Let e be an idempotent in R, and $r \in R$.

(1) Put $x = e+er(1-e)$ and $y = 1-e$ in (i)$'$ to get $0 = [1-e, (er(1-e))^{n+1} - er(1-e)] = er(1-e)$, and hence $er = ere$. Similarly, $re = ere$.

(2) Put $x = e$, $y = e+er(1-e)$ in (ii) to get $0 = [e, (e+er(1-e))^n-e] = er(1-e)$, and hence $er = ere$. Similarly, $re = ere$. The proof is similar under the hypotheses (iii) and (iv).

Lemma 6. (1) *Let R be a subdirectly irreducible ring with 1. If R satisfies (*) and* (i)$'$, *then R is a local ring with radical N and R/N is a finite field.*

(2) *Let R be a subdirectly irreducible ring different from N. If R satisfies (*) and any of (ii)–(iv), then R is a local ring with radical N and R/N is a finite field.*

Proof. Let r be an arbitrary element in $R\backslash N$. Then, by (*), $r^2 = r^3f(r)$ for some $f(t) \in \mathbf{Z}[t]$. Obviously, $e = (rf(r))^2$ is an idempotent with $r^2e = r^2$. Hence e is a non-zero central idempotent by Lemma 5. Since R is subdirectly irreducible, $e = 1$ and r is invertible. We have thus seen that R is a local ring with radical N. Finally, noting that R/N satisfies the identity $x-x^n = 0$, we see that R/N is a finite field, by Jacobson's theorem.

We are now in a position to prove our theorems.

Proof of Theorem 1. Let $x \in R$, $a \in N$, and choose a pseudo-identity e of $\{a, x\}$(see [2]). Then, by Lemma 4, $(n-1)[x, a] = (n-1)[x^n, a] = \{((e+a)x)^n-(e+a)^nx^n\} - \{(x(e+a))^n-x^n(e+a)^n\} \in C$, namely $(n-1)[y, [x, a]] = 0$ for all $y \in R$. Hence $[y, [x, a]] = 0$, and therefore R is commutative by Lemma 2.

Corollary 1. *Let R be an s-unital ring. Suppose $(x-x^2)(y-y^2) = 0$ and $(xy)^2-x^2y^2 \in C$ for all $x, y \in R$. Then R is commutative.*

Proof of Theorem 2. In view of [2, Proposition 1], we may assume that R has 1. Then, by Lemmas 4 and 6, we may assume further that R is a local ring with radical N, $N^2 = 0$ and $R/N = \mathrm{GF}(p^\alpha)$ with some prime p.

Let u, v be units in R, $a \in N$, and $x, y \in R$. By Lemma 4, $n[x^2, a] = n[x^{n+1}, a] = \{((1+a)x)^{n+1}-(1+a)^{n+1}x^{n+1}\}-\{(x(1+a))^{n+1}-x^{n+1}(1+a)^{n+1}\} \in C$. Since R/N is commutative and $N^2 = 0$, $[(uv)^n-v^nu^n, x] = [v^{-1}\{(vu)^{n+1}-v^{n+1}u^{n+1}\}u^{-1}, x] = \{(vu)^{n+1}-v^{n+1}u^{n+1}\}[v^{-1}u^{-1}, x] = 0$, and so $(uv)^n-v^nu^n \in C$. Hence $(n+1)[u, a] = (n+1)[u^n, a] = \{(u(1+a))^n-(1+a)^nu^n\} - \{((1+a)u)^n-u^n(1+a)^n\} \in C$. Since both $n[u^2, a]$ and $(n+1)[u^2, a]$ are in C, we get $[u^2, a] \in C$, and therefore $[x^2, a] \in C$. If $p \neq 2$ then $2[u, a] = [(1+u)^2-u^2, a] \in C$ and $p^2[u, a] = 0$ show that $[u, a] \in C$. On the other hand, if $p = 2$ then $u^{2^\alpha}-u \in N$, and so $[u, a] = [u, a]-[u-u^{2^\alpha}, a] = [u^{2^\alpha}, a] \in C$. Thus, in either case, $[x, a] \in C$, and therefore R is commutative by Lemma 2.

Proof of Theorem 3. In view of Lemmas 4 and 6, we may assume that R is a local ring with radical N, $N^2 = 0$ and $R/N = \mathrm{GF}(p^\alpha)$ with some prime p.

Suppose first R satisfies (ii). Then, for any $x, y, z \in R$, we have $[(xy)^{n+1}-x^{n+1}y^{n+1}, z] = [x\{(yx)^n-x^ny^n\}y, z] = \{(yx)^n-x^ny^n\}[xy, z] = 0$. Hence R satisfies (i)$'$, and R is commutative by Theorem 2.

Next, suppose R satisfies (iii). Let $a \in N$, and $x \in R$. Then, by Lemma 4, $[a, x] = [a, x^n] = (x+ax)^n-(x+xa)^n = ((1+a)x)^n-(x(1+a))^n \in C$. Hence R is commutative by Lemma 2.

Finally, suppose R satisfies (iv). Let u, v be units in R, $a \in N$, and $x, y \in R$. Then $[u^{n-1}, v^{n-1}] = [1-u^{n-1}, 1-v^{n-1}] = u^{-1}(u-u^n)(v-v^n)v^{-1} - v^{-1}(v-v^n)(u-u^n)u^{-1} = 0$. Since $N^2 = 0$ and $[a, v^{n-1}] = 0$ by Lemma 4, this proves that $[x^{n-1}, y^{n-1}] = 0$. Then, noting that $[u^n, v^n] \in C$, we get $0 = [u^{n(n-1)}, v^{n(n-1)}] = (n-1)^2u^{n(n-2)}v^{n(n-2)}[u^n, v^n]$(Lemma 1), namely $(n-1)^2[u^n, v^n] = 0$. Hence $(n-1)^2[x^n, y^n] = 0$. On the other hand, by Lemma 4, $[a, ny] = n[a, y] = n[a, y^n] = [(1+a)^n, y^n] \in C$. If $p \nmid n$ then $nR = R$, and so $[a, x] \in C$. Hence R is commutative by Lemma 2. If $p \mid n$ then $n^2[x^n, y^n] = 0$. This together with $(n-1)^2[x^n, y^n] = 0$ implies $[x^n, y^n] = 0$. Hence R is commutative, by Lemma 3.

Example 1. Let R be the ring considered in the introduction. Obviously, R is not s-unital but satisfies (*), (i) and (i)$'$ for $n = 2$. This example shows that in Theorems 1 and 2 the hypothesis that R is s-unital cannot be deleted.

Example 2. Let $R = \left\{\begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c \in \mathrm{GF}(4)\right\}$. Then R is a ring with 1 satisfying (*) and (i) for $n = 7$. Obviously, $\left[\begin{pmatrix} a & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] \neq 0$ for $a \neq 0, 1$. This example shows that in Theorem 1 the last hypothesis cannot be deleted.

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(*Received January 13, 1987*)