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SOME PERIODICITY CONDITIONS FOR RINGS

Dedicated to Professor Hisao Tominaga on his 60th birthday

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1. Introduction. Let Z_+ denote the set of positive elements of the ring Z of integers. Let R be an arbitrary associative ring, with center denoted by C. Call an element $a \in R$ *potent* if there exists an integer $n > 1$ for which $a^n = a$; call R a *J-ring* if every element is potent. Define R to be *periodic* if for each $x \in R$, there exist distinct $m, n \in Z_+$ such that $x^m = x^n$.

It is well known that if R is periodic, then $R = P + N$, where P and N denote respectively the sets of potent and nilpotent elements of R. Whether $R = P + N$ implies that R is periodic is apparently not known, except in the presence of additional hypotheses. A recent result in this area, due to Bell and Tominaga [3], is the following:

Theorem B-T. *If R is a ring in which every element has a unique representation as a sum of a potent element and a nilpotent element, then R is a direct sum of a J-ring and a nil ring. In particular, R is periodic.*

The major purpose of this paper is to study periodicity of rings in which the zero divisors satisfy conditions of Bell-Tominaga type. Specifically, letting D and E denote respectively the sets of right zero divisors and idempotents of R, we consider the following conditions:

(†) Each $x \in D$ is uniquely representable in the form $x = a + u$, where $a \in E$ and $u \in N$.

(††) Each $x \in D$ is uniquely representable in the form $x = a + u$, where $a \in P$ and $u \in N$.

Condition (†) was introduced by Abu-Khuzam and Yaqub in [1], and (††) is a natural analogue. The condition of the final theorem of [1] suggests another condition we shall explore briefly, namely $D \subseteq P \cup N$.

An indispensable tool in the study of periodicity is a result due to Chacron [4] (see also [2]): specifically, if R is a ring such that for each $x \in R$, there exist $m \in Z_+$ and $p(X) \in Z[X]$ for which $x^m = x^{m+1}p(x)$, then R is periodic. It follows at once that if $R = P + N$ and N is an ideal, then R must be periodic.

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2. Rings with (††). Before stating the first theorem, we establish two lemmas, the first of which applies to rings satisfying either (†) or (††).

Lemma 1. *Let R be any ring with the property that each $x \in D$ has at most one representation as a sum of an idempotent and a nilpotent element. Let e be an arbitrary idempotent of R.*

(i) *If $e \notin D$, then $x = xe$ for each $x \in R$; if $e \in D$, then $ex = xe$ for each $x \in R$. In particular, $eR = eRe$.*

(ii) *If $\bar{R} = R/A_r(R)$, where $A_r(R)$ is the right annihilator of R, then every idempotent of $\bar{R}$ is central.*

Proof. (i) For arbitrary $x \in R$, consider the idempotents $f_x = e-(ex-exe)$ and $g_x = e-(xe-exe)$. In case $e \notin D$, $(x-xe)e = 0$ implies $x = xe$. In case $e \in D$, by hypothesis, $e = f_x+(ex-exe) = g_x+(xe-exe)$ implies $ex-exe = xe-exe$, i.e., $ex = xe$.

(ii) Suppose, to the contrary, that there exists a non-central idempotent $\bar{f} = f+A_r(R)$ in $\bar{R}$. Then $R(f^2-f) = 0$; in particular, $f^2 = f^3$, so that $e = f^2$ is a non-central idempotent of R with $\bar{f} = \bar{e}$. By (i), $e \notin D$ and $Re = R$, and furthermore $R(ex-x) = 0$ for each $x \in R$, whence we see that $\bar{e}\bar{x} = \bar{x} = \bar{x}\bar{e}$. This is a contradiction.

Lemma 2. *Let R be any ring which satisfies (††) and has the property that $N^2 \subseteq N$. If $a \in P$ and $u \in N$, then $au \in N$.*

Proof. Suppose that $u^k = 0$ and $a^n = a$, $n > 1$; and consider $e = a^{n-1} \in E$. By Lemma 1 (i), either $e \in C$ or e is a right identity element for R.

Note that $(a^i ua^{n-i-1})^k = a^i(ue)^{k-1}ua^{n-i-1}$ $(i = 1, 2, \cdots, n-2)$, which is equal to either $a^i eu^k a^{n-i-1} = 0$ or $a^i u^k a^{n-i-1} = 0$, depending on the nature of e. In either event, $a^i ua^{n-i-1} \in N$. Since $N^2 \subseteq N$, we now get $(au)^{n-1} = (aua^{n-2})(a^2ua^{n-3})\cdots(a^{n-2}ua)u \in N$. Thus $au \in N$.

Theorem 1. *Let R be a ring satisfying (††). If $N^2 \subseteq N$, then either $N = D$ or R is a direct sum of a J-ring and a nil ring.*

Proof. Assume $N \neq D$, and choose an element x of $D\backslash N$: $yx = 0$, $y \neq 0$. We write $x = a+u$, where $a^n = a \neq 0$ and $u \in N$. We assume without loss $n \geq 3$. Then $e = a^{n-1}$ is a non-zero idempotent with $ea = ae = a$. Left-multiplying the above by a^{n-2} gives $a^{n-2}x = e+a^{n-2}u$, where $a^{n-2}u \in N \cap eR$ by Lemma 2. Hence $a^{n-2}x$ is invertible in $eR = eRe$ (Lemma 1 (i)): $a^{n-2}xw = e$ with some $w \in eR$. Our first task is to show that e is in D, and consequently in C (Lemma 1 (i)). Suppose, to the contrary, that $e \notin D$. Then e is a right identity element of R, and so $0 = yexwa^{n-2} = ya \cdot a^{n-2} xwa^{n-2} = ya \cdot ea^{n-2} = ye = y$, a contradiction.

Noting that $eR \subseteq D$ and e is central, we can easily see that eR satisfies the hypothesis of Theorem B-T, hence is a J-ring. It follows that $eN = 0$. Now consider the direct decomposition $R = eR \oplus A(e)$, where $A(e)$ is the annihilator of e. Since $A(e) \subseteq D$ and $eN = 0$, we see that $A(e)$ satisfies the hypothesis of Theorem B-T, hence is a direct sum of a J-ring and a nil ring; consequently so is R itself.

Corollary 1. *Let R satisfy* (††), *and suppose that $D \neq N$. If N is commutative, then R is commutative.*

Proof. Since N commutative implies $N^2 \subseteq N$, the result follows from Theorem 1 and the well-known fact that J-rings are commutative.

In what follows, we shall frequently find it convenient to compute in $\bar{R} = R/A_r(R)$. For $x \in R$, denote by $\bar{x}$ the element $x + A_r(R)$ of $\bar{R}$; and let $\bar{N}$ be the set of nilpotent elements of $\bar{R}$. Note that $x \in N$ if and only if $\bar{x} \in \bar{N}$, hence N is an ideal of R if and only if $\bar{N}$ is an ideal of $\bar{R}$.

Theorem 2. *Let R satisfy* (††), *and suppose that $N^2 \subseteq N$. If $R = P + N$, then R is periodic.*

Proof. If $N \neq D$, the conclusion is immediate from Theorem 1. Suppose, then, that $N = D$. Since nil rings are obviously periodic, we may assume that $R \neq N$, in which case the hypothesis $R = P + N$ guarantees the existence of a non-zero idempotent e. The proof of Lemma 1 (ii) shows that for each such e, $\bar{e}$ is a multiplicative identity element for $\bar{R}$. It follows that for any non-zero $a \in P$, $\bar{a}$ is invertible in $\bar{R}$.

Now, consider $x \in R \backslash N$. Since $x \notin N$, there exists a non-zero $a \in P$ and $u \in N$ such that $x = a + u$; and we may assume $a^n = a$ with $n > 2$. Then, as in the proof of Theorem 1, we can see that $\bar{a}^{n-2}\bar{x}$ is invertible in $\bar{R}$. Thus, as is well known, $\bar{R}$ is a local ring with radical $\bar{N}$. Therefore, N is an ideal of R, and Chacron's result implies that R is periodic.

3. Rings with (†). While (†) does not seem to yield nice direct-sum decompositions, we can establish periodicity theorems which parallel Theorem 1 and Theorem 2. The following lemma will be needed.

Lemma 3. *Let R be any ring satisfying* (†).

(i) *If $x \in D$, then there exists a positive integer m and central idempotent f in $\langle x \rangle$ such that $x^m = x^m f$. If, furthermore, $x \in yR$ (resp. $x \in Ry$), then $x^m \in y^2 R$ (resp. $x^m \in Ry^2$).*

(ii) *If $u \in N$ and $r \in R$, then $ur \in N$ and $ru \in N$.*

Proof. (i) Writing $x = e+u$, where $e \in E$ and $u \in N$, and noting that $\bar{e}$ is central (Lemma 1 (ii)), we see that $\bar{x}-\bar{x}^2 = \bar{e}+\bar{u}-(\bar{e}+\bar{u})^2 = \bar{u}-\bar{u}^2-2\bar{e}\bar{u} \in \bar{N}$. Thus $(x-x^2)^m = 0$ with some $m \in \boldsymbol{Z}_+$. By standard computation, we obtain $x^m = x^{2m}x'$ with some $x' \in \langle x \rangle$. Then $f = x^m x' \in \langle x \rangle \cap E$ and $x^m = x^m f$; and f is necessarily central by Lemma 1 (i). Now, assume further that $x = yr$ ($r \in R$). Since f is central, we see that $(yr)^m = (yr)^m f = yfr(yr)^{m-1} \in y^2 R$.

(ii) Suppose the assertion is false. Then choose $u \in N$, of minimal index of nilpotency, such that $uR \nsubseteq N$. Then $ur \notin N$ for some $r \in R$, and $u^2 R \subseteq N$. Since $ur \in D$, there exists a positive integer m such that $(ur)^m \in u^2 R \subseteq N$, by (i). This is a contradiction.

Theorem 3. *Let R be any ring satisfying* (†) *and having $N \neq D$. Then R is periodic. Moreover, if N is commutative, then R is commutative.*

Proof. Since $D \supsetneq N$, Lemma 3 (i) guarantees the existence of a nonzero central idempotent e in D. Then for any $x \in R$, $ex \in D$; hence by the proof of Lemma 3 (i), $e(x-x^2)^m = (ex-(ex)^2)^m = 0$ for some $m \in \boldsymbol{Z}_+$. Therefore $(x-x^2)^m \in D$. Again by the proof of Lemma 3 (i), $(x-x^2)^m - (x-x^2)^{2m} \in N$. It is now clear that Chacron's condition is satisfied for all $x \in R$, so that R is periodic.

Suppose now that N is commutative. Continuing with the same central idempotent e, we have $R = R_1 \oplus R_2$, where $R_1 = eR$ and $R_2 = A(e)$. Clearly $R_i \subseteq D$. Moreover, if $x \in R_i$ is represented as $f+v$ with $f \in E$ and $v \in N$, it is easy to show that $f \in D$, hence $f \in C$ (Lemma 1 (i)). It is now clear that R is commutative.

Theorem 4. *If $R = P+N$ and R satisfies* (†), *then R is periodic.*

Proof. If $N \neq D$, Theorem 3 yields the desired conclusion. If $N = D$, the proof of Theorem 2 works, the only change being the replacement of Lemma 2 by Lemma 3 (ii).

4. Rings with $D \subseteq P \cup N$. Our final theorem provides another application of Chacron's theorem.

Theorem 5. *Let R be a ring with N commutative. If each element of D is either potent or nilpotent, then N is an ideal. Moreover, if $D \neq N$, then R is periodic.*

Proof. Of course, $(N, +)$ is a subgroup of $(R, +)$. Assume N is not an ideal, and choose $u \in N$, of minimal index of nilpotency, for which $uR \nsubseteq N$. Then $u^2R \subseteq N$. Let r be an arbitrary element of R. Since $ur \in D$, ur is either potent or nilpotent, hence there exists $k \in \boldsymbol{Z}_+$ such that $e = (ur)^k$ is an idempotent, possibly 0. Since $re - ere \in N$ and $ue \in u^2R$, we have

$$(ur)^{k+2} = u[re-ere, u]r + u^2(re-ere)r + uereur \in u^2R \subseteq N,$$

contrary to the original supposition. Thus, N is an ideal.

Assume now that $N \neq D$. Let d be a fixed element of $D \backslash N$, so that d is potent and some power of d is a nonzero idempotent e in D. Let r be an arbitrary element of R, and note that both er and $r - er$ are in D.

If $er \in N$, then $e(r^k - r) \in N$ for all $k \in \boldsymbol{Z}_+$. On the other hand, if $er \in P$, there exists $m > 1$ such that $(er)^m = er$. Now $ere \equiv er$ mod. N, so $(er)^j \equiv er^j$ for all j; in particular, $(er)^m = er \equiv er^m$ mod. N and hence $e(r^m - r) \in N$.

Repeat the argument with $r - er$ instead of er. Noting that if er and $r - er$ are both potent, we can find a single m which works for both. Thus in all cases, there exists $m > 1$ such that $e(r^m - r) \in N$ and $r^m - r - e(r^m - r) \in N$. Consequently $r^m - r \in N$, and Chacron's condition is satisfied.

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