Math. J. Okayama Univ. 29 (1987), 173–177

(σ, τ)-DERIVATIONS IN PRIME RINGS*

Dedicated to Professor Hisao Tominaga on his 60th birthday

NURCAN ARGAÇ, ARIF KAYA and ALEV KISIR

Throughout, R will represent a prime ring with center C, and I a non-zero right ideal of R. Let σ, τ be ring automorphisms of R. We set $C_{\sigma,\tau} = \{c \in R \mid c\sigma(x) = \tau(x)c$ for all $x \in R\}$, $[x, y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$ and $(x, y)_{\sigma,\tau} = x\sigma(y) + \tau(y)x$; in particular, $C_\tau = C_{1,\tau}$, $[x, y]_\tau = [x, y]_{1,\tau}$ and $(x, y)_\tau = (x, y)_{1,\tau}$. Needless to say, $C_1 = C$, $[x, y]_1 = [x, y] = xy - yx$ and $(x, y)_1 = (x, y) = xy + yx$. Let $d: x \to x'$ be a (σ, τ)-derivation of R, namely an additive endomorphism of R such that $(xy)' = x'\sigma(y) + \tau(x)y'$ for all $x, y \in R$. Given a subset U of R, we set $[U]_{\sigma,\tau} = \{u \in U \mid [u', u]_{\sigma,\tau} \in C_{\sigma,\tau}\}$ and $(U)_{\sigma,\tau} = \{u \in U \mid (u', u)_{\sigma,\tau} \in C_{\sigma,\tau}\}$; in case d is a $(1, \tau)$-derivation, we write $[U]_\tau = [U]_{1,\tau}$ and $(U)_\tau = (U)_{1,\tau}$.

We consider the following conditions:

a) R is commutative and $\sigma = \tau$.

a)* R is a commutative ring of characteristic 2 and $\sigma = \tau$.

b) $[a', a]_{\sigma,\tau} = 0$ for all $a \in I$.

c) $(a', a)_{\sigma,\tau} = 0$ for all $a \in I$.

d) $I = [I]_{\sigma,\tau}$, that is, $[a', a]_{\sigma,\tau} \in C_{\sigma,\tau}$ for all $a \in I$.

e) $I = (I)_{\sigma,\tau}$, that is, $(a', a)_{\sigma,\tau} \in C_{\sigma,\tau}$ for all $a \in I$.

f) $I = [I]_{\sigma,\tau} \cup (I)_{\sigma,\tau}$.

If d is a (σ, σ)-derivation of R, then $\sigma^{-1}d$ is a usual derivation, and so the next is immediate by [3, Theorem 2]: *If d is a non-zero (σ, σ)-derivation, then* a) *and* f) *are equivalent.*

In the present paper, we shall generalize [2, Proposition 2 and Theorem 1] by proving the following theorems.

Theorem 1. *Let R be a prime ring. Let $d: x \to x'$ be a non-zero (σ, τ)-derivation of R, and I a non-zero right ideal of R. Then* a) *and* b) *are equivalent, and* a)* *and* c) *are equivalent.*

Theorem 2. *Let R be a prime ring of characteristic not 2. Let $d: x \to x'$ be a non-zero (σ, τ)-derivation of R, and I a non-zero right ideal of R. Then*

*This work has been supported by TUBITAK, Ankara (TBAG-756)

a), b), d) *and* e) *are equivalent.*

Theorem 3. *Let R be a prime ring of characteristic not* 2. *Let d: $x \rightarrow x'$ be a non-zero (σ, τ)-derivation of R, and I a non-zero ideal of R. Then* a), b), d), e) *and* f) *are equivalent.*

Proof of Theorem 1. Obviously, $\sigma^{-1}d$ is a $(1, \sigma^{-1}\tau)$-derivation of R. So, without loss of generality, we may assume that d is a $(1, \tau)$-derivation. As is easily seen, $I' \neq 0$ and a) and a)* imply b) and c), respectively. Assume b) (resp. c)). Linearlizing the identity b) (resp. c)) on I, we obtain $[a', b]_\tau+[b', a]_\tau = 0$ (resp. $(a', b)_\tau+(b', a)_\tau = 0$) for all $a, b \in I$. Replacing b by ab in the above equation, we get $0 = [a', ab]_\tau+[(ab)', a]_\tau = \tau(a)[a', b]_\tau+[a', a]_\tau b+[a'b+\tau(a)b', a]_\tau = \tau(a)([a', b]_\tau+[b', a]_\tau)+a'ba-\tau(a)a'b = a'[b, a]+[a', a]_\tau b = a'[b, a]$, that is, $a'[b, a]=0$ (resp. $a'(b, a) = 0$) for all $a, b \in I$. Replace b by bx $(x \in R)$ in the above equation to obtain $0 = a'b[x, a]$, i.e., $a'I[x, a] = 0$. This proves that either $I'I = 0$ or $I \subseteq C$. Assume now that $I \nsubseteq C$ (and so $I'I = 0$). Since b) (resp. c)) implies that $\tau(a)a' = 0$ for all $a \in I$, we get $0 = \tau(ab)(ab)' = \tau(aba)b'$ for all $a, b \in I$. Hence $axbax\tau^{-1}(b') = 0$ for all $x \in R$. Thus, either $bI = 0$ or $b' = 0$ for every $b \in I$, by [4, Lemma 2]; therefore $I^2 = 0$. This contradiction shows that $I \subseteq C$ and I is an ideal of R. Now, for the case b), we can apply the argument employed in the last part of the proof of [2, Proposition 2] to see that a) holds. On the other hand, for the case c), we can easily see that $\tau(a) = -a$ for all $a \in I$, Then $-ab = \tau(ab) = \tau(a)\tau(b) = ab$ for all $a, b \in I$, and so $2I^2 = 0$. Hence R is of characteristic 2 and $\tau = 1$.

In preparation for proving Theorems 2 and 3, we need some lemmas.

Lemma 1. *Let d be a non-zero $(1, \tau)$-derivation of R.*

(1) *Let $a, b \in [I]_\tau$ (resp. $(I)_\tau$). Then $a+b \in [I]_\tau$ (resp. $(I)_\tau$) if and only if $a-b \in [I]_\tau$ (resp. $(I)_\tau$).*

(2) *If $b \in (I)_\tau$ then $[b', b^2]_\tau = 0$.*

(3) *Let $a, b \in [I]_\tau$. If R is of characteristic not* 3 *and* f) *holds, then either $a+b \in [I]_\tau$ or $a, b, a+b, a-b \in (I)_\tau$; in particular, if $a \in I \backslash (I)_\tau$ and $b \in [I]_\tau$ then $a+b \in [I]_\tau$.*

(4) *If $I \neq [I]_\tau$, then there is no positive integer n such that $b^n = 0$ for all $b \in I \backslash [I]_\tau$.*

Proof. (1) follows from $[a'-b', a-b]_\tau = -[a'+b', a+b]_\tau + 2([a', a]_\tau + [b', b]_\tau)$ (resp. $(a'-b', a-b)_\tau = -(a'+b', a+b)_\tau + 2((a', a)_\tau + (b', b)_\tau)$), and (2) is obvious by $[x, y^2]_\tau = [(x, y)_\tau, y]_\tau$ for all $x, y \in R$. In order to see (3), it is enough to take (1) into account and follow the proof of [1, Lemma 5]. The proof of (4) is quite similar to that of [3, Lemma 2 (2)].

Lemma 2. *Let d be a non-zero* $(1, \tau)$*-derivation of R. Suppose that R is of characteristic not 2 and* f) *holds.*

(1) *If* $b \in I \setminus [I]_\tau$, *then* $(b^2)' = \tau(b^2)b' = b'b^2 = 0$ *and* $b^2 \neq 0$.

(2) *If* $C \cap I \neq 0$, *then* d) *holds.*

(3) *If* $b \in I \setminus [I]_\tau$ *and x is an element of R with* $x' = 0$, *then* $\tau(b)b'xb^3 = 0$.

Proof. (1) Since $(b^2)' = (b', b)_\tau \in C_\tau$ and $[b', b^2]_\tau = 0$ by Lemma 1 (2), we have $[(b^2+b)', b^2+b]_\tau = [(b^2-b)', b^2-b]_\tau = [b', b]_\tau \notin C_\tau$, namely $b^2+b \notin [I]_\tau$ and $b^2-b \notin [I]_\tau$. By Lemma 1 (1), these together with $(b^2+b)-(b^2-b) = 2b \notin [I]_\tau$ show that $2b^2 = (b^2+b)+(b^2-b) \in (I)_\tau$, and so $b^2 \in (I)_\tau$. Hence $2(b^2)'b^2 = ((b^2)', b^2)_\tau \in C_\tau$ by $(b^2)' \in C_\tau$, i.e., $(b^2)'b^2 = \tau(b^2)(b^2)' \in C_\tau$. Furthermore, by Lemma 1 (2), $0 = (b^2)'[(b^2+b)', (b^2+b)^2]_\tau = 2(b^2)'[b', b^3]_\tau = 2(b^2)'\tau(b^2)[b', b]_\tau$. Now, by $(b^2)'b^2 \in C_\tau$ and $(b^2)' \in C_\tau$, $0 = [(b^2)'b^2, x]_\tau = \tau([b^2, x])(b^2)'$ for all $x \in R$. Hence, as is easily seen, either $b^2 \in C$ or $(b^2)' = 0$. If $b^2 \in C$ then, by $(b^2)'\tau(b^2)[b', b]_\tau = 0$ and $(b^2)' \in C_\tau$, $\tau(b^2)\ R\ (b^2)'R[b', b]_\tau = 0$, and so $(b^2)' = 0$, in any case. Since $b^2+b \notin [U]_\tau$, we have also $0 = ((b^2+b)^2)' = ((b^2+b)', (b^2+b))_\tau = (b', b^2)_\tau = 2b'b^2$. Hence $b'b^2 = 0$ and $\tau(b^2)b' = b'b^2 - [b', b^2]_\tau = 0$.

Finally, we shall prove that $b^2 \neq 0$. Suppose, to the contrary, that $b^2 = 0$. Then $\tau(b)b'b = \tau(b)(b^2)' = 0$. Let $x \in R$. Since $c = (b+bx)'(b+bx) \mp \tau(b+bx)(b+bx)' \in C_\tau$, we see that $0 = \tau(b^2)c = \tau(b)cb = \tau(b)(b'+b'x+\tau(b)x')bxb = \tau(b)b'xbxb$, and so $\tau(b)b' = 0$ by [4, Lemma 2]. But, this forces a contradiction $[b', b]_\tau = (b^2)' - 2\tau(b)b' = 0$.

(2) First, we claim that $(c^2)' \in C_\tau$ and $\tau(c^2) = c^2$ for any $c \in C \cap I$. Let $x \in R$. Then $[c', x]_\tau = [x', c]_\tau$ by $(cx)' = (xc)'$. If $[c', c]_\tau \in C_\tau$ (resp. $(c', c)_\tau \in C_\tau$) then $0 = [[c', c]_\tau, x]_\tau = (c-\tau(c))[c', x]_\tau = (c-\tau(c))[x', c]_\tau = (c-\tau(c))^2 x'$ (resp. $0 = [(c', c)_\tau, x]_\tau = (c^2 - \tau(c^2))x'$). Now, $R' \neq 0$ shows that $\tau(c^2) = c^2$, in any case. Hence $0 = (c^2x)' - (xc^2)' = [(c^2)', x]_\tau$, and so $(c^2)' \in C_\tau$.

Now, suppose, to the contrary, that there exists $b \in I \setminus [I]_\tau$. Take any non-zero $c_0 \in C \cap I$, and put $c = c_0^2 (\neq 0)$. Then $\tau(c) = c$ and $c' \in C_\tau$ by the above claim. Hence $[(b+c)', b+c]_\tau = [b', b]_\tau \notin C_\tau$, namely $b+c \notin [I]_\tau$. Recalling that $((b+c)^2)' = 0 = (b^2)'$ by (1), we get $0 = [((b+c)^2)', b]_\tau = 2[b'c+\tau(b)c'+cc', b]_\tau = 2c[b', b]_\tau$. Since $2c$ is a non-zero central element, this forces a contradiction $[b', b]_\tau = 0$.

(3) Since $c = (b+bxb^2)'(b+bxb^2) \mp \tau(b+bxb^2)(b+bxb^2)' \in C_\tau$ and $(b^2)' = \tau(b^2)b' = 0$ by (1), we get $0 = \tau(b^2)cR = \tau(b^2)Rc$. Hence $c = 0$, because $b^2 \neq 0$ by (1). Now, $\tau(b)b'b = \tau(b)(b^2)' = 0$ and $0 = \tau(b)\,c = \tau(b)(b'+b'xb^2)(b+bxb^2) = \tau(b)b'xb^3+\tau(b)b'xb^3xb^2$. Similarly, $0 = -\tau(b)b'xb^3 + \tau(b)b'xb^3xb^2$, and therefore $2\tau(b)b'xb^3 = 0$. Hence $\tau(b)b'xb^3 = 0$.

Proof of Theorem 2. We may, and shall, assume that d is a $(1, \tau)$-derivation. In view of Theorem 1, it suffices to show that each of d) and e) implies b).

d) ⇒ b). The proof of [2, Theorem 1] still works under the present hypothesis.

e) ⇒ b). Suppose, to the contrary, that b) does not hold. For any $a \in I$, we have $2(a', a)_\tau a^2 = ((a', a)_\tau, a^2)_\tau = ((a^2)', a^2)_\tau \in C_\tau$. So, for any $x \in R$, $0 = [(a', a)_\tau a^2, x]_\tau = (a', a)_\tau[a^2, x]$, and therefore $(a', a)_\tau R[a^2, x] = 0$. Hence either $(a', a)_\tau = 0$ or $a^2 \in C$. By Theorem 1, $(b^2)' = (b', b)_\tau \neq 0$ for some $b \in I$, and hence b^2 is a non-zero element of $C \cap I$, and so d) holds, by Lemma 2 (2); hence b) holds by the above. This is a contradiction.

Lemma 3. *Let d be a $(1, \tau)$-derivation of R. If R is of characteristic 3 and I is a non-zero ideal, then* a), b), d), e) *and* f) *are equivalent.*

Proof. In view of Theorem 2, it suffices to show that f) implies d). First, we claim that $(a^3)' = 0$ for all $a \in I$. Actually, if $a \in [I]_\tau$ then $(a^3)' = a'a^2+\tau(a)a'a+\tau(a)\tau(a)a' = a'a^2+(a'a-[a', a]_\tau)a + \tau(a)(a'a-[a', a]_\tau) = 3a'a^2-3[a', a]_\tau a = 0$; if $a \notin [I]_\tau$ then $(a^3)' = (a^2)'a+\tau(a^2)a' = 0$ by Lemma 2 (1).

Now, suppose, to the contrary, that $I \neq [I]_\tau$. Then, by Lemma 1 (4), there exists $b \in I \backslash [I]_\tau$ with $b^3 \neq 0$. Recalling that $(a^3)' = 0$ for all $a \in I$, we see that $\tau(b)b'(ax)^3b^3 = 0$ for all $a \in I$ and $x \in R$ (Lemma 2 (3)). Since $b^3 \neq 0$, [5, Theorem] proves that $\tau(b)b'I = 0$. Thus $\tau(b)b' = 0$, which together with $(b^2)' = 0$ (Lemma 2 (1)) forces a contradiction $[b', b]_\tau = 0$.

We are now in a position to complete the proof of Theorem 3.

Proof of Theorem 3. We may, and shall, assume that d is a $(1, \tau)$-derivation. In view of Theorem 2 and Lemma 3, it remains only to show that f) implies d) or e) under the hypothesis that R is of characteristic not 2 or 3.

Suppose, to the contrary, that neither d) nor e) holds. Then there exist $a \in [I]_\tau \setminus (I)_\tau$ and $b \in (I)_\tau \setminus [I]_\tau$. If $a+b \in (I)_\tau$ then $(a+b)-b = a \notin (I)_\tau$ shows that $a+2b = (a+b)+b \notin (I)_\tau$ (Lemma 1 (1)), and so $2b = a+2b-a \in [I]_\tau$ (Lemma 1 (3)), a contradiction. On the other hand, if $a+b \notin (I)_\tau$ then $b = (a+b)-a \in [I]_\tau$ (Lemma 1 (3)), again a contradiction.

Acknowledgement. The authors are very much indebted to Professor H. Tominaga for valuable comments to improve the presentation of this paper.

References

[1] L. O. Chung and J. Luh : On semicommuting automorphisms of rings, Canad. Math. Bull. 21 (1978), 13–16.

[2] Y. Hirano and H. Tominaga : Some commutativity theorems for prime rings with derivations and differentially semiprime rings, Math. J. Okayama Univ. **26** (1984), 101–108.

[3] A. Kaya : Semi-centralizing derivations in prime rings, Doğa Turk. J. Math. **11** (1987), 100–105.

[4] E. C. Posner : Derivations in prime rings, Proc. Amer. Math. Soc. 8 (1957), 1093–1100.

[5] A. Richoux : A theorem for prime rings, Proc. Amer. Math. Soc. **77** (1979), 27–31.

Department of Mathematics
Ege University
Bornova, Izmir, Turkey

(*Received January 14, 1987*)