Math. J. Okayama Univ. 29 (1987), 167−171

ON THE COMPLEMENT CLASS TO THE TORSION THEORY

Dedicated to Professor Hisao Tominaga on his 60th birthday

KAZUO SHIGENAGA

In this paper, we shall define the leftover class **L** for a torsion theory (**T**, **F**) to be the class of all modules, each of which does not belong to neither **T** nor **F**, and study some basic properties on this class.

It is shown that the torsion class **T** is splitting if and only if **L** is the class consisting of all modules of the form $M \oplus N$ with M a nonzero torsion module and N a nonzero torsionfree module (Theorem 5), that the torsion class **T** is hereditary if and only if each element of **L** has no nonzero torsionfree essential submodules (Theorem 8), and that the hereditary torsion class **T** is stable if and only if **L** is closed under essential submodules (Theorem 11).

Finally, for a 3-fold torsion theory ($\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$), let $\mathbf{L}_1$ and $\mathbf{L}_2$ be the leftover classes for ($\mathbf{T}_1$, $\mathbf{T}_2$) and ($\mathbf{T}_2$, $\mathbf{T}_3$) respectively. We shall show that $\mathbf{T}_1$ is hereditary if and only if $\mathbf{L}_1 \supseteqq \mathbf{L}_2$, or equivalently, $\mathbf{T}_1 \cap \mathbf{L}_2$ is empty.

1. Preliminaries. Throughout this paper, R is a ring with identity and modules are unitary left R-modules. R-**mod** denotes the category of all R-modules. Let (**T**, **F**) be a torsion theory for R-**mod** with the associated idempotent radical t. Define the *leftover class* **L** of (**T**, **F**) to be the *class of all modules, each of which does not belong to neither* **T** *nor* **F**. Hence we have three classes of modules, namely

$$\mathbf{T} = \{M \mid t(M) = M\},$$
$$\mathbf{F} = \{M \mid t(M) = 0\} \text{ and}$$
$$\mathbf{L} = \{M \mid 0 \neq t(M) \subsetneqq M\}.$$

The modules in **T** are said to be torsion modules, and those in **F** are said to be torsionfree modules. We call the modules in **L** *leftover modules*. We will retain these notations throughout this paper.

For example, when (**T**, **F**) is trivial, i.e.(**T**, **F**) = (R-**mod**, **0**) or (**0**, R-**mod**), **L** is empty. If (**T**, **F**) is a splitting torsion theory, every element of **L** is of the form of $M \oplus N$ with $0 \neq M \in \mathbf{T}$ and $0 \neq N \in \mathbf{F}$ (Theorem 5).

For all undefined notions about torsion theories we refer to Stenstorӧm

[2].

2. Some basic properties of the class L. We are assuming that t is an idempotent radical, but some of our results hold only assuming that t is a preradical or an idempotent preradical.

Proposition 1. *The leftover class* L *is closed under group extensions, direct sums and direct products.*

Proof. Let $0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be an exact sequence with $K \in \mathbf{L}$ and $N \in \mathbf{L}$. Since $t(K) \neq 0$ and $f(t(K)) \leqq t(M)$, $t(M)$ must be nonzero. If $t(M) = M$ then $N = g(M) = g(t(M)) \leqq t(N)$; hence $N = t(N)$, a contradiction. So $0 \neq t(M) \lneqq M$, namely, M is in L.

Next, we show that L is closed under direct sums and direct products. Let $\{M_\alpha\}$ be a family of modules in L. If $t(\oplus M_\alpha) = 0$ then $t(M_\alpha) = 0$. So, $t(\oplus M_\alpha)$ must be nonzero. Since $t(\oplus M_\alpha) \leqq \oplus t(M_\alpha)$, we have $0 \neq t(\oplus M_\alpha) \lneqq \oplus M_\alpha$. Therefore $\oplus M_\alpha$ is in L. Similarly, $\Pi\, M_\alpha$ is in L.

Proposition 2. *A module K is in* L *if and only if K has a nonzero submodule in* T *and a nonzero factor module in* F.

Proof. The "only if" part is trivial. Suppose that K has a nonzero submodule K' in T and a nonzero factor module K/K'' in F. Then, $K' \leqq t(K)$ and hence $t(K) \neq 0$. Let π: $K \to K/K''$ be the natural homomorphism. Then $\pi(t(K)) \leqq t(K/K'') = 0$. Hence $t(K) \leqq K''$. Since $K/t(K) \to K/K'' \to 0$ is exact, it follows that $K/t(K) \neq 0$. Thus K is in L.

Corollary 3. *For a submodule N of a module M, M/N is in* L *if and only if there is a proper submodule N' of M such that $N \lneqq N'$, N'/N is in* T *and M/N' is in* F.

Corollary 4. *If M is a nonzero torsion module and N is a nonzero torsionfree module, then $M \oplus N$ is a leftover module.*

Proof. This is clear by Proposition 2.

Theorem 5. *The following conditions are equivalent.*

(1) T *is splitting.*

(2) L *is the class consisting of all modules of the form $M \oplus N$ with*

M a nonzero torsion module and N a nonzero torsionfree module.

Proof. (1) ⇒ (2). By Corollary 4 the module of the form $M \oplus N$, where M is nonzero torsion and N is nonzero torsionfree, is leftover. Conversely, let K be any element of **L**. Since **T** is splitting, $t(K)$ is a direct summand of K. There is a submodule N of K such that $K = t(K) \oplus N$. Since $N \cong K/t(K) \neq 0$, it follows that N is nonzero torsionfree. (2) ⇒ (1). If K is in **T** or **F**, then $t(K)$ is trivially a direct summand of K. Otherwise K is in **L** and $K = M \oplus N$ for some $M(\neq 0)$ in **T** and $N(\neq 0)$ in **F**. Then $t(K)$ must be equal to M and hence is a direct summand of K.

Proposition 6. *The following conditions are equivalent.*

(1) *If N is a submodule of a torsion module M such that $t(N) \neq 0$, then N is a torsion module.*

(2) **L** *is closed under extensions.* (*i.e., if N is in* **L** *and $N \leqq M$, then M is in* **L**.)

Proof. (1) ⇒ (2). Let N be a leftover module containd in a module M. Since $0 \neq t(N) \leqq t(M)$, $t(M) \neq 0$. If $t(M) = M$ then by (1) N is torsion, a contradiction. (2) ⇒ (1). Let M be a torsion module and N a submodule of M with $t(N) \neq 0$. If N is not torsion, then by (2) M is in **L**, a contradiction. So N is torsion.

Corollary 7. *If* **T** *is hereditary, then* **L** *is closed under extensions.*

Theorem 8. *The following conditions are equivalent.*

(1) **T** *is hereditary.*

(2) *Each element of* **L** *has no nonzero torsionfree essential submodule.*

Proof. (1) ⇒ (2). This follows from the fact that **F** is closed under essential extensions. (2) ⇒ (1). Let M be a nonzero element of **F** and $E(M)$ its injective hull. Since M is a nonzero torsionfree essential submodule of $E(M)$, $E(M)$ is not in **L**. If $E(M)$ is in **T**, then $M \oplus E(M)$ is in **L** by Corollary 4. But this is impossible because $M \oplus M$ is a nonzero torsionfree essential submodule of $M \oplus E(M)$. Thus, $E(M)$ must be in **F**, that is **T** is hereditary.

Lemma 9. *If* **L** *is closed under essential submodules, then* **T** *is stable.*

Proof. Let M be any nonzero torsion module and N an essential extension of M. Since $t(N) \neq 0$, N is not torsionfree. Since $t(N) \leqq_e N$ and $t(N)$ can not be in **L**, N is not in **L** by assumption. Thus N is in **T** and **T** is stable.

Lemma 10. *If* **T** *is hereditary and stable, then* **L** *is closed under essential submodules.*

Proof. Trivial.

Theorem 11. *If* **T** *is hereditary, then the following conditions are equivalent.*

(1) **T** *is stable.*

(2) **L** *is closed under essential submodules.*

Proof. This follows from Lemmas 9 and 10.

3. Leftover class for 3-fold torsion theory. Let $(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)$ be a 3-fold torsion theory. By $\mathbf{L}_1$ and $\mathbf{L}_2$ we denote the leftover classes for $(\mathbf{T}_1, \mathbf{T}_2)$ and $(\mathbf{T}_2, \mathbf{T}_3)$, respectively.

Proposition 12. *The following conditions are equivalent.*

(1) $\mathbf{T}_1$ *is hereditary.*

(2) $\mathbf{T}_1 \subseteqq \mathbf{T}_3$.

(3) $\mathbf{L}_1 \supseteqq \mathbf{L}_2$.

(4) $\mathbf{T}_1 \cap \mathbf{L}_2 = \emptyset$.

(5) *Each element of* $\mathbf{L}_1$ *has no nonzero torsionfree essential submodules.*

(6) $\mathbf{L}_2$ *is closed under essential submodules.*

Proof. (1) $\Leftrightarrow$ (2) is well-known. (2) $\Leftrightarrow$ (3) is clear since $\mathbf{T}_1 \subseteqq \mathbf{T}_3$ means that $(R\text{-}\mathbf{mod}-(\mathbf{T}_1 \cup \mathbf{T}_2)) \supseteqq (R\text{-}\mathbf{mod}-(\mathbf{T}_2 \cup \mathbf{T}_3))$. (3) $\Rightarrow$ (4) is clear. (4) $\Rightarrow$ (3). Since $\mathbf{L}_2 \subseteqq (R\text{-}\mathbf{mod}-\mathbf{T}_1)$, $\mathbf{L}_2 \subseteqq (R\text{-}\mathbf{mod}-\mathbf{T}_1) \cap (R\text{-}\mathbf{mod}-\mathbf{T}_2) = \mathbf{L}_1$. (1) $\Leftrightarrow$ (5) and (1) $\Leftrightarrow$ (6) follow from Theorems 8 and 11, respectively.

Corollary 13. *A 3-fold torsion theory* $(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)$ *has length* 2 *if and only if* $\mathbf{L}_1 = \mathbf{L}_2$.

References

[1] Y. Kurata: On an n-Fold Torsion Theory in the Category ${}_R M$, J. Algebra **22** (1972), 559–572.

[2] B. STENSTRÖM : Rings of Quotients, Springer Verlag, Berlin-Heidelberg-New York, 1975.

UBE TECHNICAL COLLEGE
UBE, 755 JAPAN

(*Received February 4, 1986*)