Math. J. Okayama Univ. 29 (1987), 159—165

ON THE DOUBLE CENTRALIZER OF THE INJECTIVE HULL RELATIVE TO A TORSION THEORY

Dedicated to Professor Hisao Tominaga on his 60th birthday

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1. It is a well-known result due to Lambek that the double centralizer of the injective hull of ${}_RR$ is isomorphic to the localization R_η of R with respect to the Lambek torsion radical η. Recently, A. Beg [1] has generalized this result and proved that, for a given left exact radical t, the double centralizer of the t-injective hull of ${}_RR$ is isomorphic to the localization $R_{t'}$ of R with respect to $t' = t \cap \eta$.

Let t and t' be left exact radicals for R-mod such that $t' \leqq t$ and $t'(R) = 0$. In this paper, generalizing the results due to both Lambek and Beg, we shall give necessary and sufficient conditions for the double centralizer of t-injective hull of ${}_RR$ to be isomorphic to $R_{t'}$. We shall provide an example to show that it is not necessary to assume that $t' = t \cap \eta$.

2. Let R be a ring with identity 1_R and t a left exact radical for R-mod with the corresponding left Gabriel topology $L(t)$. The t-injective hull of ${}_RR$ is given by

$$E_t(R) = \{x \in E(R) \mid (R : x) \in L(t)\},$$

where $E(R)$ denotes the injective hull of ${}_RR$. Then $E_t(R)$ is t-injective and $E_t(R)/R$ is t-torsion, while $E(R)/E_t(R)$ is t-torsion-free.

Let $S = \mathrm{End}({}_RE_t(R))$ and Q the double centralizer of ${}_RE_t(R)$. Then $E_t(R)$ may be seen as a Q-left and S-right bimodule. The canonical mapping $f: R \to Q$ defined by $a \to a_L$, the left multiplication by a, is an injective ring homomorphism and by this mapping we can regard Q as a left R-module. Also the canonical mapping $S \to E_t(R)$ defined by $s \to 1_Rs$ is a surjective S-homomorphism. Furthermore, the mapping $Q \to E_t(R)$ given by $\alpha \to \alpha(1_R)$ is an injective R-homomorphism ([1, Lemma 2.1]) and its image $Q1_R = \{\alpha(1_R) \mid \alpha \in Q\}$ is an R-submodule of $E_t(R)$ containing R. This submodule is also characterized by

$$Q1_R = \{x \in E_t(R) \mid s \in S,\ 1_Rs = 0 \Rightarrow xs = 0\}$$

([1, Proposition 2.2]).

Now let t' be a left exact radical for R-mod with the corresponding left Gabriel topology $L(t')$. Assume that $t' \leqq t$. Then the t'-injective hull $E_{t'}(R)$ of ${}_RR$ is also an R-submodule of $E_t(R)$ containing R. The following lemma connects the R-submodule $E_{t'}(R)$ with $Q1_R$.

Lemma 1. *If* $t'(R)=0$ *and* $Q1_R \leqq E_{t'}(R)$, *then* $Q1_R = E_{t'}(R)$.

Proof. Since $R \leqq Q1_R \leqq E_{t'}(R) \leqq E_t(R)$ and $E_{t'}(R)/R$ is t'-torsion, $E_{t'}(R)/Q1_R$ is also t'-torsion. Therefore, to prove the lemma we may show that $E_{t'}(R)/Q1_R$ is t'-torsion-free. Let $x \in E_{t'}(R)$ and assume that $(Q1_R : x) \in L(t')$. Then, for any $s \in S$ with $1_R s = 0$, we have $(Q1_R : x) \leqq l_R(xs)$. Hence $l_R(xs) \in L(t')$ and so $xs \in t'(E_t(R)) = 0$. This shows that $x \in Q1_R$ and that $E_{t'}(R)/Q1_R$ is t'-torsion-free.

Note that, concerning the assumption of the preceding lemma, the following conditions are equivalent :

(1) $Q1_R \leqq E_{t'}(R)$.

(2) $(R\alpha(1_R)+R)/R$ is t'-torsion for all $\alpha \in Q$.

(3) $Q1_R/R$ is t'-torsion.

(4) Coker(f) is t'-torsion.

The equivalence of the last two conditions follows from the fact that Coker (f) is R-isomorphic to $Q1_R/R$ via $\alpha+f(R) \to \alpha(1_R)+R$.

Now we prove

Lemma 2. *If* $t'(R)=0$, *then*

(1) *Q is t'-torsion-free.*

(2) *Q is t'-injective.*

Proof. (1) is obvious, since Q is isomorphic to $Q1_R$ and $Q1_R \leqq E(R)$.

(2) Suppose that $A \in L(t')$ and that an R-homomorphism $v : A \to Q$ is given. For $x \in E_t(R)$, the R-homomorphism $v_x : A \to E_t(R)$ defined by $a \to v(a)x$ can be extended uniquely to an R-homomorphism $w_x : R \to E_t(R)$. It is easily seen that, for $x, y \in E_t(R)$ and $s \in S$, we have $w_{x+y} = w_x + w_y$ and $w_{xs}(a) = w_x(a)s$ for all $a \in R$. Therefore, the mapping $\alpha : E_t(R) \to E_t(R)$ defined by $x \to w_x(1_R)$ is an S-homomorphism. For each $a \in A$ and $x \in E_t(R)$, $v(a)(x) = v_x(a) = w_x(a) = a \cdot w_x(1_R)$ and hence $v(a) = a\alpha$. This shows that $v : A \to Q$ has an extension $R \to Q$ and thus Q is t'-injective.

3. For a given left exact radical t for R-mod and any R-module ${}_RM$, define the localization of M with respect to t to be

$$M_t = \varinjlim_{A \in L(t)} \mathrm{Hom}_R(A,\ M/t(M)).$$

Then by definition, for each $A \in L(t)$, there is a Z-homomorphism

$$u_A : \mathrm{Hom}_R(A,\ M/t(M)) \to M_t$$

and the canonical mapping

$$\phi_M : M \to M_t$$

is given by $x \to u_R(\bar{x}_R)$ where $\bar{x}_R$ denotes the right multiplication by $\bar{x} = x + t(M)$.

It is well-known (cf. [4]) that

(1) both $\mathrm{Ker}(\phi_M)$ and $\mathrm{Coker}(\phi_M)$ are t-torsion,

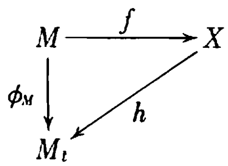
(2) M_t is t-torsion-free, and

(3) M_t is t-injective.

It is shown that these properties characterize the localization of ${}_RM$. Here we shall quote from [2] the following

Proposition 3. *Let M and X be R-modules and $f: M \to X$ an R-homomorphism. Suppose that a left exact radical t for R-mod is given and that both* $\mathrm{Ker}(f)$ *and* $\mathrm{Coker}(f)$ *are t-torsion. Then,*

(1) *there exists a unique R-homomorphism $h : X \to M_t$ making the diagram*



commutative.

(2) $\mathrm{Ker}(h) = t(X)$.

(3) *h is surjective iff $X/t(X)$ is t-injective.*

Proof. (1) Since $\mathrm{Coker}(f)$ is t-torsion, $(f(M) : x) \in L(t)$ for all $x \in X$. Define the R-homomorphism $\alpha : (f(M) : x) \to M/t(M)$ to be $\alpha(a) = m + t(M)$ for $a \in (f(M) : x)$, where $ax = f(m)$ for some $m \in M$. Then the R-homomorphism $h : X \to M_t$ given by $x \to u_{(f(M):x)}(\alpha)$ has the desired property.

Suppose that both h, $h' : X \to M_t$ satisfy the condition that $hf = \phi_M = h'f$. Then the R-homomorphism $\mathrm{Coker}(f) \to M_t$ given by $x + f(M) \to h(x) - h'(x)$ is the zero mapping, and so we have $h = h'$.

(2) Clearly $h(t(X)) \leqq t(M_t) = 0$. Conversely, suppose that $x \in \mathrm{Ker}(h)$. For each $a \in (f(M) : x)$, we can find $m \in M$ such that $ax = f(m)$. Then $m \in \mathrm{Ker}(\phi_M) = t(M)$ and hence $ax \in f(t(M)) \leqq t(X)$. Therefore, $(l_R(x) : a) \in L(t)$, which shows that $l_R(x) \in L(t)$ and thus $x \in t(X)$.

(3) Note that it suffices to prove (3) in case where $t(X) = 0$. Suppose that h is surjective. Let $A \in L(t)$ and $v : A \to X$ any R-homomorphism. Since $\mathrm{Coker}(f)$ is t-torsion, $(v^{-1}(f(M)) : a) = (f(M) : v(a)) \in L(t)$ for $a \in A$. Hence $v^{-1}(f(M)) \in L(t)$. Let $\beta : v^{-1}(f(M)) \to M/t(M)$ be the R-homomorphism given by $a \to m + t(M)$, where $v(a) = f(m)$ for some $m \in M$. Then $u_{v^{-1}(f(M))}(\beta) \in M_t$ and hence by assumption there exists $x \in X$ such that

$$u_{v^{-1}(f(M))}(\beta) = h(x) = u_{(f(M):x)}(\alpha).$$

This means that there exists $A' \in L(t)$ such that $A' \leqq v^{-1}(f(M)) \cap (f(M) : x)$ and that $\alpha(a) = \beta(a)$ for all $a \in A'$.

Let $a \in A'$. Then there are some m and m' in M such that $ax = f(m)$ and $v(a) = f(m')$. Hence $m + t(M) = \alpha(a) = \beta(a) = m' + t(M)$ and $m - m' \in t(M)$. However, $t(M) = \mathrm{Ker}(f)$ and so $v(a) = ax$. The R-homomorphism $A/A' \to X$ given by $a + A' \to v(a) - ax$ is the zero mapping and therefore $v(a) = ax$ for all $a \in A$. This shows that X is t-injective.

Conversely, suppose that X is t-injective. Let $u_A(\beta)$ be any element of M_t, where $A \in L(t)$ and $\beta : A \to M/t(M)$. By assumption there is an R-homomorphism $w : R \to X$ such that the diagram

$$\begin{array}{ccc} A & \leqq & R \\ \beta \downarrow & & \downarrow w \\ M/t(M) & \xrightarrow[f^*]{} & X \end{array}$$

is commutative, where f^* is the R-homomorphism induced by f. For $a \in A$, if we put $\beta(a) = m + t(M)$, then $a \cdot w(1_R) = w(a) = f^*(\beta(a)) = f(m)$. Hence $A \leqq (f(M) : w(1_R))$ and $h(w(1_R)) = u_{(f(M):w(1_R))}(\alpha) = u_A(\beta)$. This shows that h is surjective, which completes the proof of the proposition.

If, in particular, $M = R$, X is a ring and $f : M \to X$ is a ring homomorphism, then h is also a ring homomorphism. To see this, it is enough to note that, for any $y \in X$, the mapping $\mathrm{Coker}(f) \to M_t$ given by $x + f(M) \to h(xy)$

$-h(x)h(y)$ is an R-homomorphism and must be the zero mapping.

Let R be a subring of a ring R'. Given a left exact radical r for R-mod, we shall call R', following [3], a ring of left quotients of R with respect to r, if, for any $x\,(\neq 0) \in R'$, $(R : x) \in L(r)$ and $(R : x)x \neq 0$.

As an application of Proposition 3, we shall prove

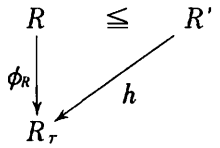
Proposition 4 (*cf.* [3, p. 99, *Proposition* 8], [1, *Proposition* 2.12]). *Let R be a subring of a ring R' and r a left exact radical for R-mod with $r(R) = 0$. Then the following conditions are equivalent :*

(1) *R' is a ring of left quotients of R with respect to r.*

(2) *R'/R is r-torsion and $R \leqq_e R'$ as left R-modules.*

(3) *R'/R is r-torsion and R' is r-torsion-free.*

(4) *There exists a unique injective ring homomorphism $h : R' \to R_r$ making the diagram*



commutative.

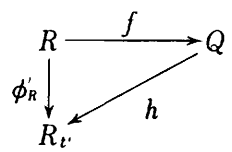
Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious and (3) $\Rightarrow$ (4) follows from Proposition 3.

(4) $\Rightarrow$ (1). Since $R'/R \cong h(R')/\phi_R(R) \leqq R_r/\phi_R(R)$, it follows that R'/R is r-torsion. Let $x\,(\neq 0) \in R'$ and assume that $(R : x)x = 0$. Then $(R : x) \leqq l_R(x)$ and so $x \in r(R')$. However, $h(r(R')) \leqq r(R_r) = 0$ implies that $r(R') = 0$. Therefore, $x = 0$, a contradiction.

4. As another application of Proposition 3, we have

Theorem 5. *Let t and t' be left exact radicals for R-mod such that $t' \leqq t$ and $t'(R) = 0$. Let $E_t(R)$ and $E_{t'}(R)$ be the t- and t'-injective hull of ${}_RR$ respectively and Q the double centralizer of ${}_RE_t(R)$. Then the following conditions are equivalent :*

(1) *There exists a ring isomorphism $h : Q \to R_{t'}$ such that the diagram*



is commutative, where ϕ'_R *denotes the canonical mapping of the localization with respect to* t'.

(2) $\mathrm{Coker}(f)$ *is* t'-*torsion.*

(3) $Q1_R \leqq E_{t'}(R)$.

(4) $Q1_R = E_{t'}(R)$.

Proof. As was already remarked in Section 2, (2), (3) and (4) are equivalent.

(1) ⇒ (2) follows from the fact that $\mathrm{Coker}(f)$ is R-isomorphic to Coker (ϕ'_R), while (2) ⇒ (1) follows from Lemma 2 and Proposition 3.

Note that if we take $t = 1$ and $t' = \eta$ in Theorem 5, then we get Lambek's result mentioned in Section 1, while in case t is any and $t' = t \cap \eta$ we get Beg' one. These facts are immediate consequences of the following

Lemma 6. *With the notation of Section* 2, $Q1_R/R$ is η-*torsion.*

Proof. First we shall show that, for each $\alpha \in Q$,

$$s \in S,\ (R : \alpha(1_R)) \cdot s = 0 \Rightarrow 1_R s = 0.$$

Indeed, by assumption the mapping $\nu : R + R \cdot \alpha(1_R) \to E_t(R)$ given by $a + b \cdot \alpha(1_R) \to bs$ is an R-homomorphism. Since $E_t(R)/(R + R \cdot \alpha(1_R))$ is t-torsion, ν can be extended to an R-homomorphism $s' \in S$. Then $1_R s' = \nu(1_R) = 0$ and thus $1_R s = \nu(\alpha(1_R)) = (\alpha(1_R))s' = \alpha(1_R s') = 0$.

Now, for each $\alpha \in Q$, $(R : \alpha(1_R))$ is dense. For assume that there are a, $b(\neq 0)$ in R such that $((R : \alpha(1_R)) : a)b = 0$. Then the mapping $R \to R$ defined by $c \to cb$ can be extended to an $s \in S$ and $(R : a \cdot \alpha(1_R))s = (R : a \cdot \alpha(1_R))b = 0$. Thus we have $b = 1_R s = 0$, as was shown above. This shows that $(R : \alpha(1_R))$ is dense.

Finally, we shall give an example of a ring R and show that to obtain the isomorphism $Q \cong R_{t'}$ in Theorem 5 it is not necessary to assume that $t' = t \cap \eta$.

Example 7. We may give a ring R for which

(1) R is left non-singular,

(2) the left exact radical t' for R-mod defined by $t'(M) = \{x \in M \mid l_R(x)$ contains a regular element in $R\}$ for each ${}_RM$, is strictly smaller than η, and

(3) $R_{t'}$ is isomorphic to R_η over R.

To do this, let ${}_DV$ be a vector space over a division ring D and let $R =$ End$({}_DV)$. Then, as is well-known, R is a regular, left self-injective ring. Since R is regular, it follows that R is left non-singular and every regular element of R is a unit in R. Therefore, $R_{t'}$ is isomorphic to R_η over R. If, in addition, ${}_DV$ is infinite dimensional over D, then $R \neq \mathrm{soc}({}_RR)$ and so $L(t') \subsetneqq L(\eta)$. Thus, we have $t' \lneqq \eta$.

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(*Received November 1, 1985*)