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ON MORITA PAIRS OF RINGS

Dedicated to Professor Hisao Tominaga on his 60th birthday

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1. Introduction. Let $(Q, R;\ S, T;\ \mu, \nu)$ be a Morita context where Q and R are rings with or without identities, $S = {}_QS_R$, $T = {}_RT_Q$, and μ and ν are homomorphisms of $S \otimes_R T$ to Q and of $T \otimes_Q S$ to R, respectively ([2]). Especially when μ and ν are surjective, we say that (Q, R) is a Morita pair of rings. In this case, rings Q and R have many common properties. For example, it was shown in [4] and [6] that there is a one-to-one correspondence between properly generated Q- and R-modules. The purpose of this note is to derive some important Morita pairs from a given Morita pair. For a ring Q, we denote by $\mathfrak{P}(Q)$ the prime radical of Q, and by $\mathfrak{J}(Q)$ the Jacobson radical of Q. We shall show that if Q and R have unities, then $(\mathfrak{P}(Q), \mathfrak{P}(R))$ and $(\mathfrak{J}(Q), \mathfrak{J}(R))$ are Morita pairs, and that, regardless of the existence of unities, $(Q/\mathfrak{P}(Q), R/\mathfrak{P}(R))$ and $(Q/\mathfrak{J}(Q), R/\mathfrak{J}(R))$ are Morita pairs. The same results hold for another kind of radicals which are the intersections of maximal ideals of rings provided the rings Q and R satisfy $Q^2 = Q$ and $R^2 = R$. These last results are slightly generalized in a general case, i.e., when rings are not idempotent as above. In a general case, we consider as a radical the intersection of all maximal non-special ideals, where ideals are called non-special if they contain no powers of the rings.

The methods used in this note are the ring-theoretic formalisms of a Morita context. A Morita context is considered as a gamma ring, which is a natural generalization of a gamma ring of homomorphisms given in [5] and [6]. The ring-theoretic formalisms of a gamma ring will be explained in 2. In 3, the correspondences of ideals of Q and R will be discussed. Then, in 4, we prove the above mentioned results.

2. Gamma rings of Morita contexts. Let (Q, R) be a Morita pair as in 1. Denote $\mu(s \otimes t)$ by st and $\nu(t \otimes s)$ by ts for $s \in S$ and $t \in T$. Then, $Q = ST = \{\sum_i s_it_i \mid s_i \in S,\ t_i \in T\}$. Also, $R = TS$. We have $(st)s' = s(ts')$ for $s, s' \in S$ and $t \in T$ due to the associative properties of a Morita context. So, we denote $(st)s'$ by sts', etc. Clearly, $STS \subseteq S$, and similarly, $TST \subseteq T$. A pair (S, T) of modules S and T satisfying the above conditions is called a gamma ring. (For more precise definitions, see [1] or [3].) A gamma ring (S, T) which is obtained from a Morita context as above is called a gamma ring of a Morita context. In the previous works [5] and [6], a gamma ring of homomorphisms was introduced and the relations between the left and right operator rings were obtained. The same results hold for a gamma ring of a Morita context where Q and R are considered as the left and right operator rings.

In the following, we consider left modules over a ring unless mentioned otherwise. Let M be a Q-module. We define an R-module TM as follows. For $t \in T$ and $m \in M$, let tm be a homomorphism of S to M defined by $tm\colon s \to (st)m$. We define TM as the submodule of $\mathrm{Hom}_Q(S, M)$ generated by all tm for $t \in T$ and $m \in M$.. For $r \in R$ and $\sum_i t_i m_i \in TM$, define $r(\sum_i t_i m_i) = \sum_i (rt_i)m_i$. We have to show that it is well defined. Let $\sum_i t_i m_i = 0$. It is enough to show that $\sum_i (rt_i)m_i = 0$. First, let $r = ts$. Then $\sum_i (tst_i)m_i$ maps any element s' of S to $\sum_i (s'ts)t_i m_i = (s'ts) \sum_i t_i m_i = 0$. Hence, $\sum_i (tst_i)m_i = 0$. Since $R = TS$, we have $\sum_i (rt_i)m_i = 0$ as required. Thus, TM is an R-submodule of $\mathrm{Hom}_Q(S, M)$. Similarly, for an R-module N, we can define SN which is a Q-submodule of $\mathrm{Hom}_R(T, N)$.

Let M be a Q-module. We say that M is properly generated (over Q) if (i) $QM = M$ and (ii) $Qm = 0$ implies $m = 0$ for $m \in M$. Assume that M is properly generated. We want to show that TM is properly generated (over R). First, we see easily that $R(TM) = (RT)M = (TQ)M = T(QM) = TM$. Next, suppose that $R(\sum_i t_i m_i) = 0$. Then, $\sum_i (tst_i)m_i = 0$ for any s and t, and hence $\sum_i (s'tst_i)m_i = 0$ for any s', which implies $Q(\sum_i st_i m_i) = 0$ as Q is generated by $s't$. Since M is properly generated, we have $\sum_i st_i m_i = 0$. Therefore, $\sum_i t_i m_i = 0$ as required.

Let M be a properly generated Q-module as above. Then $S(TM)$ is a properly generated Q-module as we can apply the above argument twice. We can show that $S(TM)$ is isomorphic with M in a natural sense. For it, consider the mapping $\sum_i s_i(\sum_j t_{ij} m_{ij}) \to \sum_{i,j} s_i t_{ij} m_{ij}$ of $S(TM)$ to M. The mapping is an isomorphism by a similar argument as above. In the following, we identify $S(TM)$ and $M = (ST)M$.

Proposition 1. *If M is an irreducible Q-module, then TM is an irreducible R-module.*

Proof. First, note that an irreducible Q-module M is a properly generated Q-module. For, $QM = M$ by the definition of an irreducible module.

Let $M' = \{m \in M \mid Qm = 0\}$. Then, M' is a proper submodule of M and hence $M' = 0$. Now, let N be a non-zero R-submodule of TM. SN is a submodule of M and hence $SN = 0$ or $SN = M$. If $SN = 0$, then $RN = TSN = 0$, which implies $N = 0$. Just note that TM is a properly generated R-module as shown early and that N is a subset of TM. Therefore, SN must coincide with M. Then, $TM = TSN = RN \subseteq N$, which implies $TM = N$. We have shown that TM is an irreducible R-module.

3. **Correspondences of ideals.** Let (Q, R) be a Morita pair. Let A be an ideal of Q. We define $A_* = TAS$ and $A^* = S^{-1}AT^{-1} = \{y \in R \mid SyT \subseteq A\}$. A_* and A^* are ideals of R. Let U be an ideal of R. We define $U_* = SUT$ and $U^* = T^{-1}US^{-1}$. Let $A_c = (A_*)_*$ and $A^c = (A^*)^*$. Similarly, we define U_c and U^c. When $A_c = A$, we say that A is closed below, and when $A^c = A$, we say that A is closed above. Then the mapping $A \to A_*$ gives a bijection of the set of lower closed ideals of Q to that of R. Similarly, $A \to A^*$ gives a bijection of the set of upper closed ideals of Q to that of R. In this note, we consider the correspondence $A \to A^*$. In this connection, note that a prime ideal is always closed above. Thus, the above mapping gives a bijection of the set of prime ideals of Q to that of R.

Next, we consider a primitive ideal of Q. Let P be a primitive ideal of Q. It is defined as $P = (0 : M) = \{x \in Q \mid xM = 0\}$ for some irreducible Q-module M. P is a prime ideal, and hence is closed above.

Proposition 2. *If P is a primitive ideal of Q, then P^* is a primitive ideal of R.*

Proof. Let $P = (0 : M)$ as above. By Proposition 1, TM is an irreducible R-module. We show that $P^* = (0 : TM)$. Let $U = (0 : TM)$. P^*TM is an R-submodule of an irreducible TM, and hence $P^*TM = 0$ or $P^*TM = TM$. $SP^*TM \subseteq PM = 0$, and hence $P^*TM \neq M$. So, $P^*TM = 0$. Hence, $P^* \subseteq U$. Since $STM = M$, we have $U^* \subseteq P$ by symmetry. Then, $U = U^{**} \subseteq P^*$. Thus, $P^* = U = (0 : TM)$, and hence P^* is a primitive ideal of R.

Let A_i $(i \in I)$ be upper closed ideals. Then $\bigcap_i A_i$ is also closed above, because we have $(\bigcap_i A_i)^* = \bigcap_i A_i^*$. Note that the latter identity holds for any ideals A_i. This fact is applied for radicals. The prime radical of a ring is defined to be the intersection of all prime ideals. In the above, let A_i range over all prime ideals of Q. Then, $\bigcap_i A_i = \mathfrak{P}(Q) =$ the prime radical of Q and we have $(\mathfrak{P}(Q))^* = \mathfrak{P}(R)$. Next, let A_i range over all primitive ideals of Q. Then $\bigcap_i A_i = \mathfrak{J}(Q)$ = the Jacobson radical of Q. We have obtained

Theorem 1. *$\mathfrak{P}(Q)$, $\mathfrak{P}(R)$, $\mathfrak{J}(Q)$ and $\mathfrak{J}(R)$ are all closed above, and we have $(\mathfrak{P}(Q))^* = \mathfrak{P}(R)$ and $(\mathfrak{J}(Q))^* = \mathfrak{J}(R)$.*

We apply the above argument for A_i which range over maximal ideals of Q. However, a maximal ideal is not necessarily closed above. We need a new concept. An ideal A of Q is said non-special if A does not contain Q^n for any integer n. Note that if $Q^2 = Q$, then Q is the only special ideal. Now, an ideal A is said maximally non-special if A is non-special and every ideal which contains A properly is special.

Proposition 3. *If A is a maximally non-special ideal of Q, then A is closed above and A^* is a maximally non-special ideal of R.*

Proof. First, we show that if B is a non-special ideal of Q, then B^* is a non-special ideal of R. For it, suppose that B^* contains R^n for some n. Then, $B \supseteq SB^*T \supseteq SR^nT = Q^{n+1}$, which is a contradiction. Thus, B^* is non-special. Then, B^c is also non-special. So, for the ideal A in Proposition 3, A^c is non-special. Since A is maximally non-special and $A^c \supseteq A$, we have $A^c = A$, or A is closed above. Next, let U be an ideal of R containing A^* properly, and assume that U is non-special. Then, U^* is non-special and contains $(A^*)^* = A$. Thus, $U^* = A$. Then, $U \subseteq (U^*)^* = A^*$, which is a contradiction. We have proved Proposition 3.

Define $\mathfrak{M}(Q)$ = the intersection of all maximally non-special ideals of Q. The following Theorem 2 is clear.

Theorem 2. *$\mathfrak{M}(Q)$ is closed above and $(\mathfrak{M}(Q))^* = \mathfrak{M}(R)$.*

4. Morita pairs of rings. First, recall that a Morita pair is a pair (Q, R) of rings Q and R such that $Q = ST$ and $R = TS$ with some modules S and T where we have $STS \subseteq S$ and $TST \subseteq T$. For example, if I and J are ideals of a ring, then (IJ, JI) is a Morita pair. Now, let (Q, R) be a Morita pair. If A is an ideal of Q, then a pair (AST, TAS) is also a Morita pair because $(AS)T(AS) \subseteq AS$ and $T(AS)T \subseteq T$. Similarly, (STA, TAS) is a Morita pair.

Proposition 4. *If A is an ideal of Q, then $(AQ, A*)$ and $(QA, A*)$ are Morita pairs. If A is closed below, then $(A, A*)$ is a Morita pair.*

Proof. Just note that $Q = ST$ and $A* = TAS$. Also note that if A is closed below then $QAQ = QA = AQ = A$.

Corollary. *(Q^n, R^n) is a Morita pair.*

Proof. Let $A = Q^{n-1}$ in Proposition 4.

Theorem 3. *Suppose that Q and R have unities. Then, $(\mathfrak{P}(Q), \mathfrak{P}(R))$, $(\mathfrak{J}(Q), \mathfrak{J}(R))$ and $(\mathfrak{M}(Q), \mathfrak{M}(R))$ are all Morita pairs.*

Proof. Theorem 3 follows from the latter part of Proposition 4. Note that if A is an ideal of Q then $A^* = S^{-1}AT^{-1} = (TS)S^{-1}AT^{-1}(TS) \subseteq TAS = A*$ and hence $A^* = A*$.

Proposition 5. *Let A be an ideal of Q. Then, $(Q/A, R/A*)$ and $(Q/A, R/A^*)$ are Morita pairs.*

Proof. Let $\bar{S} = S/AS$, $\bar{T} = T/TA$, $\bar{Q} = Q/A$ and $\bar{R} = R/A*$. $\bar{s}$ denotes an element of $\bar{S}$ which is represented by s, and $\bar{t}$ denotes an element of $\bar{T}$ represented by t. Define $\bar{s}\bar{t}$ as an element of $\bar{Q}$ represented by st. Similarly define $\bar{t}\bar{s}$ as an element of $\bar{R}$ represented by ts. We can verify that these are well defined. Then we have $\bar{S}\bar{T} = \bar{Q}$ and $\bar{T}\bar{S} = \bar{R}$. Next, define $\bar{s}\bar{t}\bar{s}'$ as an element of $\bar{S}$ represented by sts'. Similarly, define $\bar{t}\bar{s}\bar{t}'$. It is almost routine to verify that we obtain a Morita pair $(\bar{Q}, \bar{R})$. For the second part, let $\hat{S} = S/AT^{-1}$, where $AT^{-1} = \{s \in S \mid sT \subseteq A\}$, $\hat{T} = T/S^{-1}A$ where $S^{-1}A = \{t \in T \mid St \subseteq A\}$, $\hat{Q} = Q/A$ and $\hat{R} = R/A^*$. As in the first part, we can show that $(\hat{Q}, \hat{R})$ is a Morita pair.

Theorem 4. *Let (Q, R) be a Morita pair. Then, $(Q/\mathfrak{R}(Q), R/\mathfrak{R}(R))$ are Morita pairs where $\mathfrak{R} = \mathfrak{P}$, $\mathfrak{J}$ or $\mathfrak{M}$.*

Proof. Theorem 4 is a direct consequence of the latter part of Proposition 5.

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