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CLOSED IDEALS IN NON-UNITAL MATRIX RINGS

Dedicated to Professor Hisao Tominaga on his 60th birthday

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1. Introduction. Closed ideals in a non-unital ring were first introduced in [3] to study the correspondence of ideals in Morita equivalent rings. Let I be an ideal of a ring R. We say that I is *lower closed* if $RIR = I$. Every irreducible ideal of R is lower closed. On the other hand, we say that I is *upper closed* if $R^{-1}IR^{-1} = I$, where $R^{-1}IR^{-1} = \{x \in R \mid RxR \subseteq I\}$. Every prime ideal is upper closed. Some properties which are usually satisfied by ideals of a unital ring fail for general ideals of a non-unital ring. But, they are satisfied by closed ideals in the above sense. For example, it is well known that there is a one-to-one correspondence between ideals of two Morita equivalent unital rings. This is not true in case of non-unital rings. However, if we restrict to closed ideals, the same property holds. (See Theorem 3 and Theorem 5.) In this paper, we consider two types of matrix rings. One is the total matrix ring over a ring, and the other is a Morita context which is considered as a subring of a matrix ring of 2×2 over a ring. Let R be a non-unital ring, and R_n the total matrix ring of $n\times n$ over R. It is known that an arbitrary ideal of R_n is not necessarily the total matrix ring A_n over an ideal A of R (contrary to the unital ring case). In 2, we show that every closed (lower or upper) ideal of R_n is expressed as A_n with some ideal A of R. Moreover, it will be shown that the ideal of R_n is lower (or upper) closed if and only if A is lower (or upper) closed. In 3, we deal with a Morita context ring $C = C_{11} \oplus C_{22} \oplus C_{12} \oplus C_{21}$, where C_{ij} are submodules satisfying that $C_{ij}C_{jk} \subseteq C_{ik}$ and $C_{ij}C_{km} = 0$ if $j \neq k$. It is easy to see that C is considered as a subring of a total matrix ring of 2×2 over a ring. Here, we do not assume that C has the identity. However, we have to assume that $C_{12}C_{21} = C_{11}$ and $C_{21}C_{12} = C_{22}$. First, we show that every closed ideal I of C is a Morita context ring $I = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21}$, where $I_{ij} = I \cap C_{ij}$ are C_{ii}-C_{jj}-submodules of C_{ij}. We can define upper and lower closed submodules, and we will show that I is lower(or upper closed if and only if all I_{ij} are lower(or upper closed. When there exists an ideal $I = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21}$, we say that submodules I_{11}, I_{22}, I_{12} and I_{21} correspond to each other via the ideal I. We can show that the correspondence is one-to-one among all upper (or lower) closed submodules. Especially, I_{11} and I_{22} are ideals of C_{11} and C_{22}, and the correspondence between closed ideals is one-to-one. C_{11} and C_{22} are Morita equivalent in a general sense, and the above result is a generalization of the result in case of a unital ring. See [1].

2. Closed ideals in a total matrix ring. Let R_n be the total matrix ring of $n \times n$ over a non-unital ring R. Let e_{ij} be the matrix units. Note that e_{ij} do not exist in R_n. However, the formal multiplication by e_{ij} will always make sense in the following context.

Proposition 1. *Let I be an ideal of R_n. There exist ideals A and B of R such that*

$$R_nIR_n \subseteq B_n \subseteq I \subseteq A_n \subseteq R_n^{-1}IR_n^{-1}.$$

Proof. Let $I(i,j) = \{r \in R \mid r$ appears in the (i,j)-entry of some element of $I\}$. It is clear that $I(i,j)$ is an ideal of R. We have

$$(RI(i,j)R)e_{km} \subseteq I, \tag{1}$$

because $(RI(i,j)R)e_{km} = (Re_{ki})I(Re_{jm}) \subseteq R_nIR_n \subseteq I$. Let $A = \sum_{i,j} I(i,j)$. A is an ideal of R, and $I \subseteq A_n$. On the other hand, $R_nA_nR_n \subseteq I$, because $e_{kk}(R_nA_nR_n)e_{mm} = (RAR)e_{km} \subseteq I$ by (1). Therefore, $A_n \subseteq R_n^{-1}IR_n^{-1}$. We obtained $I \subseteq A_n \subseteq R_n^{-1}IR_n^{-1}$. Let $B = RAR$. Then, $B_n = R_nA_nR_n$. Since $I \subseteq A_n \subseteq R_n^{-1}IR_n^{-1}$, we have $R_nIR_n \subseteq B_n \subseteq I$. The proof of Proposition 1 is completed.

From Proposition 1, we can conclude that if I is lower (or upper) closed ideal of R_n, it is the total matrix ring over an ideal of R.

Theorem 1. *Let I be an ideal of R_n. I is upper closed if and only if $I = A_n$ with an upper closed ideal A of R. I is lower closed if and only if $I = B_n$ with a lower closed ideal B of R.*

Proof. First, suppose that I is upper closed. Then, $I = A_n$ with an ideal A as noted above. It is clear that $R_n(R^{-1}AR^{-1})_nR_n \subseteq A_n$. Since A_n is upper closed, we have $(R^{-1}AR^{-1})_n \subseteq A_n$ and $R^{-1}AR^{-1} \subseteq A$. Therefore, $R^{-1}AR^{-1} = A$ and A is upper closed. Conversely, let A be an upper closed ideal of R. We show that A_n is upper closed. Let x be an element of R_n such that $R_nxR_n \subseteq A_n$. Let r_{ij} be an element of R such that $e_{ii}xe_{jj} = r_{ij}e_{ij}$.

Then, $(Rr_{ij}R)e_{km} = (Re_{ki})x(Re_{jm}) \subseteq A_n$. So, $Rr_{ij}R \subseteq A$. Since A is upper closed, $r_{ij} \in A$. Hence, $x \in A_n$ and A_n is upper closed. Secondly, suppose that I is lower closed. Then, $I = B_n$ with an ideal B. $(RBR)_n = R_nB_nR_n = R_nIR_n = I = B_n$. So, $RBR = B$ and B is lower closed. Conversely, let B be a lower closed ideal of R. Then, $R_nB_nR_n = (RBR)_n = B_n$. Hence, B_n is lower closed. The proof of Theorem 1 is completed.

Corollary. (i) *If I is a prime ideal of R_n, then $I = P_n$ with a prime ideal P of R.*

(ii) *If $x \in R_nxR_n$ for every element x of an ideal I of R_n, then $I = A_n$ with an ideal A of R.*

Proof. (i) A prime ideal of R_n is upper closed. For, let I be a prime ideal of R_n. Then, $R_n(R_n^{-1}IR_n^{-1})R_n \subseteq I$ implies $R_n^{-1}IR_n^{-1} \subseteq I$. Thus, I is upper closed. Then, $I = P_n$ with an ideal P of R. We want to show that P is prime. Let $CD \subseteq P$ for ideals C and D of R. $C_nD_n = (CD)_n \subseteq P_n = I$ implies $C_n \subseteq I$ or $D_n \subseteq I$. If $C_n \subseteq I = P_n$, then $C \subseteq P$. If $D_n \subseteq I$, then $D \subseteq P$. P is a prime ideal.

(ii) Suppose that the condition of (ii) of Corollary is satisfied. Then, $R_nIR_n = I$, and I is lower closed. So, $I = A_n$ with an ideal A of R.

(i) of Corollary is obtained by Sands [4]. (ii) of Corollary is obtained by Luh [2].

3. Closed ideals in a Morita context ring. A subring S of R_2 is called a *Morita context ring* (or a *M. c. ring*) if $S = S_{11} \oplus S_{22} \oplus S_{12} \oplus S_{21}$, where $S_{ij} = e_{ii}Se_{jj}$. Thus, S is a M. c. ring if and only if S contains all S_{ij}. Note that $S_{ij}S_{jk}$ is contained in S_{ik} but is not necessarily equal to S_{ik}. S_{ii} are rings, and S_{ij} are S_{ii}-S_{jj}-bimodules. In the following, we fix a M. c. ring C, which satisfies the conditions $C_{12}C_{21} = C_{11}$ and $C_{21}C_{12} = C_{22}$. Under this basic assumption, we have $C_{11}C_{12} = C_{12}C_{22}$ and $C_{22}C_{21} = C_{21}C_{11}$. Let I be an ideal of C. I is not necessarily a M. c. ring, and in this direction we have Proposition 2 which is an analogue of Proposition 1.

Proposition 2. *Let I be an ideal of C. Then, there exist ideals A and B which are M. c. rings such that $CIC \subseteq B \subseteq I \subseteq A \subseteq C^{-1}IC^{-1}$.*

Proof. Let $A = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21}$. A is an ideal of C as well as a M. c. ring. Clearly, $I \subseteq A$. We have $C_{ik}I_{km}C_{mj} \subseteq C_{ik}IC_{mj} \subseteq I$. Hence,

$CAC \subseteq I$ and hence $A \subseteq C^{-1}IC^{-1}$. Next, let $B = CAC$. $B_{ij} = e_{ii}CACe_{jj} \subseteq CAC = B$. So, B is a M. c. ring. Clearly, B is an ideal of C. It is also clear that $CIC \subseteq B \subseteq I$.

Proposition 2 implies that a lower (or upper) closed ideal of C is a M. c. ring.

Lemma. *$C_{ij}C_{jk}$ is either C_{ii} or $C_{ii}C_{ik}$. Similarly, $C_{ij}C_{jk}$ is either C_{kk} or $C_{ik}C_{kk}$.*

Proof. If $i \neq j \neq k$, then $i = k$ and $C_{ij}C_{jk} = C_{ij}C_{ji} = C_{ii}$. Otherwise, $C_{ij}C_{jk} = C_{ii}C_{ik}$ due to the fact $C_{12}C_{22} = C_{11}C_{12}$ and $C_{21}C_{11} = C_{22}C_{21}$. The second part is similarly proven.

Let M_{ij} stand for a C_{ii}-C_{jj}-submodule of C_{ij} in general. We say that M_{ij} is *lower closed* if $C_{ii}M_{ij}C_{jj} = M_{ij}$.

Theorem 2. *An ideal I of C is lower closed if and only if $I = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21}$ with lower closed I_{ij}.*

Proof. Suppose that I is lower closed. Then, $I = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21}$ as above. We want to show that I_{ij} is lower closed. Since $I = CIC = C(CIC)C$, we have $I_{ij} = \sum_{s,k,m,t} C_{is}C_{sk}I_{km}C_{mt}C_{tj}$. Now, by Lemma, $C_{is}C_{sk} \cdot I_{km}C_{mj}C_{jj} \subseteq C_{ii}I_{ij}C_{jj}$. So, $I_{ij} \subseteq C_{ii}I_{ij}C_{jj}$, or I_{ij} is lower closed. Conversely, suppose that $I = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21}$ with lower closed I_{ij}. Then, $CIC \supseteq C_{11}I_{11}C_{11} \oplus C_{22}I_{22}C_{22} \oplus C_{11}I_{12}C_{22} \oplus C_{22}I_{21}C_{11} = I$. Thus, $CIC = I$ and I is lower closed.

Let π_{ij} be the mapping of the set of ideals of C to the set of C_{ii}-C_{jj}-submodules of C_{ij} such that $\pi_{ij}(I) = e_{ii}Ie_{jj}$.

Theorem 3. *π_{ij} induces a bijection of the set of lower closed ideals of C to the set of lower closed C_{ii}-C_{jj}-submodules of C_{ij}.*

Proof. Let $I = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21}$ be a lower closed ideal of C, where I_{ij} are all lower closed. We show that $I_{km} = C_{ki}I_{ij}C_{jm}$ for any i, j, k and m. For example, suppose that $i = k$ and $j \neq m$. Then, $I_{km} = I_{im} \supseteq C_{ii}I_{ij}C_{jm} \supseteq C_{ii}(I_{im}C_{mj})C_{jm} = C_{ii}I_{im}C_{mm} = I_{im} = I_{km}$. Therefore, $I_{km} = C_{ii}I_{ij}C_{jm} = C_{ki}I_{ij}C_{jm}$ as required. All the other cases are similarly proven. Now, $I_{km} = C_{ki}I_{ij}C_{jm}$ implies that I_{km} is uniquely determined by I_{ij} for any

k and m. Therefore, I is uniquely determined by I_{ij}. Conversely, let M_{ij} be a lower closed C_{ii}-C_{jj}-submodule of C_{ij}. Let $I_{km} = C_{ki}M_{ij}C_{jm}$. It is easily verified that I_{km} is lower closed. Let $I = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21}$. We can show that I is a lower closed ideal. This completes the proof of Theorem 3.

In order to discuss the upper closed case, we define the operators C_{ij}^{-1} as follows. Define $C_{ik}^{-1}M_{ij} = \{x \in C_{kj} \mid C_{ik}x \subseteq M_{ij}\}$ and $M_{ij}C_{kj}^{-1} = \{x \in C_{ik} \mid xC_{kj} \subseteq M_{ij}\}$. Now, we say that M_{ij} is *upper closed* if $C_{ii}^{-1}M_{ij}C_{jj}^{-1} = M_{ij}$.

Theorem 4. *Let I be an ideal of C. I is upper closed if and only if $I = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21}$ with upper closed I_{ij}.*

Proof. First, suppose that I is an upper closed ideal of C. Then, $I = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21}$ as above. We want to show that I_{ij} is upper closed. For it, observe that $CC(C_{ii}^{-1}I_{ij}C_{jj}^{-1})CC \subseteq I$ which follows due to Lemma. So, $C_{ii}^{-1}I_{ij}C_{jj}^{-1} \subseteq C^{-1}(C^{-1}IC^{-1})C^{-1} = I$, or $C_{ii}^{-1}I_{ij}C_{jj}^{-1} \subseteq I \cap C_{ij} = I_{ij}$. Thus, I_{ij} is upper closed. Conversely, suppose that $I = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21}$ with upper closed I_{ij}. Let x be an element of C such that $CxC \subseteq I$. Express $x = x_{11}+x_{22}+x_{12}+x_{21}$ with $x_{ij} \in C_{ij}$. Then, $C_{ii}x_{ij}C_{jj} = C_{ii}xC_{jj} \subseteq I \cap C_{ij} = I_{ij}$. Since I_{ij} is upper closed, $x_{ij} \in I_{ij}$, which implies that $x \in I$, or I is upper closed.

Theorem 5. *π_{ij} induces a bijection of the set of upper closed ideals of C to the set of upper closed C_{ii}-C_{jj}-submodules of C_{ij}.*

Proof. Let $I = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21}$ be an upper closed ideal of C, where I_{ij} are all upper closed. We show that $I_{km} = C_{ik}^{-1}I_{ij}C_{mj}^{-1}$ for any i, j, k and m. For example, suppose that $i = k$ and $j \neq m$. Then, $I_{km} = I_{im} \subseteq C_{ii}^{-1}I_{ij}C_{mj}^{-1} \subseteq C_{ii}^{-1}(I_{im}C_{jm}^{-1})C_{mj}^{-1} = C_{ii}^{-1}I_{im}C_{mm}^{-1}$ (due to the fact $C_{mj}C_{jm} = C_{mm}$) $= I_{im} = I_{km}$. So, $I_{km} = C_{ii}^{-1}I_{ij}C_{mj}^{-1} = C_{ik}^{-1}I_{ij}C_{mj}^{-1}$ as required. All the other cases are similarly proven. Thus, all I_{km} and hence I are uniquely determined by I_{ij}. Conversely, if M_{ij} is an upper closed C_{ii}-C_{jj}-submodule of C_{ij}, we let $I_{km} = C_{ik}^{-1}M_{ij}C_{mj}^{-1}$ and let $I = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21}$. I is an upper closed ideal and its projection to the (i, j)-component is M_{ij}.

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