Math. J. Okayama Univ. 29 (1987), 119–126

RADICALS OF SKEW POLYNOMIAL RINGS AND SKEW LAURENT POLYNOMIAL RINGS

Dedicated to Professor Hisao Tominaga on his 60th birthday

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Let K be a ring, ρ an automorphism of K, and D a derivation of K. We denote by $K[X;\rho]$ (resp. $K\langle X;\rho\rangle$; resp. $K[X;D]$) the skew polynomial ring of automorphism type (resp. skew Laurent polynomial ring; resp. skew polynomial ring of derivation type) over K. We write the coefficients at right. An element either of $K[X;\rho]$ or $K[X;D]$ is a polynomial $\sum_{i=0}^{n} X^i b_i$, $b_i \in K$, and an element of $K\langle X;\rho\rangle$ is of the form $\sum_{i=-n}^{n} X^i b_i$, $b_i \in K$. The addition is defined as usually and the multiplication is defined by $bX = X\rho(b)$ in $K[X;\rho]$ and $K\langle X;\rho\rangle$, and by $bX = Xb+D(b)$ in $K[X;D]$, for all $b \in K$.

In ([2], Theorem 3.1) it is proved that the Jacobson radical $J(S)$ of $S = K[X;\rho]$ is equal to $A \oplus \sum_{i\geq 1} X^i B$, where $A = J(S) \cap K$ and $B = \{b \in K : Xb \in J(S)\}$. A similar result is obtained in ([9], Theorem 3.1) for the prime radical. The Jacobson radical of $R = K\langle X;\rho\rangle$ is also obtained in ([2], Theorem 3.1). It is proved that $J(R)$ is equal to $C\langle X;\rho\rangle$, where $C = J(R) \cap K$. Finally, in ([4], Theorem 3.2) we proved that the Jacobson radical $J(T)$ of $T = K[X;D]$ is equal to $I[X;D]$, where $I = J(T) \cap K$. A similar result is obtained for the prime radical of T (Corollary 2.2).

The purpose of this paper is to give a generalization of these theorems. We will give a unified proof of these generalizations for a class of radicals which includes several of the most well-known radicals, namely, the Brown-McCoy, Jacobson, Levitzki, prime and strongly prime radicals. We will use the recent results on normalizing extensions obtained in [8] and [10]. Our proof is an adaptation of the proofs in [2] and [4], which finally are generalizations of that in [1], to the general case.

We use § 1 as an introductory section. In § 2 we consider the radicals of $K\langle X;\rho\rangle$ and $K[X;\rho]$. In § 3 we deal with a skew polynomial ring

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This paper was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil, and Programa de Matemática Pura y Aplicada de Rosario (PROMAR (CONICET-UNR)), Argentina.

of derivation type. Finally, we devote § 4 to the generalized nil radical of these rings, a particular case which is not included in the general case.

1. Prerequisites. Let α be a radical in the class of the associative rings. We remember that the radical α is said to be hereditary if every ideal of a α-ring is a α-ring again. Equivalently, α is hereditary if and only if for every ring A and for every ideal I of A, $\alpha(I) = \alpha(A) \cap I$. Every special radical is hereditary ([3], Ch. 7). Also every normal class of rings is a special class [7]. Hence every radical defined by a normal class is hereditary. Thus all the radicals considered in ([10], § 3) as well as the strongly prime radical ([8]) are hereditary radicals.

We denote by γ the upper radical defined by the class of all finite fields. Thus for every ring A, $\gamma(A) = \cap \{I : I \lhd A$ and A/I is a finite field$\}$ where $I \lhd A$ means that I is an ideal of A. We say that $\alpha \leq \gamma$ if $\alpha(A) \subseteq \gamma(A)$ for every ring A. It is clear that $\alpha \leq \gamma$ holds for all the radicals α mentioned above.

Let A be a normalizing extension of the ring B. Recently some papers have studied whether the radical α satisfies the condition $\alpha(A) \cap B = \alpha(B)$ (see [5], [6], [8] and [10]). This is the key condition for our theorems to hold. In ([10], § 3) a radical which is defined by a class $\mathscr{P}$ of prime rings is considered and a rigid class is defined. When $\mathscr{P}$ is a rigid class and α is defined by $\mathscr{P}$, i. e., $\alpha(A) = \cap \{I : I \lhd A$ and $A/I \in \mathscr{P}\}$, then $\alpha(A) \cap B = \alpha(B)$, for every normalizing extension A of B ([10], Proposition 3.3). Following this we say here that α is a *rigid radical* in this case, namely, if α is defined by a rigid class of prime rings. In case that for some radical α the equation $\alpha(A) \cap B = \alpha(B)$ holds for any normalizing extension A of B, we say that α is an *admissible radical*. Hence all the radicals considered in ([10], § 3) as well as the strongly prime radical [8] are rigid (and hence admissible) radicals.

We begin with some elementary facts.

Lemma 1.1. *Let* $\mathbf{Z}$ *be the ring of the integer numbers and* $\mathbf{Z}_p$ *the prime field of* p *elements* (p *prime*). *We have* $\gamma(\mathbf{Z}[X]) = 0$, $\gamma(\mathbf{Z}\langle X\rangle) = 0$, $\gamma(\mathbf{Z}_p[X]) = 0$ *and* $\gamma(\mathbf{Z}_p\langle X\rangle) = 0$.

Proof. It is clear since γ coincides with the Jacobson radical for these rings.

Lemma 1.2. *Let* A *be a finite Galois extension of* B *and let* α *be* *a radical with* $\alpha(A) \cap B = \alpha(B)$. *Then* $\alpha(A) = A\alpha(B) = \alpha(B)A$.

Proof. Let G be the Galois group and x_i, y_i, $i = 1, 2, \ldots, n$, the Galois coordinates of A over B. If $x \in \alpha(A)$ then $x = \sum_{i=1}^{n} \mathrm{tr}(y_i x)$, where $\mathrm{tr}(y_i x) = \sum_{\sigma \in G} \sigma(y_i x) \in \alpha(A) \cap B = \alpha(B)$. Thus $\alpha(A) = A\alpha(B)$ and similarly $\alpha(A) = \alpha(B)A$.

Corollary 1.3. *Let A be an algebra over the commutative ring C and let B be a commutative Galois extension of C. If α is an admissible radical, then* $\alpha(A \otimes_C B) = \alpha(A) \otimes_C B$.

Proof. Since $A \otimes_C B$ is a Galois extension as well as a normalizing extension of A, it follows easily from Lemma 1.2.

2. **Radicals of $K\langle X\,;\,\rho\rangle$ and $K[X\,;\,\rho]$.** Throughout this section ρ is an automorphism of K, $R = K\langle X\,;\,\rho\rangle$ is the skew Laurent polynomial ring over K and $S = K[X\,;\,\rho]$ is the skew polynomial ring. A ρ-ideal I of K is an ideal of K such that $\rho(I) = I$. If I is a ρ-ideal of K, then $I\langle X\,;\,\rho\rangle$ (resp. $I[X\,;\,\rho]$) is an ideal of R (resp. S). The automorphism ρ can be extended to R (and S) by the natural way. We denote the extension by ρ again.

Let $K^* = K \oplus \mathbf{Z}$ be the usual extension of K obtained by adjoining the identity of $\mathbf{Z}$ and ρ^* the automorphism of K^* defined by $\rho^*(a, n) = (\rho(a), n)$ for $(a, n) \in K^*$. Further we put $R^* = K^*\langle X\,;\,\rho^*\rangle$. Then we have the following.

Lemma 2.1. *Let α be a hereditary radical with $\alpha \leq \gamma$. Then*
(i) $\alpha(R^*) = \alpha(R)$;
(ii) $\alpha(R^*) \cap K^* = \alpha(R) \cap K$.

Proof. (i) Since K is a ρ^*-ideal of K^*, R is an ideal of R^* and $R^*/R \simeq \mathbf{Z}\langle X\rangle$. Thus $\alpha(R^*/R) = \alpha(\mathbf{Z}\langle X\rangle) \subseteq \gamma(\mathbf{Z}\langle X\rangle) = 0$ and one can see that $\alpha(R^*) \subseteq \alpha(R)$ by ([11], Theorem 1.12). Hence $\alpha(R) = \alpha(R^*) \cap R = \alpha(R)$.

(ii) Since $\alpha(R) \subseteq K\langle X\,;\,\rho\rangle$ we have $\alpha(R^*) \cap K^* = \alpha(R) \cap K^* = \alpha(R) \cap K$.

Lemma 2.2. *Let α be a hereditary admissible radical with $\alpha \leq \gamma$. If $I = \alpha(R) \cap K = 0$, then $\alpha(R) = 0$.*

Proof. Assume that $\alpha(R) \neq 0$ and $I = 0$. By Lemma 2.1 we may assume that K has an identity. Since X is invertible in R we may suppose that there exists a polynomial $f=\sum_{i=0}^{n} X^i a_i \in \alpha(R)$ of minimal degree $n \geq 1$ $(a_n \neq 0)$. We consider two cases.

Case (I): There exists an integer number m such that $ma_n = 0$. Then $mf=0$ and we have $pf=0$, for a prime p. Put $K_p = \{a \in K : pa = 0\}$. Then K_p is a ρ-ideal of K and hence $R_p = K_p\langle X;\rho\rangle$ is an ideal of R. Since α is hereditary $\alpha(R_p) = \alpha(R) \cap R_p$ and we have that f is a polynomial of minimal degree n in $\alpha(R_p)$.

Then we may suppose that K is a $\mathbf{Z}_p$-algebra, for a prime p. Let $K' = K \oplus \mathbf{Z}_p$ be the usual extension of K obtained by adjoining the identity of $\mathbf{Z}_p$ and ρ' the automorphism defined by $\rho'(a,n) = (\rho(a), n)$, for $(a,n) \in K'$. Then one can see that $\alpha(R') = \alpha(R)$ arguing as in Lemma 2.1, (i), and so f is a polynomial of minimal degree n in $\alpha(R')$.

Then we may suppose that K contains the field $\mathbf{Z}_p$. Let F_q be the field of $q=p^{n+1}$ elements and put $K^* = K \otimes_{Z_p} F_q$ and $\rho^*(x \otimes a) = \rho(x) \otimes a$, for all $x \otimes a \in K^*$. Thus $R^* = K^*\langle X;\rho^*\rangle \simeq R \otimes_{Z_p} F_q$ and $\alpha(R^*) \simeq \alpha(R) \otimes_{Z_p} F_q$ by Corollary 1.3. Hence f is a polynomial of minimal degree n in $\alpha(R^*)$. Let $\eta_0, \eta_1, \dots, \eta_n$ be $n+1$ distinct units of F_q and σ_j the automorphism of R^* defined by $\sigma_j(X) = X\eta_j$, $j=0,1,\dots,n$. We have $\sum_{i=0}^{n} X^i a_i \eta_j^i \in \alpha(R^*)$. So we get $X^n a_n d \in \alpha(R^*)$, where $d \in F_q$ is the value of the Vandermonde determinant of the matrix (η_j^i). Since d is invertible, $X^n a_n \in \alpha(R^*)$ and hence $a_n \in \alpha(R^*)$, a contradiction.

Case (II): a_n is not a $\mathbf{Z}$-torsion element. In this case we may suppose that K contains the ring of integers. Let ζ be a complex primitive $(n+1)$th root of the unity, $K^* = K \otimes_Z \mathbf{Z}[\zeta]$ and $\rho^* = \rho \otimes \mathrm{id}_{Z[\zeta]}$. Thus $R^* = K^*\langle X;\rho^*\rangle \simeq R \otimes_Z \mathbf{Z}[\zeta]$ and $\alpha(R^*) \cap R = \alpha(R)$, since R^* is a normalizing extension of R. Then $f \in \alpha(R^*)$.

Let $\eta_0, \eta_1, \dots, \eta_n$ be $n+1$ distinct units of $\mathbf{Z}[\zeta]$. As above, we get $X^n a_n d \in \alpha(R^*)$, where $d \in \mathbf{Z}[\zeta]$ is the determinant of the matrix (η_j^i). Let σ_i be the automorphisms of the Galois extension $\mathbf{Q}[\zeta]$ of $\mathbf{Q}$, $i=1,\dots,n$. Since $\sigma_i(d) \in \mathbf{Z}[\zeta]$ we have $X^n a_n \sigma_1(d) \cdots \sigma_n(d) \in \alpha(R^*)$, where $m = \sigma_1(d) \cdots \sigma_n(d) \in \mathbf{Z}[\zeta] \cap \mathbf{Q}[\zeta]^G = \mathbf{Z}$. Hence $a_n m \in \alpha(R^*) \cap R = \alpha(R)$, a contradiction since $a_n m \neq 0$.

Now we have our first main result.

Theorem 2.3. *Let α be a hereditary admissible radical with $\alpha \leq \gamma$. Then $\alpha(R) = I\langle X;\rho\rangle$, where $I = \alpha(R) \cap K$ is a ρ-ideal of K.*

Proof. Since I is a ρ-ideal of K, $\bar{K} = K/I$ has an automorphism $\bar{\rho}$ induced by ρ and $\bar{R} = \bar{K}\langle X ; \bar{\rho}\rangle \simeq R/I\langle X ; \rho\rangle$. Consider the natural homomorphism from R to $\bar{R}$. Then, as $I\langle X ; \rho\rangle \subseteq \alpha(R)$, $\alpha(\bar{R}) \simeq \alpha(R)/I\langle X ; \rho\rangle$ by an easy consequence of ([11], Theorem 1.12). It follows that $\alpha(\bar{R}) \cap \bar{K} = 0$ and from the former Lemma we have $\alpha(R) = I\langle X ; \rho\rangle$.

Now we consider the skew polynomial ring $S = K[X ; \rho]$. Suppose that α is a radical defined by a class $\mathscr{P}$ of prime rings, i. e., $\alpha(A) = \cap \{I \lhd A : A/I \in \mathscr{P}\}$. Define $B = \{b \in K : Xb \in \alpha(S)\}$. Then B is a ρ-ideal of K.

Lemma 2.4. *Let α be a hereditary rigid radical with $\alpha \leq \gamma$. If $B = 0$, then $\alpha(S) = 0$.*

Proof. Assume that $\alpha(S) \neq 0$. Arguing as in Lemma 2.2 we can see that there exists a polynomial of the form $X^n a \in \alpha(S)$, with $n \geq 1$ and $a \neq 0$. Suppose $n \geq 2$ and $X^{n-1}a \notin \alpha(R)$. Then there is a prime P in $\mathscr{P}$ such that $X^{n-1}a \notin P$ while $X^n a \in P$. Since $XS = SX$ we have $XSX^{n-1}aS = SX^n aS \subseteq P$ and so $X \in P$, a contradiction. Thus $Xa \in \alpha(R)$ and so $B \neq 0$.

Theorem 2.5. *Let α be a hereditary rigid radical with $\alpha \leq \gamma$. Then $\alpha(S) = A \oplus XB[X ; \rho]$, where $A = \alpha(S) \cap K$ and $B = \{b \in K : Xb \in \alpha(S)\}$.*

Proof. It is clear that $I = A \oplus XB[X ; \rho]$ is an ideal of S and $I \subseteq \alpha(S)$. The ring $\bar{K} = K/B$ has an automorphism $\bar{\rho}$ induced by ρ. Put $\bar{S} = \bar{K}[X ; \bar{\rho}] \simeq S/B[X ; \rho]$ and note that $\alpha(\bar{S}) = 0$. In fact, let $\bar{a} \in \bar{K}$ and suppose $X\bar{a} \in \alpha(\bar{S})$. If P is a prime ideal with $S/P \in \mathscr{P}$, we have $XB[X ; \rho] \subseteq \alpha(S) \subseteq P$. Then either $X \in P$ or $B[X ; \rho] \subseteq P$. In this last case, $X\bar{a} \in P/B[X ; \rho]$ because $\bar{S}/(P/B[X ; \rho]) \simeq S/P \in \mathscr{P}$. Thus $Xa + B[X ; \rho] \subseteq P$ and so we get $Xa \in P$. Hence $Xa \in \alpha(S)$ and it follows that $\bar{a} = 0$. Then $\alpha(\bar{S}) = 0$ from Lemma 2.4.

Now Theorem 1.12 in [11] gives $\alpha(S) \subseteq B[X ; \rho]$. If $\sum_{i=0}^{n} X^i b_i \in \alpha(S)$, $n \geq 1$, then $b_i \in B$ for $i = 1, \ldots, n$. We have $\sum_{i=1}^{n} X^i b_i \in \alpha(S)$ and so $b_0 \in \alpha(S) \cap K = A$. This completes the proof.

Remark 2.6. It is easy to see that $\alpha(S) \cap K \subseteq \alpha(K)$ when α is a radical defined by a class $\mathscr{P}$ of prime rings. In fact, if P is a prime ideal of K and $K/P \in \mathscr{P}$, then $P \oplus XS$ is an ideal of S and $S/P \oplus XS \simeq K/P \in \mathscr{P}$.

Hence $\alpha(S) \cap K \subseteq P$.

3. Radicals of $K[X; D]$. Throughout this section D is a derivation of K and $T = K[X; D]$ is the skew polynomial ring of derivation type. We omit the proofs of the following results because they are very similar to that in Section 2. In particular, Lemma 3.1 can be proved by the same way as it is proved Lemma 3.1 in [4]. We have

Lemma 3.1. *Let α be a hereditary admissible radical with $\alpha \leq \gamma$. If $I = \alpha(R) \cap K = 0$, then $\alpha(R) = 0$.*

Theorem 3.2. *Let α be a hereditary admissible radical with $\alpha \leq \gamma$. Then $\alpha(R) = I[X; D]$, where $I = \alpha(R) \cap K$ is a D-ideal of K.*

4. The Generalized Nil Radical. An ideal P of a ring K is said to be *completely prime* if K/P has no zero divisors. The intersection $N(K)$ of all the completely prime ideals of K is called the generalized nil radical of K ([3], § 7.8). Then N is a special radical and so it is hereditary. Also $N \leq \gamma$ is clear. But N is not an admissible radical. In fact, a field F is N-semisimple and the full matrix ring $M_n(F)$, $n \geq 2$, is a N-radical ring. Thus the generalized nil radical cannot be obtained as a consequence of our theorems. Nevertheless, we can obtain directly similar results. To see this we denote by $N_\rho(K)$ (resp. $N_D(K)$) the intersection of all the ρ-ideals (resp. D-ideals) of K which are also completely prime ideals. We have the following

Proposition 4.1. *Let ρ be an automorphism of K. Then*
(i) $N(K\langle X; \rho\rangle) = N_\rho(K)\langle X; \rho\rangle$;
(ii) $N(K[X; \rho]) = N(K) \oplus XN_\rho(K)[X; \rho]$.

Proof. (i) Put $R = K\langle X; \rho\rangle$. If P is a completely prime ideal of R, then $P \cap K$ is a ρ-ideal as well as a completely prime ideal of K, because $K/P \cap K \subseteq R/P$. Then $N_\rho(K) \subseteq P$ and so we get $N_\rho(K)\langle X; \rho\rangle \subseteq N(R)$. Conversely, if Q is a ρ-ideal of K which is completely prime, then $R/Q\langle X; \rho\rangle \simeq K/Q\langle X; \bar{\rho}\rangle$ (where $\bar{\rho}$ is the automorphism induced by ρ) has no zero divisors as it is easy to see. Hence $N(R) \subseteq Q\langle X; \rho\rangle$ and so $N(R) \subseteq N_\rho(K)\langle X; \rho\rangle$.

(ii) Let S be $K[X; \rho]$. If P is a completely prime ideal of S, then

$P \cap K$ is a completely prime ideal of K. Hence $N(K) \subseteq N(S) \cap K$. Conversely, if Q is a completely prime ideal of K, then $Q+XS$ is a completely prime ideal of S since $S/Q+XS \simeq K/Q$. Thus $N(S) \cap K \subseteq Q$ and so we get $N(S) \cap K = N(K)$.

Let now P be a completely prime ideal of S. If $X \notin P$, since P is prime, $P \cap K$ is a ρ-ideal ([9], Proposition 1.2) as well as a completely prime ideal of K. Then we have $XN_\rho(K)[X ; \rho] \subseteq P$ and also $N(K) \oplus XN_\rho(K)[X ; \rho] \subseteq N(S)$. On the other hand, suppose that $f = \sum_{i=0}^{n} X^i b_i \in N(S)$. Hence $f \in Q+XS$ for every completely prime ideal Q of K. It follows that $b_0 \in N(K) \subseteq N(S)$ and so $X\sum_{i=1}^{n} X^{i-1}b_i \in N(S)$. Further, if I is a completely prime ideal as well as a ρ-ideal of K, it can be easily seen that $I[X ; \rho]$ is a completely prime ideal of S. Thus $X\sum_{i=1}^{n} X^{i-1}b_i \in I[X ; \rho]$ and hence $\sum_{i=1}^{n} X^{i-1}b_i \in I[X ; \rho]$. This gives $b_i \in I$ and so $b_i \in N_\rho(K)$, for $i \geq 1$. The proof is then completed.

Example 4.2. This is an example in which $N_\rho(K) \neq N(K)$. Let E be a field and $A = E[X_i : i \in \mathbf{Z}]$ a polynomial ring in indeterminates X_i, $i \in \mathbf{Z}$. Define $\rho : A \to A$ by $\rho | E = \mathrm{id}_E$ and $\rho(X_i) = X_{i+1}$, for all $i \in \mathbf{Z}$, and let H be the ideal of A generated by $\{X_i X_j : i \neq J\}$. Then ρ induces an automorphism on the ring $K = A/H$, which we denote by ρ again. We write $x_i = X_i + H \in K$ and we have $x_i x_j = 0$ if $i \neq J$ and $\rho(x_i) = x_{i+1}$, for all $i \in \mathbf{Z}$. We can easily see that $N(K) = 0$ and $N_\rho(K)$ is the ideal of K generated by $\{x_i : i \in \mathbf{Z}\}$. Thus $N(K[X ; \rho]) = XQ[X ; \rho]$.

Finally we have

Proposition 4.3. *Let D be a derivation of K. Then $N(K[X ; D]) = N_D(K)[X ; D]$.*

Proof. It is similar to Proposition 4.1, (i).

Acknowledgement. This paper was written while the author was visiting the Departamento de Matemática, Facultad de Ciencias Exactas y Ingeniería, Universidad Nacional de Rosario (Argentina). He wishes to express his thanks to the members of the Department for their hospitality. The author is also grateful to Prof. K. Kishimoto for his valuable advice.

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(Received December 21, 1985)