Math. J. Okayama Univ. 29 (1987), 103—107

ON H-SEPARABLE EXTENSIONS IN AZUMAYA ALGEBRAS

Dedicated to Professor Hisao Tominaga on his 60th birthday

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Throughout the present paper, A/B will represent a ring extension with common identity 1, C the center of A, and $V_A(B)$ the centralizer of B in A. If an A-B-bimodule M is A-B-isomorphic to some A-B-direct summand of a finite direct sum of copies of an A-B-bimodule N, we write ${}_AM_B \mid {}_AN_B$. Needless to say, ${}_AM \mid {}_AA$ means that ${}_AM$ is finitely generated projective. Also, it is clear that ${}_BB_B \mid {}_BA_B$ (resp. ${}_BB \mid {}_BA$) if and only if B is B-B-isomorphic (resp. B-isomorphic) to a direct summand of ${}_BA_B$ (resp. ${}_BA$). An extension A/B is called a separable extension if the A-A-map $A \otimes_B A \to A$ defined by $x \otimes y \to xy\,(x,\ y \in A)$ splits. It is clear that A/B is separable if and only if ${}_AA_A \mid {}_AA \otimes_B A_A$. Following [7], A/B is called an H-separable extension if ${}_AA \otimes_B A_A \mid {}_AA_A$; that is, if there exist $v_i \in V_A(B)$ and $\sum_j x_{ij} \otimes y_{ij} \in (A \otimes_B A)^A = \{ \sum_k a_k \otimes b_k \in A \otimes_B A \mid\ a \sum_k a_k \otimes b_k = \sum_k a_k \otimes b_k a$ for all $a \in A\}$ $(i = 1, 2, \dots, n)$ such that $\sum_{i,j} x_{ij} \otimes y_{ij} v_i = 1 \otimes 1$. Such a system $\{v_i, \sum_j x_{ij} \otimes y_{ij}\}_i$ is called an H-system for the H-separable extension A/B. It is well known that any H-separable extension is a separable extension (see, e.g., [2, Theorem 2.2] or [6, (4)]), and that if A is an Azumaya C-algebra then A/C is an H-separable extension (see, e.g., [7, Proposition 1.1]).

The main purpose of this paper is to prove the following theorem.

Theorem 1. *Let A be an Azumaya C-algebra, B a C-subalgebra of A, and $\Delta = V_A(B)$.*

(1) *B is a separable C-algebra if and only if ${}_BB_B \mid {}_BA_B$. When this is the case, A/B is an H-separable extension with $V_A(\Delta) = B$, and ${}_BA$ is finitely generated projective.*

(2) *B is an Azumaya C-algebra if and only if ${}_BA_B \mid {}_BB_B$.*

In preparation for proving Theorem 1, we state first the next lemma (see [3, Proposition 4.7] and [7, Proposition 1.2]).

Lemma 1. *Let A/B be an H-separable extension, and $\Delta = V_A(B)$.*

(1) *The map $\eta: \Delta \otimes_C \Delta \to \mathrm{Hom}({}_BA_B, {}_BA_B)$ defined by $\eta(x \otimes y)(a) = xay$ $(x, y \in \Delta,\ a \in A)$ is a Δ-Δ-isomorphism.*

(2) *If ${}_BB \mid {}_BA$ then $V_A(\Delta) = B$.*

(3) *If ${}_BB_B \mid {}_BA_B$, then Δ is a separable C-algebra.*

Proof. (1) Let $\{v_i, \sum_j x_{ij} \otimes y_{ij}\}_i$ be an H-system for A/B. Then the map ϕ: $\mathrm{Hom}({}_BA_B, {}_BA_B) \to \Delta \otimes_C \Delta$ defined by $\phi(g) = \sum_i \sum_j g(x_{ij}) y_{ij} \otimes v_i = \sum_i v_i \otimes \sum_j x_{ij} g(y_{ij})$ $(g \in \mathrm{Hom}({}_BA_B, {}_BA_B))$ is the inverse map of η (see [6, p. 296]).

(2) See [6, (5)].

(3) Assume that ${}_BB_B \mid {}_BA_B$. Then, by (1), $\Delta \simeq \mathrm{Hom}({}_BB_B, {}_BA_B) \mid \mathrm{Hom}({}_BA_B, {}_BA_B) \simeq \Delta \otimes_C \Delta$ as Δ-Δ-module. Hence Δ is a separable C-algebra.

Lemma 2. *Let A be an Azumaya C-algebra, and B a C-subalgebra of A. Then A/B is H-separable if and only if $(A \otimes_B A)^A$ is a projective C-module.*

Proof. We claim first that $(A \otimes_B A)^A$ is a finitely generated C-module. Since ${}_AA \otimes_C A_A \mid {}_AA_A$, we see that $(A \otimes_C A)^A \simeq \mathrm{Hom}({}_AA_A, {}_AA \otimes_C A_A)$ is a finitely generated projective C-module. In virture of [1, p. 52, Theorem 3.4], $(A \otimes_C A)^A \simeq \mathrm{Hom}({}_AA_A, {}_AA \otimes_C A_A) \to \mathrm{Hom}({}_AA_A, {}_AA \otimes_B A_A) \simeq (A \otimes_B A)^A$ is a C-epimorphism. Hence $(A \otimes_B A)^A$ is a finitely generated C-module.

Assume that A/B is H-separable: ${}_AA \otimes_B A_A \mid {}_AA_A$. Then $(A \otimes_B A)^A \simeq \mathrm{Hom}({}_AA_A, {}_AA \otimes_B A_A) \mid \mathrm{Hom}({}_AA_A, {}_AA_A) \simeq C$ as C-module, that is, $(A \otimes_B A)^A$ is a finitely generated projective C-module. Conversely, assume that $(A \otimes_B A)^A$ is a projective C-module. Then, as was claimed above, $(A \otimes_B A)^A$ is a finitely generated projective C-module: $(A \otimes_B A)^A_C \mid C_C$. Since $A \otimes_B A \simeq (A \otimes_B A)^A \otimes_C A$ as A-A-bimodule by [1, p. 54, Corollary 3.6], we get ${}_AA \otimes_B A_A \mid {}_AA_A$.

As a direct consequence of Lemma 2, we have the following corollary which is interesting in itself.

Corollary 1. *If A is an Artinian semisimple Azumaya C-algebra, then A/B is an H-separable extension for every C-subalgebra B of A. In particular, if A is a finite dimensional central simple C-algebra, then A/B is an H-sepa-*

rable extension for every C-subalgebra B of A.

Recently, K. Hirata proved the following ([4, Proposition 6]): Let A be the group ring $K[G]$ of a finite group G with a coefficient field K whose characteristic does not divide the order of G. Let B be the group ring $K[H]$ of a subgroup H of G with the coefficient field K, and $B' = V_A(V_A(B))$. Then A/B' is an H-separable extension. As a matter of fact, this is immediate by Corollary 1.

We are now ready to complete the proof of Theorem 1.

Proof of Theorem 1. (1) The only if part has been proved in [8, Proposition 1.5]. Assume now that ${}_BB_B \mid {}_BA_B$. Let $\mathfrak{m}$ be an arbitrary maximal ideal of C, $\bar{A} = A/\mathfrak{m}A$, $\bar{B} = B/\mathfrak{m}B$, and $\bar{C} = C/\mathfrak{m}C$. Then $\bar{A}$ is a finite dimensional central simple $\bar{C}$-algebra, and $\bar{B}$ is a $\bar{C}$-subalgebra of $\bar{A}$ such that ${}_{\bar{B}}\bar{B}_{\bar{B}} \mid {}_{\bar{B}}\bar{A}_{\bar{B}}$. Hence, by Corollary 1, $\bar{A}/\bar{B}$ is an H-separable extension. Then $\bar{D} = V_{\bar{A}}(\bar{B})$ is a separable $\bar{C}$-algebra with $V_{\bar{A}}(\bar{D}) = \bar{B}$, by Lemma 1. By [8, Proposition 1.5], we have ${}_{\bar{D}}\bar{D}_{\bar{D}} \mid {}_{\bar{D}}\bar{A}_{\bar{D}}$. Since $\bar{A}/\bar{D}$ is an H-separable extension (Corollary 1), we see that $V_{\bar{A}}(\bar{D}) = \bar{B}$ is a separable $\bar{C}$-algebra, by Lemma 1 (3). Hence by [1, p. 72, Theorem 7.1], B is a separable C-algebra. Since ${}_BB_B \mid {}_BB \otimes_C B_B$ and ${}_AA \otimes_C A_A \mid {}_AA_A$, we obtain ${}_AA \otimes_B A_A \simeq {}_AA \otimes_B B \otimes_B A_A \mid {}_AA \otimes_B B \otimes_C B \otimes_B A_A \simeq {}_AA \otimes_C A_A \mid {}_AA_A$. Furthermore, noting that ${}_CA \mid {}_CC$ (see, e. g., [1, p. 52, Theorem 3.4]), we see that ${}_BA \simeq {}_BB \otimes_B A \mid {}_BB \otimes_C B \otimes_B A \simeq {}_BB \otimes_C A \mid {}_BB \otimes_C C \simeq {}_BB$, that is, ${}_BA$ is finitely generated projective. Now, $V_A(\Delta) = B$ is obvious by Lemma 1 (2).

(2) Assume that ${}_BA_B \mid {}_BB_B$. It is well known that ${}_BA_B \mid {}_BB_B$ implies ${}_BB_B \mid {}_BA_B$ (see, e. g., [3, Proposition 5.6]). Hence, by (1), B is a separable C-algebra with $V_A(\Delta) = B$ and A/B is H-separable. Then, by Lemma 1 (1), $\Delta \otimes_C \Delta \simeq \mathrm{Hom}({}_BA_B, {}_BA_B) \mid \mathrm{Hom}({}_BB_B, {}_BA_B) \simeq V_A(B) = \Delta$ as Δ-Δ-module, that is, Δ is an H-separable C-algebra. Since Δ is a finitely generated C-module (see, e. g., [6, (3)]), Δ is an Azumaya C-algebra, by [7, Corollary 1.2]. Since $V_B(B) = V_A(B) \cap B = V_A(B) \cap V_A(V_A(B)) = V_\Delta(\Delta) = C$, B is an Azumaya C-algebra. Conversely, assume that B is an Azumaya C-algebra. Then by [1, p. 57, Theorem 4.3], $A = B \otimes_C \Delta$ and Δ is an Azumaya C-algebra. Hence, we see that ${}_BA_B = {}_BB \otimes_C \Delta_B \mid {}_BB \otimes_C C_B \simeq {}_BB_B$.

The following corollary may be regarded as a sharpening of [4, Proposition 6].

Corollary 2. *Let A be a separable (faithful) R-algebra, B a separable R-subalgebra of A, $\Delta = V_A(B)$, and $B' = V_A(V_A(B))$. Then A/B' (resp. A/Δ) is an H-separable extension and ${}_{B'}A$ (resp. ${}_{\Delta}A$) is finitely generated projective.*

Proof. By [1, p. 55, Theorem 3.8], A is an Azumaya C-algebra and C is a separable R-algebra. Then BC is a separable R-algebra as a homomorphic image of $B \otimes_R C$, by [1, p. 43, Proposition 1.6]. Then BC is a separable C-algebra, by [1, p. 46, Proposition 1.12]. Since $V_A(BC) = V_A(B)$, our assertion follows from Theorem 1 (1) and [1, p. 57, Theorem 4.3].

We shall conclude this paper with giving two examples of H-separable extensions.

Examples. Let K be a field.

(1) Let $A = \mathrm{M}_3(K)$, and $B = \begin{pmatrix} K & K & K \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix}$. Then A/B is an H-separable extension (Corollary 1) and $V_A(B) = C$. As is easily seen, ${}_BA$ is not projective, but A_B is projective. Needless to say, both ${}_BA$ and A_B are finitely generated. In [9], H. Tominaga proved that if A/B is an H-separable extension and ${}_BA$ is projective, then ${}_BA$ is finitely generated. This example shows that the converse need not be true.

(2) Let $A = \mathrm{M}_4(K)$, and $B = \left\{ \begin{pmatrix} a & 0 & b & c \\ 0 & a & d & e \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \middle| \, a, b, c, d, e, \in K \right\}$. Then both ${}_BA$ and A_B are finitely generated and A/B is an H-separable extension with $V_A(B) = B$. But, neither ${}_BA$ nor A_B is projective.

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(Received December 30, 1986)