Math. J. Okayama Univ. 29 (1987), 83—90

ON THE HIGHER DERIVATIONS OF COMMUNITIVE RINGS

Dedicated to Professor H. Tominaga on the occasion of his 60th birthday

SADI ABU-SAYMEH and MASATOSHI IKEDA

1. Introduction. In the paper [3], Y. Nakai pointed out that the n-th term of any Hasse-Schmidt sequence of higher derivations [2] (an H-S sequence for short) of a commtative ring A is an n-th order derivation of A in the sense of H. Osborn [4], but the converse is not always the case. He further raised the question how to characterize the terms of an H-S sequence. The first author [1] has recently given an answer to this question, under the assumption that A is a commutative algebra over a field k of characteristic zero, by giving an explicit formula of the n-th term of an H-S sequence of A as a non-commutative polynomial of a special type in k-derivations of A. He has further shown that there is a bijection between the set of all H-S sequences of A and the direct product of $\mathrm{Der}_k(A)$, the Lie algebra of all k-derivations of A. The aim of this note is two-fold: We first give another explicit formula for the terms of an H-S sequence of A over a field k of characteristic zero and discuss its relation with the formula given in [1]. We then consider a special class of k-algebras which includes the case of separably generated fields, and we show that, for such an algebra A/k, any finite sequence $\Delta_n = \{D_r : r = 0, 1, \ldots, n\}$ with $D_r \in \mathrm{Hom}_k(A, A)$ satisfying conditions (i) and (ii) imposed upon H-S sequences can be completed to an H-S sequence of A. This fact can be interpreted in terms of the embeddings of A into the formal power series ring $A[[T]]$.

Throughout this note we always understand by a k-algebra a commutative algebra with unity over a commutative ring k.

A sequence $\{D_r : r = 0, 1, 2, \ldots\}$ with $D_r \in \mathrm{Hom}_k(A, A)$ is called an H-S sequence of the k-algebra A if (i) $D_0 = \mathrm{id}_A$, and (ii) for every $r \geq 1$, $D_r(xy) = \sum_{s=0}^{r} D_s(x)D_{r-s}(y)$ hold for all $x, y \in A$.

2. The case where k is a field of characteristic zero.

Proposition 1. *Let A be an algebra over a field k of characteristic zero, and $\{D_r : r = 0, 1, 2, \ldots\}$ an H-S sequence of A. Then there is a unique* *sequence* $\{d_r: r=1,2,\dots\}$ *of k-derivations of A such that, for every* $r \geq 1$, *the following equality holds :*

$$(1) \qquad D_r = \sum_{N_r} \frac{1}{N_r!!} d_1^{n_1} d_2^{n_2} \dots d_r^{n_r},$$

where $N_r = (n_1, n_2, \dots, n_r)$ *is a solution of* $\sum_{i=1}^{r} in_i = r$, n_i*'s are non-negative integers,* $N_r!! = n_1! n_2! \dots n_r!$, *and the summation is taken over all solutions* N_r. *Conversely, if* $\{d_r: r=1,2,\dots\}$ *is any sequence of k-derivations of A, then the sequence* $\{D_r: r=0,1,2,\dots\}$ *defined by* (1) *is an H-S sequence of A.*

Proof. We shall prove the first half by induction on r. The assertion is true for $r=1$. Assume that we have already found a sequence $\{d_1, \dots, d_{r-1}\}$ such that (1) holds for all $s \leq r-1$. Observe that N_r is either $(0,\dots 0,1)$ or of the form $N_r' = (n_1, \dots, n_{r-1}, 0)$, satisfying $\sum_{i=1}^{r-1} in_i = r$.

Set

$$P_r = \sum_{N_r'} \frac{1}{N_r'!!} d_1^{n_1} d_2^{n_2} \dots d_{r-1}^{n_{r-1}},$$
$$d_r = D_r - P_r$$

Further set $\delta d_r(x, y) = d_r(xy) - xd_r(y) - yd_r(x)$ for every $x, y \in A$ and similarly $\delta D_r(x, y)$ and $\delta P_r(x, y)$. So to prove the assertion it is sufficient to show that d_r is a k-derivation of A. First we have

$$\delta D_r(x, y) = D_r(xy) - xD_r(y) - yD_r(x) = \sum_{\lambda=1}^{r-1} D_\lambda(x) D_{r-\lambda}(y),$$

then by induction hypothesis we get

$$(*) \qquad \delta D_r(x, y) = \sum_{\lambda=1}^{r-1} \left(\sum_{M_\lambda} \frac{1}{M_\lambda!!} d_1^{m_1} \dots d_\lambda^{m_\lambda} \right) \left(\sum_{L_{r-\lambda}} \frac{1}{L_{r-\lambda}!!} d_1^{l_1} \dots d_{r-\lambda}^{l_{r-\lambda}} \right)$$

where $M_\lambda = (m_1, \dots, m_\lambda)$ is a solution of $\sum_{i=1}^{\lambda} im_i = \lambda$ and $L_{r-\lambda} = (l_1, \dots, l_{r-\lambda})$ is a solution of $\sum_{i=1}^{r-\lambda} il_i = r-\lambda$. Notice that M_λ and $L_{r-\lambda}$ are not trivial, i.e. $\sum m_i \neq 0$ and $\sum l_i \neq 0$, and $\sum_{i=1}^{r-1} i(m_i + l_i) = r$ if we set $m_i = 0$ for every $\lambda+1 \leq i \leq r-1$ and $l_i = 0$ for every $r-\lambda+1 \leq i \leq r-1$.

On the other hand by Leibniz formula we have

$$P_r(xy) = \sum_{N_r} \sum_{\substack{m_i + l_i = n_i \\ 1 \le i \le r-1}} \left[\frac{1}{M!!} d_1^{m_1} \dots d_{r-1}^{m_{r-1}}(x)\right]\left[\frac{1}{L!!} d_1^{l_1} \dots d_{r-1}^{l_{r-1}}(y)\right].$$

Note that $M = (m_1, \dots, m_{r-1})$ and $L = (l_1, \dots, l_{r-1})$ may be trivial. So, separating the terms corresponding to the trivial M or L and the non-trivial M and L, we get

(**) $$P_r(xy) = xP_r(y) + yP_r(x) + \sum_{N_r} \sum_{\substack{m_i + l_i = n_i \\ 1 \le i \le r-1}}' \left[\frac{1}{M!!} d_1^{m_1} \dots d_{r-1}^{m_{r-1}}(x)\right]\left[\frac{1}{L!!} d_1^{l_1} \dots d_{r-1}^{l_{r-1}}(y)\right],$$

where $\sum'$ stands for the sum over the non-trivial M and L.

From (*) and (**) it is easily seen that $\delta d_r(x, y) = 0$, which proves that d_r is a k-derivation of A.

To prove the second half we verify condition (ii).

(I) $$D_r(xy) = \sum_{N_r} \frac{1}{N_r!!} d_1^{n_1} \dots d_r^{n_r}(xy)$$
$$= \sum_{N_r} \sum_{\substack{m_i + l_i = n_i \\ 1 \le i \le r}} \left[\frac{1}{M_r!!} d_1^{m_1} \dots d_r^{m_r}(x)\right]\left[\frac{1}{L_r!!} d_1^{l_1} \dots d_r^{l_r}(y)\right]$$

On the other hand,

(II) $$\sum_{s=0}^{r} D_s(x) D_{r-s}(y) = \sum_{s=1}^{r-1} \left[\sum_{M_s} \frac{1}{M_s!!} d_1^{m_1} \dots d_s^{m_s}(x)\right]\left[\sum_{L_{r-s}} \frac{1}{L_{r-s}!!} d_1^{l_1} \dots d_{r-s}^{l_{r-s}}(y)\right] + xD_r(y) + yD_r(x)$$

where $M_s = (m_1, \dots, m_s)$ is a solution of $\sum_{i=1}^{s} im_i = s$ and $L_{r-s} = (l_1, \dots, l_{r-s})$ is a solution of $\sum_{i=1}^{r-s} il_i = r-s$. It is clear that $xD_r(y)$ and $yD_r(x)$ appear in (I). Setting $M_r = (m_1, \dots, m_s, 0, \dots, 0)$ and $L_r = (l_1, \dots, l_{r-s}, 0, \dots, 0)$ we see that each term in (II) appears in (I). Conversely if $M_r = (m_1, \dots, m_r)$ and $L_r = (l_1, \dots, l_r)$ are given such that $\sum_{i=1}^{r} m_i \neq 0$, $\sum_{i=1}^{r} l_i \neq 0$ and $\sum_{i=1}^{r} i(m_i + l_i) = r$. Setting $\sum_{i=1}^{r} im_i = s$ and $\sum_{i=1}^{r} il_i = r-s$, we see that $1 \le s \le r-1$,

$m_i = 0$ for every $s+1 \leq i \leq r$, and $l_i = 0$ for every $r-s+1 \leq i \leq r$. Hence each term in (I) also appears in (II).

Remark. Notice that if $1 \leq r_1 \leq \cdots \leq r_q$ such that $\sum_{i=1}^{q} r_i = r$ then $d_{r_1}d_{r_2}\ldots d_{r_q}$ can be written uniquely as $d_{m_1}^{e_1}d_{m_2}^{e_2}\ldots d_{m_s}^{e_s}$ where $1 \leq m_1 < m_2 < \cdots < m_s$, e_i's ≥ 1, $\sum_{i=1}^{s} e_i = q$ and $\sum_{i=1}^{s} e_i m_i = r$. Hence equation (1) in proposition 1 can be written as

$$D_r = \sum_{q=1}^{r} \sum_{\substack{r_1+\cdots+r_p=r \\ 1\leq r_1\leq\cdots\leq r_q}} E(r_1, \ldots, r_q) d_{r_1}d_{r_2}\ldots d_{r_q} \tag{2}$$

where $E(r_1, \ldots, r_q) = \dfrac{1}{e_1!e_2!\ldots e_s!}$.

Thus to each H-S sequence $\{D_r: r \geq 0\}$ we can associate a unique sequence $\{d_r: r \geq 1\}$ of k-derivations such that the equality (2) holds and by the main theorem in [1] we can also associate to it the sequence $\{\delta_r: r \geq 1\}$ of k-derivations given by

$$\delta_r = \sum_{s=1}^{r} \frac{(-1)^{s+1}}{s} \sum_{\substack{r_1+r_2+\cdots+r_s=r \\ r_i\geq 1}} D_{r_1}D_{r_2}\ldots D_{r_s}.$$

Hence

$$\delta_r = \sum_{q=1}^{r} \sum_{\substack{r_1+\cdots+r_q=r \\ r_i\geq 1}} C(r_1, \ldots, r_q) d_{r_1}d_{r_2}\ldots d_{r_q} \text{ for every } r \geq 1$$

where, if $1 \leq r_1 \leq r_2 \leq \cdots \leq r_q$, we have

$$C(r_1, \ldots, r_q) =$$
$$\sum_{s=1}^{q} \frac{(-1)^{s+1}}{s} \sum_{q_1+\cdots+q_s=q} E(r_1, \ldots, r_{q_1}) \ldots E(r_{q_1+\cdots+q_{s-1}+1}, \ldots, r_q)$$

and if $(r_1, \ldots, r_q) = (r_{11}, \ldots, r_{1p_1}, \ldots, r_{2p_2}, \ldots, r_{l1}, \ldots, r_{lp_l})$ such that $r_{i1} \leq \ldots \leq r_{ip_i}$ for every $1 \leq i \leq l$ and $r_{ip_i} > r_{i+1,1}$ for every $1 \leq i \leq l-1$, then we have

$$C(r_1, \ldots, r_q) = \prod_{i=1}^{l} C(r_{i1}, \ldots, r_{ip_i})$$

It is easily seen that for $q = 1$ we have $C(r) = 1$ for every $r \geq 1$ and for $q \geq 2$ such that $r_1 = r_2 = \cdots r_q = m$ we have

$$C(m,\dots,m) = \sum_{s=1}^{q} \frac{(-1)^{s+1}}{s} \sum_{q_1+\cdots+q_s=q} \frac{1}{q_1!\dots q_s!}$$
$$= \text{coefficient of } x^q \text{ in the Taylor's expansion of } x = ln[1+(e^x-1)].$$

Thus $C(m,\dots,m) = 0$ and $C(r_1,\dots,r_q) = 0$ if there is $1 \leq i \leq l$ such that $r_{i1} = \cdots = r_{ip_i}$.

3. Algebras of type H_2. Let A be a k-algebra, and let M be a unitary A-bimodule satisfying the condition $am = ma$ for all $a \in A$, $m \in M$. By a symmetric 2-cochain of A to M, we understand a k-bilinear map f from $A \times A$ to M such that $f(x, y) = f(y, x)$ for all $x, y \in A$. A symmetric 2-cochain f is 2-cocycle if $\delta_2 f = 0$, where

$$\delta_2 f(x, y, z) = xf(y, z) - f(xy, z) + f(x, yz) - f(x, y)z.$$

Note that, for any 1-cochain g, i.e. for any $g \in \mathrm{Hom}_k(A, M)$, $\delta_1 g$ is a symmetric 2-cocycle where $\delta_1 g(x, y) = xg(y) - g(xy) + g(x)y$. So we can speak of the symmetric 2-cohomology group $H_S{}^2(A, M)$ which is the factor group of the group of all symmetric 2-cocycles modulo the subgroup of all coboundries $\delta_1 g$ with $g \in \mathrm{Hom}_k(A, M)$.

Now if $H_S{}^2(A, A) = 0$ for a k-algebra A, we say that A is of *type* H_2. Of course this type of algebras includes (commutative) algebras of cohomological dimension one. Another example of this type is any field separably generated over another field. Although this is well-known, we shall give an elementary proof of this fact.

Before entering into the proof, we insert a remark about symmetric 2-cocycles: If f is a symmetric 2-cocycle, we have $f(x, 1) = f(1, x) = xf(1, 1)$. Putting $g(x) = xf(1, 1)$ for all $x \in A$, we get $(\delta_1 g)(x, y) = xy(1, 1)$, hence, for $f' = f - \delta_1 g$, we get $f'(x, 1) = f'(1, x) = 0$. We say that f' is normalized.

Lemma 1. *Let K be a field separably generated over a field k. Then every symmetric 2-cocycle f of K to K splits, that is, there is a $g \in \mathrm{Hom}_k(K, K)$ satisfying $f = \delta_1 g$.*

Proof. We may assume that f is normalized, i.e., $f(x, 1) = f(1, x) = 0$ for all $x \in K$. Now let M be a K-bimodule isomorphic to the K-bimodule K.

Then we have $xm = mx$ for all $x \in K$ and $m \in M$. We can put a k-algebra structure on $L = K \times M$ by defining operations:

$$(x, m)+(x', m') = (x+x', m+m');$$
$$(x, m)(x', m') = (xx', xm'+x'm+f(x, x'));$$
$$\alpha(x, m) = (\alpha x, m) \text{ for } \alpha \in k.$$

Since f is a symmetric 2-cocycle, L is a commutative k-algebra. $(1, 0)$ is the unity of L, because f is normalized. Furthermore, $\widetilde{M} = \{(0, m) : m \in M\}$ is an ideal of L satisfying $\widetilde{M}^2 = 0$ and $L/\widetilde{M} \cong K$. Note that if an element of L is not contained in $\widetilde{M}$, it is invertible in L.

Now if we can construct a subfield $\widetilde{K}$ of L in such a way that $\widetilde{K}$ is isomorphic to K by the projection Π_K: $L \to K$, then we are done. To this end, first choose a maximal purely transcendental subfield $k(X) = k(x_\alpha: \alpha \in \Lambda)$ of K. Then $K/k(X)$ is separable algebraic. Now $\widetilde{X} = \{\tilde{x}_\alpha = (x_\alpha, 0) : \alpha \in \Lambda\}$ generates a purely transcendental extension $k(\widetilde{X})$ in L which is isomorphic to $k(X)$ by the projection Π_K. Let $w \in K$, and let $f(T) \in k(X)[T]$ be an irreducible polynominal satisfied by w. Since w is separable the formal derivative $f'(T)$ is not satisfied by w, i.e. $f'(w) \neq 0$. By $\tilde{f}(T)$ we understand the polynominal in $k(\widetilde{X})[T]$ obtained by replacing the coefficients of $f(T)$ by the corresponding elements in $k(\widetilde{X})$. Then, for $\tilde{w} = (w, 0)$, we have $\tilde{f}(\tilde{w}) \in \widetilde{M}$. Furthermore, $\tilde{f}'(\tilde{w}) \notin \widetilde{M}$, hence it is invertible in L. Now we wish to adjust $\tilde{w}$ by an element $\tilde{m} = (0, m) \in \widetilde{M}$ so that $\tilde{f}(\tilde{w}+\tilde{m}) = 0$. If this is possible then $\tilde{w}+\tilde{m} = (w, m)$ generates a finite algebraic extension of $k(\widetilde{X})$ which is isomorphic to $k(X)(w)$ under the projection Π_K. Then repeating this process, we arrive at a subfield $\widetilde{K}$ isomorphic to K by Π_K. Now the condition for our $\tilde{m} = (0, m)$ is $\tilde{f}(\tilde{w}+\tilde{m}) = 0$. But, since $\widetilde{M}^2 = 0$, this condition just takes the form $\tilde{f}(\tilde{w})+\tilde{f}'(\tilde{w})\cdot\tilde{m} = 0$. Because $\tilde{f}'(\tilde{w})$ is invertible, we have a unique $\tilde{m} \in \widetilde{M}$ satisfying this condition. This completes the proof.

Lemma 2. *Let A be a k-algebra, and $D_n = \{D_r: r = 0, 1, 2, \dots, n\}$ a sequence with $D_r \in \mathrm{Hom}_k(A, A)$ satisfying the conditions (i) and (ii). Then $f(x, y) = \sum_{r=1}^{n} D_r(x)D_{n+1-r}(y)$ is a symmetric 2-cocycle of A to A.*

Proof. $f(x, y) = f(y, x)$ is trivial. So we show that $\delta_2 f(x, y, z) = xf(y, z)-f(xy, z)+f(x, yz)-f(x, y)z$ vanishes.

By the condition (ii) we can write $\delta_2 f(x, y, z)$ in the form

$$\begin{aligned}
&\delta_2 f(x, y, z) \\
&= \sum_{r=1}^{n} xD_r(y)D_{n+1-r}(z) - \Big[\sum_{r=1}^{n} xD_r(y)D_{n+1-r}(z) \\
&\quad + \sum_{r=1}^{n} D_r(x)yD_{n+1-r}(z) + \sum_{r=2}^{n}\sum_{s=1}^{r-1} D_s(x)D_{r-s}(y)D_{n+1-r}(z)\Big] \\
&\quad + \Big[\sum_{r=1}^{n} D_r(x)yD_{n+1-r}(z) + \sum_{r=1}^{n} D_r(x)D_{n+1-r}(y)z \\
&\quad + \sum_{r=1}^{n-1}\sum_{s=1}^{n-r} D_r(x)D_s(y)D_{n+1-r-s}(z)\Big] - \sum_{r=1}^{n} D_r(x)D_{n+1-r}(y)z.
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{r=2}^{n}\sum_{s=1}^{r-1} D_s(x)D_{r-s}(y)D_{n+1-r}(z) \\
&\qquad = \sum_{\substack{u+v+w=n+1 \\ u\neq 0, v\neq 0, w\neq 0}} D_u(x)D_v(y)D_w(z) \\
&\qquad = \sum_{r=1}^{n-1}\sum_{s=1}^{n-r} D_r(x)D_s(y)D_{n+1-r-s}(z)
\end{aligned}$$

we have $\delta_2 f = 0$.

Proposition 2. *Let A be a k-algebra of type H_2, and $\Delta_n = \{D_r\colon r = 0, 1, 2, \ldots, n\}$ a sequence with $D_r \in \mathrm{Hom}_k(A, A)$ satisfying the conditions* (i) *and* (ii). *Then one can find a $D_{n+1} \in \mathrm{Hom}_k(A, A)$ so that $\Delta_{n+1} = \{D_r\colon r = 0, 1, 2, \ldots, n+1\}$ still satisfies the conditions* (i) *and* (ii). *The choice of D_{n+1} is unique up to k-derivations of A. Hence any such D_n can be completed to an H-S sequence D of A.*

Proof. By Lemma 2, $f(x, y) = \sum_{r=1}^{n} D_r(x)D_{n+1-r}(y)$ is a symmetric 2-cocycle, hence there is a $D_{n+1} \in \mathrm{Hom}_k(A, A)$ satisfying $f(x, y) = (\delta_1 D_{n+1})(x, y)$. But this shows nothing but that the condition (ii) is satisfied by Δ_{n+1}.

Corollary. *Let A be a k-algebra of type H_2, and $A[[T]]$ the ring of formal power series over A. Then, for any $n \geq 1$, any embedding ϕ of A into $A[[T]]/\langle T^n\rangle$ such that $\mathrm{Im}\,\phi(\mathrm{mod}\,\langle T\rangle) = A$ can be lifted to an embedding of A into $A[[T]]$.*

REFERENCES

[1] S. ABU-SAYMEH : On Hasse-Schmidt higher derivations, to appear in Osaka J. Math.
[2] H. HASSE and F. K. SCHMIDT : Noch eine Begründung der Theorie der höheren Differentialquotienten einem algebraischen Funktionenkörper einer Unbestimmten, J. Reine Angew Math. **177** (1937), 215-237.
[3] Y. NAKAI : High order derivations I, Osaka J. Math. **7** (1970), 1-27.
[4] H. OSBORN : Modules of differentials I, Math. Ann. **170** (1967), 221-224.

DEPARTMENT OF MATHEMATICS
YARMOUK UNIVERSITY
IRBID, JORDAN
DEPARTMENT OF MATHMATICS
MIDDLE EAST TECHNICAL UNIVERSITY
ANKARA, TURKEY

(Received November 30, 1985)