Math. J. Okayama Univ. 29 (1987), 53—75

ON THE LOEWY STRUCTURE OF THE PROJECTIVE INDECOMPOSABLE MODULES FOR A 3-SOLVABLE GROUP II

Dedicated to Professor Hisao Tominaga on his 60th birthday

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Let $GF(3^3)$ be the finite field with 3^3 elements, and we define $\Gamma = \Gamma(3^3)$ to be the set of transformations of the form

$$x \rightarrow ax^{3^i}+b;\ \ a, b \in GF(3^3),\ a \neq 0,\ i = 0, 1, 2,$$

on $GF(3^3)$ (see [1, Kap II, 1.18(b)]). Then Γ is a solvable group of order $2\cdot 3^4\cdot 13$, and has a normal subgroup $\mathfrak{G}$ with $[\Gamma : \mathfrak{G}] = 2$. In the previous paper [4], we completely determined the Loewy series of the projective indecomposable modules for $\mathfrak{G}$ in characteristic 3. In this paper, by making use of the result and methods in [4], we shall determine the Loewy series of the projective indecomposable modules for Γ in characteristic 3.

Let K be an algebraically closed field of characteristic 3, and let I, M_1, M_2, M_3, M_4 be the simple $K\mathfrak{G}$-modules defined in [4]. Then for each i, $1 \leqq i \leqq 4$, there exist non-isomorphic simple $K\Gamma$-modules L_i, N_i (to be defined more precisely in § 1) with $L_i|_{\mathfrak{G}} \simeq N_i|_{\mathfrak{G}} \simeq M_i$. Further $K\Gamma$ has a non-trivial module of dimension 1, which will be denoted by J, while the trivial simple $K\Gamma$-module will be denoted by K. The modules K, J, L_i, N_i $(1 \leqq i \leqq 4)$ are all the simple $K\Gamma$-modules (to within isomorphism). Given a simple $K\Gamma$-module X, we denote by $\boldsymbol{P}_X$ the projective indecomposable $K\Gamma$-module for which $\boldsymbol{P}_X/J(K\Gamma)\boldsymbol{P}_X \simeq X$. Our result is stated as

Theorem. *The Loewy series of the projective indecomposable $K\Gamma$-modules are as follows:*

$$\boldsymbol{P}_K = \begin{matrix} K \\ K\ N_1 \\ K\ L_2\ L_3\ N_1 \\ J\ L_2\ L_3\ N_1\ N_3\ N_4 \\ J\ L_1\ L_2\ L_3\ L_4\ N_3\ N_4 \\ J\ L_1\ L_4\ N_2\ N_3\ N_4 \\ K\ L_1\ L_4\ N_2 \\ K\ N_2 \\ K \end{matrix} \qquad \boldsymbol{P}_J = \begin{matrix} J \\ J\ L_1 \\ J\ L_1\ N_2\ N_3 \\ K\ L_1\ L_3\ L_4\ N_2\ N_3 \\ K\ L_3\ L_4\ N_1\ N_2\ N_3\ N_4 \\ K\ L_2\ L_3\ L_4\ N_1\ N_4 \\ J\ L_2\ N_1\ N_4 \\ J\ L_2 \\ J \end{matrix}$$

$$P_{L_1} = \begin{array}{c} L_1 \\ N_2\ N_2\ N_3 \\ K\ L_1\ L_3\ L_3\ L_4\ L_4 \\ K\ N_1\ N_2\ N_3\ N_4\ N_4 \\ K\ L_1\ L_2\ L_4\ N_1 \\ J\ L_2\ N_1 \\ J\ L_2\ L_3 \\ J\ N_3\ N_4 \\ L_1 \end{array} \qquad P_{N_1} = \begin{array}{c} N_1 \\ L_2\ L_2\ L_3 \\ J\ N_1\ N_3\ N_3\ N_4\ N_4 \\ J\ L_1\ L_2\ L_3\ L_4\ L_4 \\ J\ L_1\ N_1\ N_2\ N_4 \\ K\ L_1\ N_2 \\ K\ N_2\ N_3 \\ K\ L_3\ L_4 \\ N_1 \end{array}$$

$$P_{L_2} = \begin{array}{c} L_2 \\ J\ N_3\ N_4 \\ J\ L_1\ L_2\ L_3\ L_4 \\ J\ L_1\ N_1\ N_2\ N_4 \\ K\ L_1\ L_2\ N_2\ N_3 \\ K\ L_3\ L_4\ N_2\ N_3 \\ K\ L_3\ L_4\ N_1 \\ N_1\ N_4 \\ L_2 \end{array} \qquad P_{N_2} = \begin{array}{c} N_2 \\ K\ L_3\ L_4 \\ K\ N_1\ N_2\ N_3\ N_4 \\ K\ L_1\ L_2\ L_4\ N_1 \\ J\ L_2\ L_3\ N_1\ N_2 \\ J\ L_2\ L_3\ N_3\ N_4 \\ J\ L_1\ N_3\ N_4 \\ L_1\ L_4 \\ N_2 \end{array}$$

$$P_{L_3} = \begin{array}{c} L_3 \\ N_1\ N_3\ N_4 \\ L_1\ L_2\ L_2\ L_3\ L_4\ L_4 \\ J\ N_1\ N_2\ N_2\ N_3\ N_4\ N_4 \\ K\ J\ L_2\ L_3\ L_4 \\ K\ J\ L_1 \\ K\ L_1 \\ N_1\ N_2\ N_3 \\ L_3 \end{array} \qquad P_{N_3} = \begin{array}{c} N_3 \\ L_1\ L_3\ L_4 \\ N_1\ N_2\ N_2\ N_3\ N_4\ N_4 \\ K\ L_1\ L_2\ L_2\ L_3\ L_4\ L_4 \\ K\ J\ N_2\ N_3\ N_4 \\ K\ J\ N_1 \\ J\ N_1 \\ L_1\ L_2\ L_3 \\ N_3 \end{array}$$

$$P_{L_4} = \begin{array}{c} L_4 \\ N_1\ N_2\ N_4 \\ K\ L_1\ L_2\ L_2\ L_3\ L_4 \\ K\ J\ N_2\ N_3\ N_3\ N_4 \\ K\ J\ L_4\ N_1 \\ J\ L_1\ L_3\ N_1 \\ L_1\ L_2\ L_3 \\ N_2\ N_3\ N_4 \\ L_4 \end{array} \qquad P_{N_4} = \begin{array}{c} N_4 \\ L_1\ L_2\ L_4 \\ J\ N_1\ N_2\ N_2\ N_3\ N_4 \\ K\ J\ L_2\ L_3\ L_3\ L_4 \\ K\ J\ L_1\ N_4 \\ K\ L_1\ N_1\ N_3 \\ N_1\ N_2\ N_3 \\ L_2\ L_3\ L_4 \\ N_4 \end{array}$$

Concerning notations and terminology we refer to our previous paper [4].

1. Preliminaries. Let $\mathfrak{U} = \langle a, b, c \rangle$, $\mathfrak{V} = \langle v \rangle$ and $\mathfrak{W} = \langle s \rangle$ be the subgroups of $\mathfrak{G}$ defined in [4], and let $t \in \Gamma$ denote the transformation defined by

$$t:\ x \to -x, \qquad x \in GF(3^3).$$

We set $\mathfrak{T} = \langle t \rangle$. Then $\mathfrak{G} = \langle \mathfrak{U}, \mathfrak{B}, \mathfrak{W} \rangle$ and $\Gamma = \langle \mathfrak{U}, \mathfrak{B} \times \mathfrak{T}, \mathfrak{W} \rangle$. Since $\mathfrak{T}$ is of order 2, $K\mathfrak{T}$ has two non-isomorphic simple modules Z_0, Z_1, where Z_0 is the trivial module. Let $V_0, V_1, \dots, V_{12}$ be the simple $K\mathfrak{H}$-modules ($\mathfrak{H} = \langle \mathfrak{U}, \mathfrak{B} \rangle$) defined in [4]. Regard these modules V_i as $K\mathfrak{B}$-modules. Then

$$U_i = V_i \otimes_K Z_0,\ W_i = V_i \otimes_K Z_1 \qquad (0 \leqq i \leqq 12)$$

are all the simple $K\mathfrak{B} \times \mathfrak{T}$-modules (to within isomorphism). Set $\mathfrak{J} = \langle \mathfrak{U}, \mathfrak{B} \times \mathfrak{T} \rangle$. As $\mathfrak{J}/\mathfrak{U} \simeq \mathfrak{B} \times \mathfrak{T}$, U_i and W_i induce simple $K\mathfrak{J}$-modules, also denoted by U_i and W_i, by defining

$$uhx = hx,\ uhy = hy, \qquad u \in \mathfrak{U},\ h \in \mathfrak{B} \times \mathfrak{T},\ x \in U_i,\ y \in W_i.$$

The induced modules U_i^Γ, W_i^Γ of these $K\mathfrak{J}$-modules are simple $K\Gamma$-modules, and for $i = 1, 4, 7, 10$,

$$U_{i+k}^\Gamma \simeq U_i^\Gamma,\ W_{i+k}^\Gamma \simeq W_i^\Gamma, \qquad k = 1, 2.$$

So we set

$$L_1 = U_1^\Gamma,\ L_2 = U_4^\Gamma,\ L_3 = U_7^\Gamma,\ L_4 = U_{10}^\Gamma,$$
$$N_1 = W_1^\Gamma,\ N_2 = W_4^\Gamma,\ N_3 = W_7^\Gamma,\ N_4 = W_{10}^\Gamma.$$

Then K, J, L_i, N_i $(1 \leqq i \leqq 4)$ represent all types of simple $K\Gamma$-modules. From the definition of L_i, N_i and M_i, we see at once that $L_i|_{\mathfrak{G}} \simeq N_i|_{\mathfrak{G}} \simeq M_i$. In what follows we denote by $\tilde{P}_{U_i}$ and $\tilde{P}_{W_i}$ the projective indecomposable $K\mathfrak{J}$-modules for which $\tilde{P}_{U_i}/J(K\mathfrak{J})\tilde{P}_{U_i} \simeq U_i$ and $\tilde{P}_{W_i}/J(K\mathfrak{J})\tilde{P}_{W_i} \simeq W_i$.

Lemma 1.1. (1) *$U_i^* \simeq U_{i+3}$ and $W_i^* \simeq W_{i+3}$ for $i = 1, 2, 3, 7, 8, 9$.*

(2) $L_1^* \simeq L_2$, $L_3^* \simeq L_4$, $N_1^* \simeq N_2$, $N_3^* \simeq N_4$.

(3) $U_0^\Gamma = \begin{matrix} K \\ K \\ K \end{matrix}$, $W_0^\Gamma = \begin{matrix} J \\ J \\ J \end{matrix}$.

(4) $\tilde{P}_{U_0}^\Gamma \simeq \boldsymbol{P}_K$, $\tilde{P}_{U_1}^\Gamma \simeq \boldsymbol{P}_{L_1}$, $\tilde{P}_{U_4}^\Gamma \simeq \boldsymbol{P}_{L_2}$, $\tilde{P}_{U_7}^\Gamma \simeq \boldsymbol{P}_{L_3}$, $\tilde{P}_{U_{10}}^\Gamma \simeq \boldsymbol{P}_{L_4}$,
$\tilde{P}_{W_0}^\Gamma \simeq \boldsymbol{P}_J$, $\tilde{P}_{W_1}^\Gamma \simeq \boldsymbol{P}_{N_1}$, $\tilde{P}_{W_4}^\Gamma \simeq \boldsymbol{P}_{N_2}$, $\tilde{P}_{W_7}^\Gamma \simeq \boldsymbol{P}_{N_3}$, $\tilde{P}_{W_{10}}^\Gamma \simeq \boldsymbol{P}_{N_4}$.

(5) $K|_{\mathfrak{J}} \simeq U_0$, $J|_{\mathfrak{J}} \simeq W_0$ *and for every i, $1 \leqq i \leqq 4$,*

$$L_i|_{\mathfrak{J}} \simeq \oplus_{j=0}^{2} U_{3i+j-2},\ N_i|_{\mathfrak{J}} \simeq \oplus_{j=0}^{2} W_{3i+j-2}.$$

(6) $I^\Gamma \simeq K \oplus J$, $M_i^\Gamma \simeq L_i \oplus N_i$, $\qquad 1 \leqq i \leqq 4$.

(7) $P_I^\Gamma \simeq \boldsymbol{P}_K \oplus \boldsymbol{P}_J,\ P_{M_i}^\Gamma \simeq \boldsymbol{P}_{L_i} \oplus \boldsymbol{P}_{N_i},\qquad 1 \leqq i \leqq 4.$

(8) *For every positive integer k,*

$$L_k(P_I)^\Gamma \simeq L_k(\boldsymbol{P}_K) \oplus L_k(\boldsymbol{P}_J),$$
$$L_k(P_{M_i})^\Gamma \simeq L_k(\boldsymbol{P}_{L_i}) \oplus L_k(\boldsymbol{P}_{N_i}),\qquad 1 \leqq i \leqq 4.$$

(9) *For every positive integer k,*

$$\dim_K L_k(P_{M_i}) = \dim_K L_k(\boldsymbol{P}_{L_i}) = \dim_K L_k(\boldsymbol{P}_{N_i}),\qquad 1 \leqq i \leqq 4.$$

Proof. (1) is clear by the definition of the modules U_i and W_i, and (2) follows at once from (1).

(3) Since $J(K\Gamma) = K\Gamma J(K\mathfrak{G})$, by [4, Lemma 3.1] we have

$$J(K\Gamma) = K\Gamma J(K\mathfrak{U}) + K\Gamma\hat{\mathfrak{B}}J(K\mathfrak{W}) = K\Gamma J(K\mathfrak{U}) + K\mathfrak{T}\hat{\mathfrak{B}}J(K\mathfrak{W}).$$

Since U_0 is isomorphic to the left ideal $K\hat{\mathfrak{J}}$ of $K\mathfrak{J}$, we have

$$U_0^\Gamma \simeq \hat{\mathfrak{J}}K\mathfrak{W}.$$

Hence

$$J(K\Gamma)U_0^\Gamma \simeq (K\Gamma J(K\mathfrak{U}) + K\mathfrak{T}\hat{\mathfrak{B}}J(K\mathfrak{W}))\hat{\mathfrak{J}}K\mathfrak{W} = K\mathfrak{T}\hat{\mathfrak{B}}J(K\mathfrak{W})\hat{\mathfrak{J}}K\mathfrak{W}.$$

Noting that t commutes with both $\hat{\mathfrak{B}}$ and elements in $\mathfrak{W}$, we have

$$K\mathfrak{T}\hat{\mathfrak{B}}J(K\mathfrak{W})\hat{\mathfrak{J}}K\mathfrak{W} = \hat{\mathfrak{B}}J(K\mathfrak{W})K\mathfrak{T}\hat{\mathfrak{J}}K\mathfrak{W} = \hat{\mathfrak{B}}J(K\mathfrak{W})\hat{\mathfrak{J}}K\mathfrak{W} = J(K\mathfrak{W})\hat{\mathfrak{B}}\hat{\mathfrak{J}}K\mathfrak{W} = J(K\mathfrak{W})\hat{\mathfrak{J}}K\mathfrak{W} = \hat{\mathfrak{J}}J(K\mathfrak{W}).$$

Thus we have proved that

$$J(K\Gamma)U_0^\Gamma \simeq \hat{\mathfrak{J}}J(K\mathfrak{W}).$$

From this we have

$$J(K\Gamma)^2U_0^\Gamma \simeq (K\Gamma J(K\mathfrak{U}) + K\mathfrak{T}\hat{\mathfrak{B}}J(K\mathfrak{W}))\hat{\mathfrak{J}}J(K\mathfrak{W}) = K\mathfrak{T}\hat{\mathfrak{B}}J(K\mathfrak{W})\hat{\mathfrak{J}}J(K\mathfrak{W}) = \hat{\mathfrak{B}}J(K\mathfrak{W})\hat{\mathfrak{J}}J(K\mathfrak{W}) = J(K\mathfrak{W})\hat{\mathfrak{B}}\hat{\mathfrak{J}}J(K\mathfrak{W}) = J(K\mathfrak{W})\hat{\mathfrak{J}}J(K\mathfrak{W}) = \hat{\mathfrak{J}}J(K\mathfrak{W})^2 = K\hat{\Gamma}.$$

Thus we see at once that

$$L_1(U_0^\Gamma) \simeq L_2(U_0^\Gamma) \simeq L_3(U_0^\Gamma) \simeq K.$$

Next, since W_0 is isomorphic to the left ideal $K\hat{\mathfrak{U}}\hat{\mathfrak{B}}(t-1)$ of $K\mathfrak{J}$, the following can be obtained by the method used in the above:

$$W_0^\Gamma \simeq \hat{\mathfrak{U}}\hat{\mathfrak{V}}(t-1)K\mathfrak{W},$$
$$J(K\Gamma)W_0^\Gamma \simeq \hat{\mathfrak{U}}\hat{\mathfrak{V}}(t-1)J(K\mathfrak{W}),$$
$$J(K\Gamma)^2W_0^\Gamma \simeq K\hat{\mathfrak{U}}\hat{\mathfrak{V}}(t-1)\hat{\mathfrak{W}}.$$

This shows that

$$L_1(W_0^\Gamma) \simeq L_2(W_0^\Gamma) \simeq L_3(W_0^\Gamma) \simeq K\hat{\mathfrak{G}}(t-1) \simeq J.$$

Thus (3) is proved.

(5) is clear by the definition of the modules K, J, L_i and N_i.

(4) Clearly $\tilde{P}_{U_i}^\Gamma$ and $\tilde{P}_{W_i}^\Gamma$ are projective $K\Gamma$-modules. Further by (5) and Frobenius reciprocity theorem, we have

$$\tilde{P}_{U_i}^\Gamma/J(K\Gamma)\tilde{P}_{U_i}^\Gamma \simeq \begin{cases} K, & i = 0, \\ L_{(i+2)/3}, & i = 1, 4, 7, 10, \end{cases}$$
$$\tilde{P}_{W_i}^\Gamma/J(K\Gamma)\tilde{P}_{W_i}^\Gamma \simeq \begin{cases} J, & i = 0, \\ N_{(i+2)/3}, & i = 1, 4, 7, 10, \end{cases}$$

and so (4) follows at once.

(6) By Frobenius reciprocity theorem,

$$\mathrm{Hom}_{K\Gamma}(K, I^\Gamma) \simeq \mathrm{Hom}_{K\Gamma}(J, I^\Gamma) \simeq \mathrm{Hom}_{K\mathfrak{G}}(I, I),$$
$$\mathrm{Hom}_{K\Gamma}(L_i, M_i^\Gamma) \simeq \mathrm{Hom}_{K\Gamma}(N_i, M_i^\Gamma) \simeq \mathrm{Hom}_{K\mathfrak{G}}(M_i, M_i).$$

This shows that $K \oplus J$ (resp. $L_i \oplus N_i$) is isomorphic to a submodule of I^Γ (resp. M_i^Γ). But $\dim_K I^\Gamma = 2$ and $\dim_K M_i^\Gamma = 6$. Hence the result follows.

(7) By virtue of Frobenius reciprocity theorm, we can see that

$$P_I^\Gamma/J(K\Gamma)P_I^\Gamma \simeq K \oplus J, \; P_{M_i}^\Gamma/J(K\Gamma)P_{M_i}^\Gamma \simeq L_i \oplus N_i.$$

As P_I^Γ and $P_{M_i}^\Gamma$ are projective $K\Gamma$-modules, the result follows from the above.

(8) Given a $K\mathfrak{G}$-module M, we see that M is completely reducible if and only if M^Γ is completely reducible, because $[\Gamma: \mathfrak{G}] = 2$. Hence (8) follows at once from (7).

(9) Let e_i be the primitive idempotent in $K\mathfrak{G}$ corresponding to P_{M_i}. Then $-e_i(t+1)$ and $e_i(t-1)$ are the primitive idempotents in $K\Gamma$ corresponding to P_{L_i} and P_{N_i} respectively. Noting that

$$J(K\Gamma) = K\Gamma J(K\mathfrak{G}) \text{ and } \Gamma = \langle \mathfrak{G}, \mathfrak{T} \rangle,$$

we have

$$\begin{aligned} L_k(P_{L_i}) &\simeq J(K\Gamma)^{k-1}e_i(t+1)/J(K\Gamma)^k e_i(t+1) \\ &= J(K\mathfrak{G})^{k-1}e_i(t+1)/J(K\mathfrak{G})^k e_i(t+1), \\ L_k(P_{N_i}) &\simeq J(K\Gamma)^{k-1}e_i(t-1)/J(K\Gamma)^k e_i(t-1) \\ &= J(K\mathfrak{G})^{k-1}e_i(t-1)/J(K\mathfrak{G})^k e_i(t-1). \end{aligned}$$

From this it follows at once that

$$\begin{aligned} \dim_K L_k(P_{L_i}) &= \dim_K L_k(P_{N_i}) \\ &= \dim_K J(K\mathfrak{G})^{k-1}e_i/J(K\mathfrak{G})^k e_i \\ &= \dim_K L_k(P_{M_i}), \end{aligned}$$

proving (9).

Remark. If the Loewy series of P_K (resp. P_{L_i}) is known, then by [4, Theorem] and Lemma 1.1, (6), (8), we can obtain the Loewy series of P_J (resp. P_{N_i}). Hence in order to prove our theorem, it suffices to determine the Loewy series of P_K and P_{L_i}.

If a_i and b_i are the primitive idempotents in $K\mathfrak{Z}$ corresponding to $\tilde{P}_{U_i}$ and $\tilde{P}_{W_i}$ respectively, then

$$\begin{aligned} \mathrm{Hom}_{K\mathfrak{Z}}(\tilde{P}_{U_i}, \tilde{P}_{U_j}) &\simeq a_i K\mathfrak{U} a_j, \\ \mathrm{Hom}_{K\mathfrak{Z}}(\tilde{P}_{U_i}, \tilde{P}_{W_j}) &\simeq a_i K\mathfrak{U} b_j, \\ \mathrm{Hom}_{K\mathfrak{Z}}(\tilde{P}_{W_i}, \tilde{P}_{U_j}) &\simeq b_i K\mathfrak{U} a_j, \\ \mathrm{Hom}_{K\mathfrak{Z}}(\tilde{P}_{W_i}, \tilde{P}_{W_j}) &\simeq b_i K\mathfrak{U} b_j. \end{aligned}$$

But we easily see that

$$\begin{aligned} a_i K\mathfrak{U} a_j &= \begin{cases} \langle a_i,\ a_i a a_i\rangle, & i = j, \\ \langle a_i a a_j\rangle, & i \neq j, \end{cases} \\ a_i K\mathfrak{U} b_j &= \langle a_i a b_j\rangle, \\ b_i K\mathfrak{U} a_j &= \langle b_i a a_j\rangle, \\ b_i K\mathfrak{U} b_j &= \begin{cases} \langle b_i,\ b_i a b_i\rangle, & i = j, \\ \langle b_i a b_j\rangle, & i \neq j. \end{cases} \end{aligned}$$

Hence we have the following

Lemma 1.2. *The Cartan matrix of $K\mathfrak{Z}$ is given by*

$$\begin{pmatrix} 2 & 1 & \cdots\cdots & 1 \\ 1 & 2 & \cdots\cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots\cdots & 2 \end{pmatrix}.$$

The preceding lemma together with Lemma 1.1(3), (4) implies the following

Corollary 1.3. *The Cartan matrix of $K\Gamma$ is given by*

$$\begin{matrix} P_K \\ P_J \\ P_{L_1} \\ P_{N_1} \\ \vdots \\ P_{L_4} \\ P_{N_4} \end{matrix}\begin{pmatrix} 6 & 3 & 3 & 3 & \cdots\cdots & 3 \\ 3 & 6 & 3 & 3 & \cdots\cdots & 3 \\ 3 & 3 & 4 & 3 & \cdots\cdots & 3 \\ 3 & 3 & 3 & 4 & \cdots\cdots & 3 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 3 & 3 & 3 & 3 & \cdots\cdots & 3 \\ 3 & 3 & 3 & 3 & \cdots\cdots & 4 \end{pmatrix}.$$

2. The Loewy structure of the projective indecomposable $K\mathfrak{J}$-modules. Using an argument similar to the one used in the proof of [4, Proposition 2.3], we shall prove the following

Proposition 2.1. (1) *$\tilde{P}_{U_0}$ has the Loewy and socle series given by*

$$\begin{matrix} U_0 \\ W_1\ W_2\ W_3 \\ U_4\ U_5\ U_6\ U_7\ U_8\ U_9 \\ W_0\ W_7\ W_8\ W_9\ W_{10}\ W_{11}\ W_{12}. \\ U_1\ U_2\ U_3\ U_{10}\ U_{11}\ U_{12} \\ W_4\ W_5\ W_6 \\ U_0 \end{matrix}$$

(2) *For every i, $1 \leqq i \leqq 12$, $L_k(\tilde{P}_{U_i}) \simeq S_{8-k}(\tilde{P}_{U_i}) \simeq L_k(\tilde{P}_{U_0}) \otimes_K U_i$ $(1 \leqq k \leqq 7)$.*

(3) *For every i, $0 \leqq i \leqq 12$, $L_k(\tilde{P}_{W_i}) \simeq S_{8-k}(\tilde{P}_{W_i}) \simeq L_k(\tilde{P}_{U_0}) \otimes_K W_i$ $(1 \leqq k \leqq 7)$.*

Proof. We shall identify $\tilde{P}_{U_0}$ with the left ideal $K\mathfrak{U}\sigma$ of $K\mathfrak{J}$, where $\sigma = -\hat{\mathfrak{Y}}\hat{\mathfrak{X}} = -\sum_{x\in\mathfrak{Y}\times\mathfrak{X}} x$ is the primitive idempotent in $K\mathfrak{J}$. As $J(K\mathfrak{J}) = K\mathfrak{J}J(K\mathfrak{U})$, we have $J(K\mathfrak{J})^i\tilde{P}_{U_0} = J(K\mathfrak{U})^i\sigma$. View $K\mathfrak{U}$ as a $K\mathfrak{J}$-module via conjugation of $\mathfrak{J}$ on $\mathfrak{U}$. Then the above implies that there exists a $K\mathfrak{J}$-isomorphism:

$$J(K\mathfrak{J})^i\tilde{P}_{U_0}/J(K\mathfrak{J})^{i+1}\tilde{P}_{U_0} = J(K\mathfrak{U})^i\sigma/J(K\mathfrak{U})^{i+1}\sigma \simeq J(K\mathfrak{U})^i/J(K\mathfrak{U})^{i+1}.$$

At first we shall prove that

$$L_2(\tilde{P}_{U_0}) \simeq W_1 \oplus W_2 \oplus W_3. \tag{2.2}$$

In order to prove (2.2), it suffices to show that the $K\mathfrak{Z}$-module $J(K\mathfrak{U})/J(K\mathfrak{U})^2$ is isomorphic to $W_1 \oplus W_2 \oplus W_3$. It is easy to see that $J(K\mathfrak{U})/J(K\mathfrak{U})^2$ has a K-basis consisting of the elements:

$$(a-1)+J(K\mathfrak{U})^2,\ (b-1)+J(K\mathfrak{U})^2,\ (c-1)+J(K\mathfrak{U})^2.$$

Operating v and t on $(a-1)$, $(b-1)$ and $(c-1)$, we obtain the following congruences relative to mod $J(K\mathfrak{U})^2$:

$$\begin{aligned}
(a-1)^v &\equiv (c-1),\\
(b-1)^v &\equiv -(a-1)+(c-1),\\
(c-1)^v &\equiv -(a-1)-(b-1)+(c-1),\\
(a-1)^t &\equiv -(a-1),\\
(b-1)^t &\equiv -(b-1),\\
(c-1)^t &\equiv -(c-1).
\end{aligned}$$

This shows that $J(K\mathfrak{U})/J(K\mathfrak{U})^2$ affords the matrix representation

$$S:\ v \to S(v) = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix},\ t \to S(t) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The characteristic polynomial of the matrix $S(v)$ is X^3-X^2-X-1, and so (2.2) follows. Next we show that

$$L_3(\tilde{P}_{U_0}) \simeq U_4 \oplus U_5 \oplus U_6 \oplus U_7 \oplus U_8 \oplus U_9. \tag{2.3}$$

By (2.2), we can choose a K-basis $\{m_1, m_2, m_3\}$ of $J(K\mathfrak{U})/J(K\mathfrak{U})^2$ such that $m_i^v = \zeta^{3^{i-1}} m_i$, $m_i^t = -m_i$, where ζ is a root of the polynomial X^3-X^2-X-1. Now we express m_i as

$$m_i = \alpha_i + J(K\mathfrak{U})^2 \qquad (\alpha_i \in J(K\mathfrak{U})).$$

Then $\{\alpha_i\alpha_j + J(K\mathfrak{U})^3 \mid 1 \leqq i \leqq j \leqq 3\}$ forms a K-basis of $J(K\mathfrak{U})^2/J(K\mathfrak{U})^3$. Operating v and t on the elements $\alpha_i\alpha_j$, we obtain the following congruences relative to mod $J(K\mathfrak{U})^3$:

$$\begin{aligned}
(\alpha_1^2)^v &\equiv \zeta^2\alpha_1^2, & (\alpha_1^2)^t &\equiv \alpha_1^2,\\
(\alpha_2^2)^v &\equiv \zeta^6\alpha_2^2, & (\alpha_2^2)^t &\equiv \alpha_2^2,\\
(\alpha_3^2)^v &\equiv \zeta^5\alpha_3^2, & (\alpha_3^2)^t &\equiv \alpha_3^2,\\
(\alpha_1\alpha_2)^v &\equiv \zeta^4\alpha_1\alpha_2, & (\alpha_1\alpha_2)^t &\equiv \alpha_1\alpha_2,\\
(\alpha_1\alpha_3)^v &\equiv \zeta^{10}\alpha_1\alpha_3, & (\alpha_1\alpha_3)^t &\equiv \alpha_1\alpha_3,\\
(\alpha_2\alpha_3)^v &\equiv \zeta^{12}\alpha_2\alpha_3, & (\alpha_2\alpha_3)^t &\equiv \alpha_2\alpha_3.
\end{aligned}$$

From this we obtain (2.3)

By [2, Corollary], the Loewy and socle series of $\tilde{P}_{U_0}$ coincide. Hence noting that the Loewy length of $\tilde{P}_{U_0}$ is 7 and that $\tilde{P}_{U_0}$ is self-dual, we conclude from Lemma 1.1(1), Lemma 1.2 and the above results (2.2), (2.3), that $\tilde{P}_{U_0}$ has the Loewy and socle series given in (1).

(2) and (3) are obtained from the following isomorphism:

$$\tilde{P}_{U_i} \simeq \tilde{P}_{U_0} \otimes_K U_i \, (1 \leqq i \leqq 12), \; \tilde{P}_{W_j} \simeq \tilde{P}_{U_0} \otimes_K W_j \, (0 \leqq j \leqq 12).$$

Thus Proposition 2.1 is proved.

Corollary 2.4. (1) $L_2(\boldsymbol{P}_{L_1}) \simeq N_2 \oplus N_2 \oplus N_3$.

(2) $L_3(\boldsymbol{P}_{L_1}) \simeq K \oplus L_1 \oplus L_3 \oplus L_3 \oplus L_4 \oplus L_4$.

(3) $L_2(\boldsymbol{P}_{L_2}) \simeq J \oplus N_3 \oplus N_4$.

(4) $L_2(\boldsymbol{P}_{L_3}) \simeq N_1 \oplus N_3 \oplus N_4$.

(5) $L_3(\boldsymbol{P}_{L_3}) \simeq L_1 \oplus L_2 \oplus L_2 \oplus L_3 \oplus L_4 \oplus L_4$.

(6) $L_4(\boldsymbol{P}_{L_3}) \simeq J \oplus N_1 \oplus N_2 \oplus N_2 \oplus N_3 \oplus N_4 \oplus N_4$.

(7) $L_2(\boldsymbol{P}_{L_4}) \simeq N_1 \oplus N_2 \oplus N_4$.

(8) $L_3(\boldsymbol{P}_{L_4}) \simeq K \oplus L_1 \oplus L_2 \oplus L_2 \oplus L_3 \oplus L_4$.

Proof. (1) From the isomorphism:

$$\begin{aligned} \boldsymbol{P}_{L_1}/J(K\mathfrak{Y})\boldsymbol{P}_{L_1} &\simeq \tilde{P}^{\Gamma}_{U_1}/J(K\mathfrak{Y})\tilde{P}^{\Gamma}_{U_1} \simeq \tilde{P}^{\Gamma}_{U_1}/(J(K\mathfrak{Y})\tilde{P}_{U_1})^{\Gamma} \\ &\simeq (\tilde{P}_{U_1}/J(K\mathfrak{Y})\tilde{P}_{U_1})^{\Gamma} \simeq U_1^{\Gamma} = L_1, \end{aligned}$$

it follows that $J(K\Gamma)\boldsymbol{P}_{L_1} = J(K\mathfrak{Y})\boldsymbol{P}_{L_1}$. Therefore

$$\begin{aligned} J(K\Gamma)\boldsymbol{P}_{L_1}/J(K\mathfrak{Y})^2\boldsymbol{P}_{L_1} &= J(K\mathfrak{Y})\boldsymbol{P}_{L_1}/J(K\mathfrak{Y})^2\boldsymbol{P}_{L_1} \\ &\simeq J(K\mathfrak{Y})\tilde{P}^{\Gamma}_{U_1}/J(K\mathfrak{Y})^2\tilde{P}^{\Gamma}_{U_1} \simeq (J(K\mathfrak{Y})\tilde{P}_{U_1})^{\Gamma}/(J(K\mathfrak{Y})^2\tilde{P}_{U_1})^{\Gamma} \\ &\simeq (J(K\mathfrak{Y})\tilde{P}_{U_1}/J(K\mathfrak{Y})^2\tilde{P}_{U_1})^{\Gamma} \simeq (W_5 \oplus W_6 \oplus W_7)^{\Gamma} \\ &= N_2 \oplus N_2 \oplus N_3. \end{aligned}$$

Hence $J(K\Gamma)\boldsymbol{P}_{L_1}/J(K\mathfrak{Y})^2\boldsymbol{P}_{L_1}$ is completely reducible, and so we have

$$J(K\mathfrak{Y})^2\boldsymbol{P}_{L_1} = J(K\Gamma)^2\boldsymbol{P}_{L_1} \text{ and } L_2(\boldsymbol{P}_{L_1}) \simeq N_2 \oplus N_2 \oplus N_3,$$

proving (1).

(2) Since $J(K\Gamma)^2\boldsymbol{P}_{L_1} = J(K\mathfrak{Y})^2\boldsymbol{P}_{L_1}$, we have

$$\begin{aligned} J(K\Gamma)^2\boldsymbol{P}_{L_1}/J(K\mathfrak{Y})^3\boldsymbol{P}_{L_1} &= J(K\mathfrak{Y})^2\boldsymbol{P}_{L_1}/J(K\mathfrak{Y})^3\boldsymbol{P}_{L_1} \\ &\simeq J(K\mathfrak{Y})^2\tilde{P}^{\Gamma}_{U_1}/J(K\mathfrak{Y})^3\tilde{P}^{\Gamma}_{U_1} \simeq (J(K\mathfrak{Y})^2\tilde{P}_{U_1})^{\Gamma}/(J(K\mathfrak{Y})^3\tilde{P}_{U_1})^{\Gamma} \\ &\simeq (J(K\mathfrak{Y})^2\tilde{P}_{U_1}/J(K\mathfrak{Y})^3\tilde{P}_{U_1})^{\Gamma} \\ &\simeq (U_0 \oplus U_2 \oplus U_8 \oplus U_9 \oplus U_{10} \oplus U_{11})^{\Gamma} \end{aligned}$$

$$\begin{array}{l} \quad K\ L_1\ L_3\ L_3\ L_4\ L_4 \\ = K \\ \quad K. \end{array}$$

As $J(K\mathfrak{J})^3\boldsymbol{P}_{L_1} \subset J(K\Gamma)^3\boldsymbol{P}_{L_1}$, the above implies that

$$L_3(\boldsymbol{P}_{L_1}) \simeq K \oplus L_1 \oplus L_3 \oplus L_3 \oplus L_4 \oplus L_4,$$

proving (2).

(3) through (8) can be obtained by the method used for the proof of (1) and (2), and we omit the proof.

3. The Loewy structure of $\boldsymbol{P}_K$ and $\boldsymbol{P}_J$. In this section, by making use of an argument similar to the one used in [4, § 3], we shall determine the Loewy series of $\boldsymbol{P}_K$. Then the Loewy series of $\boldsymbol{P}_J$ is obtained (see § 1, Remark). We set $\mathfrak{R} = \langle \mathfrak{B}\times\mathfrak{T}, \mathfrak{W}\rangle$, and we let $\hat{K}$ be the trivial simple $K\mathfrak{R}$-module and $\hat{P}_{\hat{K}}$ the projective indecomposable $K\mathfrak{R}$-module for which $\hat{P}_{\hat{K}}/J(K\mathfrak{R})\hat{P}_{\hat{K}} \simeq \hat{K}$. As in the proof of Proposition 2.1, we denote by σ the primitive idempoptent $-\hat{\mathfrak{V}}\hat{\mathfrak{T}}$ in $K\mathfrak{J}$, and set

$$A = K\mathfrak{U}\sigma\hat{\mathfrak{W}},\ B = K\mathfrak{U}\sigma J(K\mathfrak{W}),\ C = K\mathfrak{U}\sigma K\mathfrak{W}.$$

Since $\hat{P}_{\hat{K}}^{\Gamma} \simeq \boldsymbol{P}_K$, $\hat{P}_{\hat{K}} \simeq K\mathfrak{R}\sigma = \sigma K\mathfrak{W}$ and $J(K\mathfrak{R}) = K\mathfrak{T}\hat{\mathfrak{V}}J(K\mathfrak{W})$, we have

$$A \simeq (J(K\mathfrak{R})^2\hat{P}_{\hat{K}})^{\Gamma},\ B \simeq (J(K\mathfrak{R})\hat{P}_{\hat{K}})^{\Gamma},\ C \simeq \hat{P}_{\hat{K}}^{\Gamma} \simeq \boldsymbol{P}_K.$$

At first we determine the Loewy series of A.

Lemma 3.1.

$$A = \begin{array}{c} K \\ N_1 \\ L_2\ L_3 \\ J\ N_3\ N_4. \\ L_1\ L_4 \\ N_2 \\ K \end{array}$$

Proof. Let C_{a^i} $(i = 1, 2)$ be the conjugate class in $\mathfrak{U}\mathfrak{B}$ containing a^i. Then we have

$$\hat{\mathfrak{B}}K\mathfrak{U}\sigma = (\hat{\mathfrak{B}}K\mathfrak{U}\hat{\mathfrak{B}})\sigma = \langle\hat{\mathfrak{B}}, \hat{C}_a\hat{\mathfrak{B}}, \hat{C}_{a^2}\hat{\mathfrak{B}}\rangle\sigma = \langle\sigma, \hat{C}_a\sigma, \hat{C}_{a^2}\sigma\rangle.$$

From this we obtain

$$\begin{aligned}\hat{\mathfrak{V}}J(K\mathfrak{W})A &= \hat{\mathfrak{V}}J(K\mathfrak{W})K\mathfrak{U}\sigma\hat{\mathfrak{W}} \\ &= J(K\mathfrak{W})(\hat{\mathfrak{V}}K\mathfrak{U}\sigma)\hat{\mathfrak{W}} \\ &= J(K\mathfrak{W})\langle \sigma, \hat{C}_a\sigma, \hat{C}_{a^2}\sigma\rangle\hat{\mathfrak{W}} \\ &= \langle \sigma, \hat{C}_a\sigma, \hat{C}_{a^2}\sigma\rangle J(K\mathfrak{W})\hat{\mathfrak{W}} = 0.\end{aligned}$$

Hence, recalling that $J(K\Gamma)$ is expressed as

$$J(K\Gamma) = K\Gamma J(K\mathfrak{U}) + K\mathfrak{T}\hat{\mathfrak{V}}J(K\mathfrak{W}),$$

we obtain

$$J(K\Gamma)^i A = K\Gamma J(K\mathfrak{U})^i A = J(K\mathfrak{U})^i\sigma\hat{\mathfrak{W}}.$$

Thus it holds that

$$J(K\Gamma)^i A|_{\mathfrak{Z}} \simeq J(K\mathfrak{U})^i\sigma \simeq J(K\mathfrak{Z})^i\tilde{P}_{U_0},$$

and so

$$(J(K\Gamma)^i A/J(K\Gamma)^{i+1}A)|_{\mathfrak{Z}} \simeq J(K\mathfrak{Z})^i\tilde{P}_{U_0}/J(K\mathfrak{Z})^{i+1}\tilde{P}_{U_0}.$$

Therefore the result follows from Proposition 2.1(1) and Lemma 1.1(5).

Next, we determine the Loewy structure of B.

Lemma 3.2. (1) $L_1(B) \simeq L_1(A)$.
(2) $L_i(B) \simeq L_i(A) \oplus L_{i-1}(A)$ *for* $2 \leqq i \leqq 7$.
(3) $L_8(B) = L_7(A)$.

Proof. At first we shall show, by induction, that

(3.3) $J(K\Gamma)^i B = J(K\mathfrak{U})^i\sigma J(K\mathfrak{W}) + J(K\Gamma)^{i-1}A$, where $J(K\Gamma)^0 = K\Gamma$.

Since $J(K\Gamma) = K\Gamma J(K\mathfrak{U}) + K\mathfrak{T}\hat{\mathfrak{V}}J(K\mathfrak{W})$, we have

$$\begin{aligned}J(K\Gamma)B &= J(K\mathfrak{U})\sigma J(K\mathfrak{W}) + K\mathfrak{T}\hat{\mathfrak{V}}J(K\mathfrak{W})K\mathfrak{U}\sigma J(K\mathfrak{W}) \\ &= J(K\mathfrak{U})\sigma J(K\mathfrak{W}) + K\mathfrak{T}J(K\mathfrak{W})(\hat{\mathfrak{V}}K\mathfrak{U}\sigma)J(K\mathfrak{W}) \\ &= J(K\mathfrak{U})\sigma J(K\mathfrak{W}) + K\mathfrak{T}J(K\mathfrak{W})\langle \sigma, \hat{C}_a\sigma, \hat{C}_{a^2}\sigma\rangle J(K\mathfrak{W}) \\ &= J(K\mathfrak{U})\sigma J(K\mathfrak{W}) + K\mathfrak{T}\langle \sigma, \hat{C}_a\sigma, \hat{C}_{a^2}\sigma\rangle J(K\mathfrak{W})^2 \\ &= J(K\mathfrak{U})\sigma J(K\mathfrak{W}) + K\mathfrak{T}\langle \sigma, \hat{C}_a\sigma, \hat{C}_{a^2}\sigma\rangle\hat{\mathfrak{W}}.\end{aligned}$$

But $t\sigma = \sigma$, $t\hat{C}_a = \hat{C}_{a^2}t$ and $t\hat{C}_{a^2} = \hat{C}_a t$, and so

$$K\mathfrak{T}\langle \sigma, \hat{C}_a\sigma, \hat{C}_{a^2}\sigma\rangle = \langle \sigma, \hat{C}_a\sigma, \hat{C}_{a^2}\sigma\rangle = \hat{\mathfrak{V}}K\mathfrak{U}\sigma.$$

Thus we have

$$J(K\Gamma)B = J(K\mathfrak{U})\,\sigma J(K\mathfrak{W}) + (\hat{\mathfrak{B}}K\mathfrak{U}\,\sigma)\hat{\mathfrak{W}},$$

which implies that

$$J(K\Gamma)B \subset J(K\mathfrak{U})\,\sigma J(K\mathfrak{W}) + A.$$

On the other hand, given $u \in \mathfrak{U}$, $u \neq 1$, we have

$$\begin{aligned} u\sigma\hat{\mathfrak{W}} &= (1-\sigma)u\sigma\hat{\mathfrak{W}} + \sigma u\sigma\hat{\mathfrak{W}} \\ &= (u + \textstyle\sum_{x\in\mathfrak{Y}\mathfrak{T}} u^x)\,\sigma\hat{\mathfrak{W}} - (1+t)\hat{\mathfrak{B}}u\sigma\hat{\mathfrak{W}} \\ &= (\hat{\mathfrak{U}} - 1 + u)\,\sigma\hat{\mathfrak{W}} - \hat{\mathfrak{B}}u\sigma\hat{\mathfrak{W}} - \hat{\mathfrak{B}}u^t\sigma\hat{\mathfrak{W}} \\ &\in J(K\mathfrak{U})\,\sigma J(K\mathfrak{W}) + (\hat{\mathfrak{B}}K\mathfrak{U}\sigma)\hat{\mathfrak{W}} = J(K\Gamma)B; \end{aligned}$$

and

$$\sigma\hat{\mathfrak{W}} = \hat{\mathfrak{B}}\sigma\hat{\mathfrak{W}} \in (\hat{\mathfrak{B}}K\mathfrak{U}\sigma)\hat{\mathfrak{W}} \subset J(K\Gamma)B.$$

This shows that $A \subset J(K\Gamma)B$. Thus (3.3) is proved for $i=1$. Next, assume that (3.3) holds for some i. Then

$$\begin{aligned} &J(K\Gamma)^{i+1}B \\ &\quad = J(K\Gamma)(J(K\mathfrak{U})^i\sigma J(K\mathfrak{W}) + J(K\Gamma)^{i-1}A) \\ &\quad = J(K\mathfrak{U})^{i+1}\sigma J(K\mathfrak{W}) + K\mathfrak{T}\hat{\mathfrak{B}}J(K\mathfrak{W})J(K\mathfrak{U})^i\sigma J(K\mathfrak{W}) + J(K\Gamma)^iA \\ &\quad = J(K\mathfrak{U})^{i-1}\sigma J(K\mathfrak{W}) + K\mathfrak{T}J(K\mathfrak{W})(\hat{\mathfrak{B}}J(K\mathfrak{U})^i\sigma)J(K\mathfrak{W}) + J(K\Gamma)^iA \\ &\quad = J(K\mathfrak{U})^{i-1}\sigma J(K\mathfrak{W}) + K\mathfrak{T}(\hat{\mathfrak{B}}J(K\mathfrak{U})^i\sigma)J(K\mathfrak{W})^2 + J(K\Gamma)^iA \\ &\quad = J(K\mathfrak{U})^{i+1}\sigma J(K\mathfrak{W}) + (\hat{\mathfrak{B}}J(K\mathfrak{U})^i\sigma)K\mathfrak{T}\hat{\mathfrak{W}} + J(K\Gamma)^iA \\ &\quad = J(K\mathfrak{U})^{i+1}\sigma J(K\mathfrak{W}) + (\hat{\mathfrak{B}}J(K\mathfrak{U})^i\sigma)\hat{\mathfrak{W}} + J(K\Gamma)^iA. \end{aligned}$$

But $\hat{\mathfrak{B}}J(K\mathfrak{U})^i\sigma\hat{\mathfrak{W}} \subset J(K\mathfrak{U})^i\sigma\hat{\mathfrak{W}} = J(K\Gamma)^iA$. Therefore (3.3) holds for every i.

From (3.3) it follows that

$$\begin{aligned} &L_1(B) \\ &\quad = B/J(K\Gamma)B = K\mathfrak{U}\sigma J(K\mathfrak{W})/(J(K\mathfrak{U})\,\sigma J(K\mathfrak{W}) + A) \\ &\quad \simeq (K\mathfrak{U}\sigma J(K\mathfrak{W})/K\mathfrak{U}\sigma\hat{\mathfrak{W}})/((J(K\mathfrak{U})\,\sigma J(K\mathfrak{W}) + K\mathfrak{U}\sigma\hat{\mathfrak{W}})/K\mathfrak{U}\sigma\hat{\mathfrak{W}}) \\ &\quad \simeq K\mathfrak{U}\sigma\hat{\mathfrak{W}}/J(K\mathfrak{U})\,\sigma\hat{\mathfrak{W}} \simeq K\hat{\Gamma} \simeq L_1(A), \end{aligned}$$

proving (1).

Because of (3.3), we have the following inclusions:

$$J(K\Gamma)^iB \subset J(K\Gamma)^iB + J(K\Gamma)^{i-2}A \subset J(K\Gamma)^{i-1}B, \qquad 2 \leqq i \leqq 7.$$

By replacing, in the proof of [4, (3.5), (3.6)], X, Y, J and ε by A, B,

$J(K\Gamma)$ and σ respectively, we can prove

$$J(K\Gamma)^{i-1}B/(J(K\Gamma)^{i}B+J(K\Gamma)^{i-2}A) \simeq L_i(A),$$
$$(J(K\Gamma)^{i}B+J(K\Gamma)^{i-2}A)/J(K\Gamma)^{i}B \simeq L_{i-1}(A).$$

Hence

$$L_i(B) \simeq L_i(A) \oplus L_{i-1}(A).$$

Thus (2) is proved.

Further by (3.3), we have $J(K\Gamma)^7B = J(K\Gamma)^6A = L_7(A)$, proving (3).

Finally, we determine the Loewy structure of C.

Lemma 3.4. (1) $L_1(C) \simeq L_1(A)$.
(2) $L_i(C) \simeq L_i(A) \oplus L_{i-1}(B)$ *for* $2 \leqq i \leqq 7$.
(3) $L_8(C) = L_7(B)$.
(4) $L_9(C) = L_8(B)$.

Proof. First we shall prove, by induction, that

$$(3.5) \quad J(K\Gamma)^iC = J(K\mathfrak{U})^i\sigma K\mathfrak{W}+J(K\Gamma)^{i-1}B, \qquad \text{where } J(K\Gamma)^0 = K\Gamma.$$

As $J(K\Gamma) = K\Gamma J(K\mathfrak{U})+K\mathfrak{T}\hat{\mathfrak{V}}J(K\mathfrak{W})$, we have

$$\begin{aligned} J(K\Gamma)C &= J(K\mathfrak{U})\,\sigma K\mathfrak{W}+K\mathfrak{T}\hat{\mathfrak{V}}J(K\mathfrak{W})K\mathfrak{U}\,\sigma K\mathfrak{W} \\ &= J(K\mathfrak{U})\,\sigma K\mathfrak{W}+K\mathfrak{T}J(K\mathfrak{W})(\hat{\mathfrak{V}}K\mathfrak{U}\,\sigma)K\mathfrak{W} \\ &= J(K\mathfrak{U})\,\sigma K\mathfrak{W}+K\mathfrak{T}(\hat{\mathfrak{V}}K\mathfrak{U}\,\sigma)J(K\mathfrak{W}) \\ &= J(K\mathfrak{U})\,\sigma K\mathfrak{W}+(\hat{\mathfrak{V}}K\mathfrak{U}\,\sigma)K\mathfrak{T}J(K\mathfrak{W}) \\ &= J(K\mathfrak{U})\,\sigma K\mathfrak{W}+(\hat{\mathfrak{V}}K\mathfrak{U}\,\sigma)J(K\mathfrak{W}). \end{aligned}$$

Hence $J(K\Gamma)C \subset J(K\mathfrak{U})\,\sigma K\mathfrak{W}+B$. But given $u \in \mathfrak{U}$, $u \neq 1$ and $w \in J(K\mathfrak{W})$, we have

$$\begin{aligned} u\sigma w &= (1-\sigma)u\sigma w+\sigma u\sigma w \\ &= (u+\textstyle\sum_{x\in\mathfrak{V}\mathfrak{T}} u^x)\,\sigma w-(1+t)\hat{\mathfrak{V}}u\sigma w \\ &= (\hat{\mathfrak{U}}-1+u)\,\sigma w-\hat{\mathfrak{V}}u\sigma w-\hat{\mathfrak{V}}u^t\sigma w \\ &\in J(K\mathfrak{U})\,\sigma K\mathfrak{W}+(\hat{\mathfrak{V}}K\mathfrak{U}\,\sigma)J(K\mathfrak{W}) = J(K\Gamma)C; \end{aligned}$$

and

$$\sigma w = \hat{\mathfrak{V}}\sigma w \in (\hat{\mathfrak{V}}K\mathfrak{U}\,\sigma)J(K\mathfrak{W}) \subset J(K\Gamma)C.$$

This shows that $B = K\mathfrak{U}\,\sigma J(K\mathfrak{W}) \subset J(K\Gamma)C$. Thus (3.5) is proved for $i = 1$. Next, assume that (3.5) holds for some i. Then we have

$$
\begin{aligned}
&J(K\Gamma)^{i+1}C\\
&\quad = J(K\mathfrak{U})^{i+1}\sigma K\mathfrak{W}+K\mathfrak{T}\hat{\mathfrak{B}}J(K\mathfrak{W})J(K\mathfrak{U})^{i}\sigma K\mathfrak{W}+J(K\Gamma)^{i}B\\
&\quad = J(K\mathfrak{U})^{i+1}\sigma K\mathfrak{W}+K\mathfrak{T}J(K\mathfrak{W})(\hat{\mathfrak{B}}J(K\mathfrak{U})^{i}\sigma)K\mathfrak{W}+J(K\Gamma)^{i}B\\
&\quad = J(K\mathfrak{U})^{i+1}\sigma K\mathfrak{W}+K\mathfrak{T}(\hat{\mathfrak{B}}J(K\mathfrak{U})^{i}\sigma)J(K\mathfrak{W})+J(K\Gamma)^{i}B\\
&\quad = J(K\mathfrak{U})^{i+1}\sigma K\mathfrak{W}+(\hat{\mathfrak{B}}J(K\mathfrak{U})^{i}\sigma)K\mathfrak{T}J(K\mathfrak{W})+J(K\Gamma)^{i}B\\
&\quad = J(K\mathfrak{U})^{i+1}\sigma K\mathfrak{W}+(\hat{\mathfrak{B}}J(K\mathfrak{U})^{i}\sigma)J(K\mathfrak{W})+J(K\Gamma)^{i}B.
\end{aligned}
$$

But

$$\hat{\mathfrak{B}}J(K\mathfrak{U})^{i}\sigma J(K\mathfrak{W}) \subset J(K\mathfrak{U})^{i}\sigma J(K\mathfrak{W}) \subset J(K\Gamma)^{i}B.$$

Hence

$$J(K\Gamma)^{i+1}C = J(K\mathfrak{U})^{i+1}\sigma K\mathfrak{W}+J(K\Gamma)^{i}B.$$

Thus (3.5) holds for every i.

By (3.5), we have

$$
\begin{aligned}
&L_1(C)\\
&\quad = C/J(K\Gamma)C = K\mathfrak{U}\sigma K\mathfrak{W}/(J(K\mathfrak{U})\sigma K\mathfrak{W}+K\mathfrak{U}\sigma J(K\mathfrak{W}))\\
&\quad \simeq (K\mathfrak{U}\sigma K\mathfrak{W}/K\mathfrak{U}\sigma J(K\mathfrak{W}))/((J(K\mathfrak{U})\sigma K\mathfrak{W}+K\mathfrak{U}\sigma J(K\mathfrak{W}))/K\mathfrak{U}\sigma J(K\mathfrak{W}))\\
&\quad \simeq K\mathfrak{U}\sigma\hat{\mathfrak{W}}/J(K\mathfrak{U})\sigma\hat{\mathfrak{W}} \simeq K\hat{\Gamma} \simeq L_1(A),
\end{aligned}
$$

proving (1).

Further, from (3.5), we have the following inclusions:

$$J(K\Gamma)^{i}C \subset J(K\Gamma)^{i}C+J(K\Gamma)^{i-2}B \subset J(K\Gamma)^{i-1}C, \quad 2 \leqq i \leqq 7.$$

In the proof of [4, (3.9), (3.10)], replace X, Y, Z, J and ε by A, B, C, $J(K\Gamma)$ and σ respectively. Then we have

$$
\begin{aligned}
J(K\Gamma)^{i-1}C/(J(K\Gamma)^{i}C+J(K\Gamma)^{i-2}B) &\simeq L_i(A),\\
(J(K\Gamma)^{i}C+J(K\Gamma)^{i-2}B)/J(K\Gamma)^{i}C &\simeq L_{i-1}(B),
\end{aligned}
$$

and so

$$L_i(C) \simeq L_i(A) \oplus L_{i-1}(B).$$

Thus (2) is proved.

By (3.5), $J(K\Gamma)^{7}C = J(K\Gamma)^{6}B$ and $J(K\Gamma)^{8}C = J(K\Gamma)^{7}B$, and so

$$
\begin{aligned}
L_8(C) = J(K\Gamma)^{7}C/J(K\Gamma)^{8}C = J(K\Gamma)^{6}B/J(K\Gamma)^{7}B = L_7(B),\\
L_9(C) = J(K\Gamma)^{7}B = L_8(B).
\end{aligned}
$$

Thus (3) and (4) are proved.

Since $C \simeq \boldsymbol{P}_K$, combining Lemmas 3.1, 3.2, 3.4, we obtain the Loewy series of $\boldsymbol{P}_K$, and so also obtain that of $\boldsymbol{P}_J$. Thus we have proved the following

Proposition 3.6. *$\boldsymbol{P}_K$ and $\boldsymbol{P}_J$ have the Loewy series given in Theorem.*

4. The Loewy structure of $\boldsymbol{P}_{L_3}$ and $\boldsymbol{P}_{N_3}$. In this section, we shall prove the following

Proposition 4.1. *$\boldsymbol{P}_{L_3}$ and $\boldsymbol{P}_{N_3}$ have the Loewy series given in Theorem.*

Proof. By Proposition 2.1, $J(K\mathfrak{J})^3\tilde{P}_{U_7}/J(K\mathfrak{J})^5\tilde{P}_{U_7}$ has factor modules

$$X = \begin{matrix} W_0\ W_3\ W_5\ W_6\ W_7\ W_{11}\ W_{12} \\ U_0\ U_5\ U_9\ U_{10} \end{matrix}, \quad Y = \begin{matrix} W_0\ W_3\ W_5\ W_6\ W_7\ W_{11}\ W_{12} \\ U_2\ U_3 \end{matrix}.$$

Since $[S_2(\tilde{P}_{U_i}), W_0] = 0$ for $i = 0, 5, 9, 10$ (Proposition 2.1), $[S_2(X), W_0] = 0$, and so we have

$$X = W_0 \oplus \begin{matrix} W_3\ W_5\ W_6\ W_7\ W_{11}\ W_{12} \\ U_0\ U_5\ U_9\ U_{10} \end{matrix}.$$

From this we obtain

$$X^\Gamma = \begin{matrix} J \\ J \\ J \end{matrix} \oplus \begin{matrix} N_1\ N_2\ N_2\ N_3\ N_4\ N_4 \\ K\ L_2\ L_3\ L_4 \\ K \\ K \end{matrix}.$$

Since X^Γ is a homomorphic image of $(J(K\mathfrak{J})^3\tilde{P}_{U_7})^\Gamma$, noting that $(J(K\mathfrak{J})^3\tilde{P}_{U_7})^\Gamma \simeq J(K\Gamma)^3\boldsymbol{P}_{L_3}$, we obtain

$$L_5(\boldsymbol{P}_{L_3}) \supset L_2(X^\Gamma) \simeq K \oplus J \oplus L_2 \oplus L_3 \oplus L_4.$$

But by [4, Theorem] and Lemma 1.1(9), $\dim_K L_5(\boldsymbol{P}_{L_3}) = 11$. Hence

$$(4.2) \qquad L_5(\boldsymbol{P}_{L_3}) \simeq K \oplus J \oplus L_2 \oplus L_3 \oplus L_4.$$

Since there exists a $K\mathfrak{J}$-monomorphism

$$J(K\mathfrak{J})^3\tilde{P}_{U_7}/J(K\mathfrak{J})^5\tilde{P}_{U_7} \to X \oplus Y$$

and $[S_2(J(K\mathfrak{J})^3\tilde{P}_{U_7}/J(K\mathfrak{J})^5\tilde{P}_{U_7}), W_0] = 1$, $[S_2(X), W_0] = 0$, it follows that $[S_2(Y), W_0] = 1$, and so the Loewy series of Y^Γ is one of the following:

(a)	(b)	(c)
$J\ N_1\ N_2\ N_2\ N_3\ N_4\ N_4$	$J\ N_1\ N_2\ N_2\ N_3\ N_4\ N_4$	$J\ N_1\ N_2\ N_2\ N_3\ N_4\ N_4$
J	J	$J\ L_1$
J	$J\ L_1$	J
$L_1\ L_1$	L_1	L_1

If (a) were the Loewy series of Y^Γ, Y^Γ would have a submodule

$$\begin{matrix} J \\ L_1\ L_1 \end{matrix}.$$

This module must be a homomorphic image of P_J. But by Proposition 3.6, P_J does not have such a homomorphic image. This contradiction shows that the Loewy series of Y^Γ is (b) or (c). If (c) were the Loewy series of Y^Γ, L_1 would appear as a composition factor of $L_5(P_{L_3})$. This contradicts (4.2), and so (b) is the Loewy series of Y^Γ. Therefore we have

$$L_6(P_{L_3}) \supset L_1,\ L_7(P_{L_3}) \supset L_1.$$

Since $L_3^* \simeq L_4$, Landrock's lemma (see [4, § 1]) together with Proposition 3.6 implies that

$$L_6(P_{L_3}) \supset K \oplus J,\ L_7(P_{L_3}) \supset K.$$

Further by [4, Theorem] and Lemma 1.1(9), $\dim_K L_6(P_{L_3}) = 5$ and $\dim_K L_7(P_{L_3}) = 4$. Therefore we obtain

$$L_6(P_{L_3}) \simeq K \oplus J + L_1,\ L_7(P_{L_3}) \simeq K \oplus L_1. \tag{4.3}$$

Thus by Corollaries 1.3, 2.4, and (4.2), (4.3), $L_8(P_{L_3})$ is determined, and so we conclude that P_{L_3} has the Loewy series given in Theorem. Hence P_{N_3} also has the Loewy series given in Theorem.

5. The Loewy structure of P_{L_1} and P_{N_1}. Here, we shall determine the Loewy series of P_{L_1} and P_{N_1}. To begin with, we shall prove the following

Lemma 5.1. $L_4(P_{L_1}) \simeq K \oplus N_1 \oplus N_2 \oplus N_3 \oplus N_4 \oplus N_4$.

Proof. (1) By Proposition 2.1, $J(K\mathfrak{Y})^2\tilde{P}_{U_1}/J(K\mathfrak{Y})^4\tilde{P}_{U_1}$ has factor modules

$$X = \begin{matrix} U_0\ U_2\ U_8\ U_9\ U_{10}\ U_{11} \\ W_4\ W_8\ W_{11}\ W_{12} \end{matrix},\quad Y = \begin{matrix} U_0\ U_2\ U_8\ U_9\ U_{10}\ U_{11} \\ W_1\ W_2\ W_3 \end{matrix}.$$

Since $[S_2(\tilde{P}_{W_i}), U_0] = 0$ for $i = 4, 8, 11, 12$, we have

$$X = U_0 \oplus \begin{matrix} U_2\ U_8\ U_9\ U_{10}\ U_{11} \\ W_4\ W_8\ W_{11}\ W_{12} \end{matrix},$$

and so

$$X^\Gamma = \begin{matrix} K \\ K \\ K \end{matrix} \oplus \begin{matrix} L_1\ L_3\ L_3\ L_4\ L_4 \\ N_2\ N_3\ N_4\ N_4 \end{matrix}.$$

Since X^Γ is a homomorphic image of $(J(K\mathfrak{J})^2\tilde{P}_{U_1})^\Gamma \simeq J(K\Gamma)^2\boldsymbol{P}_{L_1}$, the above implies that

$$(5.2) \qquad L_4(\boldsymbol{P}_{L_1}) \supsetneq L_2(X^\Gamma) \simeq K \oplus N_2 \oplus N_3 \oplus N_4 \oplus N_4.$$

There exists a $K\mathfrak{J}$-monomorphism

$$J(K\mathfrak{J})^2\tilde{P}_{U_1}/J(K\mathfrak{J})^4\tilde{P}_{U_1} \rightarrow X \oplus Y.$$

Hence, $[S_2(Y), U_0] = 1$ because $[S_2(X), U_0] = 0$. Therefore the Loewy series of Y^Γ is one of the following:

(a)	(b)	(c)
$K\ L_1\ L_3\ L_3\ L_4\ L_4$	$K\ L_1\ L_3\ L_3\ L_4\ L_4$	$K\ L_1\ L_3\ L_3\ L_4\ L_4$
K	K	$K\ N_1$
K	$K\ N_1$	K
$N_1\ N_1\ N_1$	$N_1\ N_1$	$N_1\ N_1$

(d)	(e)	(f)
$K\ L_1\ L_3\ L_3\ L_4\ L_4$	$K\ L_1\ L_3\ L_3\ L_4\ L_4$	$K\ L_1\ L_3\ L_3\ L_4\ L_4$
K	$K\ N_1$	$K\ N_1\ N_1$
$K\ N_1\ N_1$	$K\ N_1$	K
N_1	N_1	N_1

But in view of Proposition 3.6, we see that the Loewy series of Y^Γ is (e) or (f). If (f) were the Loewy series of Y^Γ, N_1 would appear at least twice as a composition factor of $L_4(\boldsymbol{P}_{L_1})$. This together with (5.2) implies that $\dim_K L_4(\boldsymbol{P}_{L_1}) \geqq 19$. But by [4, Theorem] and Lemma 1.1(9), $\dim_K L_4(\boldsymbol{P}_{L_1}) = 16$. This contradiction shows that the Loewy series of Y^Γ is (e). Hence $L_4(\boldsymbol{P}_{L_1}) \supsetneq N_1$, and

$$L_4(\boldsymbol{P}_{L_1}) \simeq K \oplus N_1 \oplus N_2 \oplus N_3 \oplus N_4 \oplus N_4.$$

Thus Lemma 5.1 is proved.

In the proof of the preceding lemma, we have proved that (e) is the Loewy series of Y^{Γ}. Hence it follows that

$$L_5(\boldsymbol{P}_{L_1}) \supsetneq K \oplus N_1,\ L_6(\boldsymbol{P}_{L_1}) \supsetneq N_1. \tag{5.3}$$

Lemma 5.4. $L_6(\boldsymbol{P}_{L_1}) \simeq J \oplus L_2 \oplus N_1$.

Proof. The preceding lemma together with Corollary 2.4 implies that $[\boldsymbol{P}_{L_1}/J(K\Gamma)^4\boldsymbol{P}_{L_1}, N_2] = 3$. Hence by [4, Theorem], Lemma 1.1(6), (8) and Corollary 1.3, we have

$$[J(K\Gamma)^4\boldsymbol{P}_{L_1}/J(K\Gamma)^7\boldsymbol{P}_{L_1}, L_2] = 3.$$

Therefore noting that

$$L_i(\boldsymbol{P}_{L_1}) \supsetneq L_i(P_{M_1})^{\Gamma},\ [L_i(P_{M_1}), M_2] = 1, \qquad i = 5, 6, 7,$$

we obtain

$$[L_i(\boldsymbol{P}_{L_1}), L_2] = 1 \qquad \text{for } i = 5, 6, 7. \tag{5.5}$$

From (5.3) and (5.5), it follows that

$$L_6(\boldsymbol{P}_{L_1}) \supsetneq L_2 \oplus N_1.$$

Further, Landrock's lemma together with Proposition 3.6 implies that $L_6(\boldsymbol{P}_{L_1}) \supsetneq J$. On the other hand, by [4, Theorem] and Lemma 1.1(9), we have $\dim_K L_6(\boldsymbol{P}_{L_1}) = 7$. Hence the lemma is proved.

Lemma 5.6. $L_8(\boldsymbol{P}_{L_1}) \simeq J \oplus N_3 \oplus N_4$.

Proof. Since $L_8(\boldsymbol{P}_{L_1}) \subsetneq S_2(\boldsymbol{P}_{L_1})$ and, by [3, Chap. I, Lemma 8.4], $S_2(\boldsymbol{P}_{L_1}) \simeq L_2(\boldsymbol{P}_{L_2})^*$, from Corollary 2.4, we have

$$L_8(\boldsymbol{P}_{L_1}) \subsetneq J \oplus N_3 \oplus N_4.$$

But $\dim_K L_8(\boldsymbol{P}_{L_1}) = 7$. Hence the result follows:

Lemma 5.7. $L_7(\boldsymbol{P}_{L_1}) \simeq J \oplus L_2 \oplus L_3$.

Proof. We have already shown that

$$[L_2(\boldsymbol{P}_{L_1}), N_3] = [L_4(\boldsymbol{P}_{L_1}), N_3] = [L_8(\boldsymbol{P}_{L_1}), N_3] = 1,$$
$$[L_3(\boldsymbol{P}_{L_1}), L_3] = 2.$$

Hence by Corollary 1.3 and Lemma 1.1(8), L_3 appears as a composition factor of $L_7(\boldsymbol{P}_{L_1})$. This together with (5.5) implies that

$$L_7(\boldsymbol{P}_{L_1}) \supset L_2 \oplus L_3.$$

Further noting that $\dim_K L_7(\boldsymbol{P}_{L_1}) = 7$ and $[L_7(\boldsymbol{P}_{L_1}), K] = 0$, we obtain

$$L_7(\boldsymbol{P}_{L_1}) \simeq J \oplus L_2 \oplus L_3.$$

Thus the lemma is proved.

By Corollaries 1.3, 2.4, and the above lemmas, $L_5(\boldsymbol{P}_{L_1})$ is determined, and so the Loewy series of $\boldsymbol{P}_{L_1}$ is established. Hence we also obtain the Loewy series of $\boldsymbol{P}_{N_1}$. Thus we have proved the following

Proposition 5.8. *$\boldsymbol{P}_{L_1}$ and $\boldsymbol{P}_{N_1}$ have the Loewy series given in Theorem.*

6. The Loewy structure of $\boldsymbol{P}_{L_4}$ and $\boldsymbol{P}_{N_4}$. In this section, we shall determine the Loewy series of $\boldsymbol{P}_{L_4}$ and $\boldsymbol{P}_{N_4}$.

Lemma 6.1. (1) $L_4(\boldsymbol{P}_{L_4}) \simeq K \oplus J \oplus N_2 \oplus N_3 \oplus N_3 \oplus N_4$.
(2) $L_5(\boldsymbol{P}_{L_4}) \simeq K \oplus J \oplus L_4 \oplus N_1$.

Proof. (1) By Proposition 2.1, $J(K\mathfrak{J})^2\tilde{P}_{U_{10}}/J(K\mathfrak{J})^4\tilde{P}_{U_{10}}$ has a factor module

$$X = \frac{U_0\ U_2\ U_5\ U_6\ U_7\ U_{12}}{W_0\ W_6\ W_8\ W_9\ W_{10}}.$$

Since $[S_2(\tilde{P}_{W_i}), U_0] = 0$ $(i = 0, 6, 8, 9, 10)$, we have

$$X = U_0 \oplus \frac{U_2\ U_5\ U_6\ U_7\ U_{12}}{W_0\ W_6\ W_8\ W_9\ W_{10}},$$

and so

$$X^\Gamma = \begin{matrix} K \\ K \\ K \end{matrix} \oplus \begin{matrix} L_1\ L_2\ L_2\ L_3\ L_4 \\ J\ N_2\ N_3\ N_3\ N_4 \\ J \\ J \end{matrix}.$$

Hence, noting that X^Γ is a homomorphic image of $(J(K\mathfrak{J})^2\tilde{P}_{U_{10}})^\Gamma \simeq J(K\Gamma)^2\boldsymbol{P}_{L_4}$, we obtain

$$L_4(\boldsymbol{P}_{L_4}) \supset L_2(X^\Gamma) \simeq K \oplus J \oplus N_2 \oplus N_3 \oplus N_3 \oplus N_4.$$

But $\dim_K L_4(\boldsymbol{P}_{L_4}) = 14$. Hence $L_4(\boldsymbol{P}_{L_4}) \simeq L_2(X^\Gamma)$, proving (1).

(2) $J(K\mathfrak{J})^2\tilde{P}_{U_{10}}/J(K\mathfrak{J})^4\tilde{P}_{U_{10}}$ has another factor module

$$Y = \begin{matrix} U_0\ U_2\ U_5\ U_6\ U_7\ U_{12} \\ W_2\ W_3 \end{matrix}.$$

Since there exists a $K\mathfrak{J}$-monomorphism

$$J(K\mathfrak{J})^2\tilde{P}_{U_{10}}/J(K\mathfrak{J})^4\tilde{P}_{U_{10}} \to X \oplus Y,$$

and $[S_2(X), U_0] = 0$, we have $[S_2(Y), U_0] = 1$. Hence the Loewy series of Y^Γ is one of the following:

(a)	(b)	(c)
$K\ L_1\ L_2\ L_2\ L_3\ L_4$	$K\ L_1\ L_2\ L_2\ L_3\ L_4$	$K\ L_1\ L_2\ L_2\ L_3\ L_4$
K	K	$K\ N_1$
K	$K\ N_1$	K
$N_1\ N_1$	N_1	N_1

In view of Proposition 3.6, we see that (a) is not the Loewy series of Y^Γ. If (c) were the Loewy series of Y^Γ, N_1 would appear as a composition factor of $L_4(\boldsymbol{P}_{L_4})$. This contradicts (1). Hence (b) is the Loewy series of Y^Γ. Thus we obtain

$$L_5(\boldsymbol{P}_{L_4}) \supsetneq K \oplus N_1.$$

Further Landrock's lemma together with Propositions 3.6, 4.1 implies that

$$L_5(\boldsymbol{P}_{L_4}) \supsetneq J \oplus L_4.$$

Hence we obtain (2) because $\dim_K L_5(\boldsymbol{P}_{L_4}) = 8$.

Lemma 6.2. $L_8(\boldsymbol{P}_{L_4}) \simeq N_2 \oplus N_3 \oplus N_4$.

Proof. We have

$$L_8(\boldsymbol{P}_{L_4}) \subsetneq S_2(\boldsymbol{P}_{L_4}) \simeq L_2(\boldsymbol{P}_{L_3})^* \simeq N_2 \oplus N_3 \oplus N_4.$$

Hence the result follows from the fact that $\dim_K L_8(\boldsymbol{P}_{L_4}) = 9$.

Lemma 6.3. (1) $L_6(\boldsymbol{P}_{L_4}) \simeq J \oplus L_1 \oplus L_3 \oplus N_1$.
(2) $L_7(\boldsymbol{P}_{L_4}) \simeq L_1 \oplus L_2 \oplus L_3$.

Proof. From Corollaries 1.3, 2.4, and the above lemmas, it follows that

$$[J(K\Gamma)^5\boldsymbol{P}_{L_4}/J(K\Gamma)^7\boldsymbol{P}_{L_4}, L_3] = 2.$$

But by [4, Theorem] and Lemma 1.1(6), (8),

$$[L_6(P_{L_4}), L_3] \leqq 1, \ [L_7(P_{L_4}), L_3] \leqq 1.$$

Hence we obtain

$$[L_6(P_{L_4}), L_3] = [L_7(P_{L_4}), L_3] = 1.$$

Further, from Landrock's lemma and Propositions 3.6, 5.8 we obtain

$$L_6(P_{L_4}) \supset J, \ L_7(P_{L_4}) \supset L_2.$$

On the other hand, in the proof of Lemma 6.1(2), we have proved that the Loewy series of of Y^{Γ} is (b), and so

$$L_6(P_{L_4}) \supset N_1.$$

Thus we obtain

$$L_6(P_{L_4}) \supset J \oplus L_3 \oplus N_1, \ L_7(P_{L_4}) \supset L_2 \oplus L_3.$$

Further, by Corollaries 1.3, 2.4 and Lemmas 6.1, 6.2, L_1 appears twice as a composition factor of $J(K\Gamma)^5 P_{L_4}/J(K\Gamma)^7 P_{L_4}$. Hence from $\dim_K L_6(P_{L_4}) = 10$ and $\dim_K L_7(P_{L_4}) = 9$, it follows that

$$L_6(P_{L_4}) \simeq J \oplus L_1 \oplus L_3 \oplus N_1, \ L_7(P_{L_4}) \simeq L_1 \oplus L_2 \oplus L_3,$$

and the lemma is proved.

Thus the Loewy series of P_{L_4} is established, and so the Loewy series of P_{N_4} is also obtained.

Proposition 6.4. *P_{L_4} and P_{N_4} have the Loewy series given in Theorem.*

7. The Loewy structure of P_{L_2} and P_{N_2}. In this section, we shall establish the Loewy series of P_{L_2} and P_{N_2}. At first, we note that the next result follows from Landrock's lemma and Propositons 3.6, 4.1, 5.8, 6.4.

Lemma 7.1. (1) $L_3(P_{L_2}) \supset J \oplus L_2 \oplus L_3 \oplus L_4$.
(2) $L_4(P_{L_2}) \supset J \oplus N_2 \oplus N_4$.
(3) $L_5(P_{L_2}) \supset K \oplus L_2 \oplus N_2 \oplus N_3$.
(4) $L_6(P_{L_2}) \supset K \oplus L_3 \oplus L_4 \oplus N_2 \oplus N_3$.
(5) $L_7(P_{L_2}) \supset K \oplus L_3 \oplus L_4$.
(6) $L_8(P_{L_2}) \supset N_4$.

Lemma 7.1(4) together with the fact that $\dim_K L_6(P_{L_2}) = 13$ implies the following

Lemma 7.2. $L_6(P_{L_2}) \simeq K \oplus L_3 \oplus L_4 \oplus N_2 \oplus N_3$.

Now we shall prove the following

Lemma 7.3. (1) $L_3(P_{L_2}) \simeq J \oplus L_1 \oplus L_2 \oplus L_3 \oplus L_4$.
(2) $L_5(P_{L_2}) \simeq K \oplus L_1 \oplus L_2 \oplus N_2 \oplus N_3$.

Proof. By Proposition 2.1, $J(K\mathfrak{J})\tilde{P}_{U_4}/J(K\mathfrak{J})^3\tilde{P}_{U_4}$ has factor modules

$$X = \begin{matrix} W_0\ W_7\ W_{12} \\ U_6\ U_9\ U_{10} \end{matrix}, \quad Y = \begin{matrix} W_0\ W_7\ W_{12} \\ U_1\ U_2\ U_3 \end{matrix}.$$

Since $[S_2(\tilde{P}_{U_i}), W_0] = 0$ for $i = 6, 9, 10$, we have $[S_2(X), W_0] = 0$. Therefore from the existence of a $K\mathfrak{J}$-monomorphism

$$J(K\mathfrak{J})\tilde{P}_{U_4}/J(K\mathfrak{J})^3\tilde{P}_{U_4} \to X \oplus Y,$$

we obtain $[S_2(Y), W_0] = 1$, and so the Loewy series of Y^Γ is one of the following

(a)	(b)	(c)	(d)	(e)	(f)
$J\ N_3\ N_4$	$J\ N_3\ N_4$	$J\ N_3\ N_4$	$J\ N_3\ N_4$	$J\ N_3\ N_4$	$J\ N_3\ N_4$
J	J	$J\ L_1$	J	$J\ L_1$	$J\ L_1\ L_1$
J	$J\ L_1$	J	$J\ L_1\ L_1$	$J\ L_1$	J
$L_1\ L_1\ L_1$	$L_1\ L_1$	$L_1\ L_1$	L_1	L_1	L_1

But in view of Proposition 3.6, we can see that the Loewy series of Y^Γ is (e) or (f). Since Y^Γ is a homomorphic image of $(J(K\mathfrak{J})\tilde{P}_{U_4})^\Gamma \simeq J(K\Gamma)P_{L_2}$, we have

$$L_3(P_{L_2}) \supset L_2(Y^\Gamma).$$

From this we may show that (e) is the Loewy series of Y^Γ. For otherwise L_1 would appear at least twice as a composition factor of $L_3(P_{L_2})$. Then by Lemma 7.1(1), we have

$$L_3(P_{L_2}) \supset J \oplus L_1 \oplus L_1 \oplus L_2 \oplus L_3 \oplus L_4.$$

But this contradicts the fact that $\dim_K L_3(P_{L_2}) = 13$, and this shows that (e) is the Loewy series of Y^Γ. Therefore we obtain

$$(7.4) \qquad L_3(P_{L_2}) \supset L_1, \quad L_5(P_{L_2}) \supset L_1.$$

Since $\dim_K L_3(P_{L_2}) = \dim_K L_5(P_{L_2}) = 13$, the result follows from Lemma 7.1(1), (3) and (7.4).

Lemma 7.5. $L_8(P_{L_2}) \simeq N_1 \oplus N_4$.

Proof. We have

$$L_8(P_{L_2}) \subsetneq S_2(P_{L_2}) \simeq L_2(P_{L_1})^* \simeq N_1 \oplus N_1 \oplus N_4.$$

Hence the result follows Lemma 7.1(6) because $\dim_K L_8(P_{L_2}) = 6$.

Lemma 7.6. (1) $L_4(P_{L_2}) \simeq J \oplus L_1 \oplus N_1 \oplus N_2 \oplus N_4$.
(2) $L_7(P_{L_2}) \simeq K \oplus L_3 \oplus L_4 \oplus N_1$.

Proof. In the proof of Lemma 7.3, we have proved that (e) is the Loewy series of Y^{Γ}, and so L_1 appears as a composition factor of $L_4(P_{L_2})$. Hence the result follows at once from Corollaries 1.3, 2.4, Lemma 7.1(2), (5) and Lemmas 7.2, 7.3, 7.5 because $\dim_K L_4(P_{L_2}) = 13$ and $\dim_K L_7(P_{L_2}) = 10$.

Thus we have established the Loewy series of P_{L_2}, and so the Loewy series of P_{N_2} is also obtained.

Proposition 7.7. *P_{L_2} and P_{N_2} have the Loewy series given in Theorem.*

Thus we complete the proof of Theorem.

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(*Received September 3, 1986*)