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ON THE LOEWY STRUCTURE OF THE PROJECTIVE INDECOMPOSABLE MODULES FOR A 3-SOLVABLE GROUP I

Dedicated to Professor Hisao Tominaga on his 60th birthday

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Let p be a prime number and r a positive integer. We set $q = p^r$ and denote by F the finite field $GF(q^p)$ with q^p elements. Let λ be a generator of the multiplicative group of F. Then the set

$$\mathfrak{G}_{p,r} = \{F \ni x \to ax^{q^n}+b \mid a \in \langle \lambda^{q-1} \rangle,\ b \in F,\ n = 0, 1, \ldots, p-1\}$$

of permutations on F forms a p-solvable group of p-length 2, and its order is $p^{rp+1}l$, where $l = (q^p-1)/(q-1)$. The purpose of this paper is to determine the Loewy series of the projective indecomposable modules for a 3-solvable group $\mathfrak{G}_{3,1}$ in characteristic 3.

A number of authors have established the Loewy series of the projective indecomposable modules for various groups. However, we can find very few examples of p-solvable groups of p-length > 1 for which the Loewy series of the projective indecomposable modules in characteristic p are determined. For example, the Loewy series of the projective indecomposable modules for S_4, the symmetric group on four letters, in characteristic 2 are given in [1, Examples 15.10], and the Loewy series of the projective indecomposable modules for the group which is the semidirect product of the elementary abelian group of order 3^2 by $SL(2, 3)$ in characteristic 3, are given in [2]. These are all the examples that we know. In the meantime, D. A. R. Wallace [8] proved that if $\mathfrak{G}$ is a p-solvable group of order $p^a m$, $(p, m) = 1$, and K is a field of characteristic p, then the Loewy length of the group algebra $K\mathfrak{G}$, namely, the nilpotecy index $t(\mathfrak{G})$ of the radical of $K\mathfrak{G}$ is greater than or equal to $a(p-1)+1$. In regard to this fact, we see that if a Sylow p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ is elementary abelian then $t(\mathfrak{G}) = a(p-1)+1$ ([7]). But, in [6], K. Motose proved that $t(\mathfrak{G}_{p,r}) = a(p-1)+1$, where $a = rp+1$. This shows that there exists a p-solvable group $\mathfrak{G}$ with $\mathfrak{P}$ a non abelian group but with $t(\mathfrak{G}) = a(p-1)+1$. Up to the present, the structure of a group $\mathfrak{G}$ with $t(\mathfrak{G}) = a(p-1)+1$ is not quite determined yet. So we wish to find general information about the structure of the group algebra of such a group. From these points of view, it seems meaningful to establish the Loewy series of the projective indecomposable modules for $\mathfrak{G}_{3,1}$ in characteristic 3. Note: Since $\mathfrak{G}_{2,1} \simeq S_4$, the Loewy series of the projective indecomposable modules for $\mathfrak{G}_{2,1}$ in characteristic 2 are known as stated above.

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1. Result and notation. Throughout this paper, we set $\mathfrak{G} = \mathfrak{G}_{3,1}$ and let K be an algebraically closed field of characteristic 3. Then λ is a primitive 26th root of 1, and so we may assume that $\lambda^3 = \lambda^2 - 1$ (see [5, Chap. 10, Table C]). Now we choose the elements a, b, c, v, s of $\mathfrak{G}$ defined by

$$\begin{aligned} &a:\ x \to x+1,\ b:\ x \to x+\lambda,\ c:\ x \to x+\lambda^2; \\ &v:\ x \to \lambda^2 x; \\ &s:\ x \to x^3; \qquad\qquad x \in F \text{ (cf. [6])}. \end{aligned}$$

Then $\mathfrak{G}$ is generated by these elements. We set

$$\mathfrak{U} = \langle a, b, c\rangle,\ \mathfrak{V} = \langle v\rangle,\ \mathfrak{W} = \langle s\rangle.$$

Then $\mathfrak{U}$ is an elementary abelian group of order 3^3, $\mathfrak{V}$ is a cyclic group of order 13 and $\mathfrak{W}$ is a cyclic group of order 3. Further $\mathfrak{V}$ acts on $\mathfrak{U}$, and $\mathfrak{W}$ acts on $\mathfrak{U}$ and $\mathfrak{V}$ in the following rule:

$$\begin{aligned} &vav^{-1} = c,\ vbv^{-1} = a^2c,\ vcv^{-1} = a^2b^2c; \\ &sas^{-1} = a,\ sbs^{-1} = a^2c,\ scs^{-1} = b^2c^2; \\ &svs^{-1} = v^3. \end{aligned}$$

We now let ζ be a primitive 13th root of 1 in K. Since

$$\begin{aligned} X^{13}-1 = &(X-1)(X^3-X^2-X-1)(X^3+X^2+X-1) \\ &(X^3-X-1)(X^3+X^2-1) \end{aligned}$$

is the factorization of the polynomial $X^{13}-1$ into the irreducible polynomials over $GF(3)$ ([5, Chap. 10, Table C]), we may assume that ζ is a root of the polynomial X^3-X^2-X-1. Then it is easy to see that $\{\zeta, \zeta^3, \zeta^9\}$, $\{\zeta^{12}, \zeta^{10}, \zeta^4\}$, $\{\zeta^2, \zeta^6, \zeta^5\}$ and $\{\zeta^{11}, \zeta^7, \zeta^8\}$ are the sets of roots of the polynomials X^3-X^2-X-1, X^3+X^2+X-1, X^3-X-1 and X^3+X^2-1 respectively. We set $\mathfrak{H} = \langle \mathfrak{U}, \mathfrak{V}\rangle = \mathfrak{U}\mathfrak{V}$. Since $\mathfrak{H}/\mathfrak{U} \simeq \mathfrak{V}$, there exist 13 non isomorphic simple (left) $K\mathfrak{H}$-modules, say, $V_0, V_1, \dots, V_{12}$, and they afford inequivalent one-dimensional representations of $\mathfrak{V}$, which we may identify with their characters, say, $\xi_0, \xi_1, \dots, \xi_{12}$. We may, and shall assume that these characters satisfy the following:

$$\begin{array}{llll} \xi_0(\nu) = 1, & \xi_1(\nu) = \zeta, & \xi_2(\nu) = \zeta^3, & \xi_3(\nu) = \zeta^9, \\ \xi_4(\nu) = \zeta^{12}, & \xi_5(\nu) = \zeta^{10}, & \xi_6(\nu) = \zeta^4, & \xi_7(\nu) = \zeta^2, \\ \xi_8(\nu) = \zeta^6, & \xi_9(\nu) = \zeta^5, & \xi_{10}(\nu) = \zeta^{11}, & \xi_{11}(\nu) = \zeta^7, \\ \xi_{12}(\nu) = \zeta^8. & & & \end{array}$$

Then V_2, V_3 are conjugate under $\mathfrak{G}$ to V_1; V_5, V_6 are conjugate under $\mathfrak{G}$ to V_4; V_8, V_9 are conjugate under $\mathfrak{G}$ to V_7 and V_{11}, V_{12} are conjugate under $\mathfrak{G}$ to V_{10}. Further V_4, V_5, V_6, V_{10}, V_{11}, V_{12} are isomorphic to the dual modules of V_1, V_2, V_3, V_7, V_8, V_9 respectively.

We now let I be the trivial simple $K\mathfrak{G}$-module, and we set $M_1 = K\mathfrak{G} \otimes_{K\mathfrak{H}} V_1$, $M_2 = K\mathfrak{G} \otimes_{K\mathfrak{H}} V_4$, $M_3 = K\mathfrak{G} \otimes_{K\mathfrak{H}} V_7$ and $M_4 = K\mathfrak{G} \otimes_{K\mathfrak{H}} V_{10}$. Then I, M_1, M_2, M_3, M_4 are all types of simple $K\mathfrak{G}$-modules. Further, M_2 and M_4 are isomorphic to the dual modules of M_1 and M_3 respectively. Given a simple $K\mathfrak{G}$-module M, we denote by P_M the projective indecomposable $K\mathfrak{G}$-module for which $P_M/J(K\mathfrak{G})P_M \simeq M$, where $J(K\mathfrak{G})$ is the radical of $K\mathfrak{G}$.

We are now in a position to state our result.

Theorem. *The Loewy series of the projective indecomposable $K\mathfrak{G}$-modules are as follows:*

$$P_I = \begin{array}{c} I \\ I\ M_1 \\ I\ M_1\ M_2\ M_3 \\ I\ M_1\ M_2\ M_3\ M_3\ M_4 \\ I\ M_1\ M_2\ M_3\ M_3\ M_4\ M_4 \\ I\ M_1\ M_2\ M_3\ M_4\ M_4 \\ I\ M_1\ M_2\ M_4 \\ I\ M_2 \\ I \end{array} \qquad P_{M_1} = \begin{array}{c} M_1 \\ M_2\ M_2\ M_3 \\ I\ M_1\ M_3\ M_3\ M_4\ M_4 \\ I\ M_1\ M_2\ M_3\ M_4\ M_4 \\ I\ M_1\ M_1\ M_2\ M_4 \\ I\ M_1\ M_2 \\ I\ M_2\ M_3 \\ I\ M_3\ M_4 \\ M_1 \end{array}$$

$$P_{M_2} = \begin{array}{c} M_2 \\ I\ M_3\ M_4 \\ I\ M_1\ M_2\ M_3\ M_4 \\ I\ M_1\ M_1\ M_2\ M_4 \\ I\ M_1\ M_2\ M_2\ M_3 \\ I\ M_2\ M_3\ M_3\ M_4 \\ I\ M_1\ M_3\ M_4 \\ M_1\ M_4 \\ M_2 \end{array} \qquad P_{M_3} = \begin{array}{c} M_3 \\ M_1\ M_3\ M_4 \\ M_1\ M_2\ M_2\ M_3\ M_4\ M_4 \\ I\ M_1\ M_2\ M_2\ M_3\ M_4\ M_4 \\ I\ I\ M_2\ M_3\ M_4 \\ I\ I\ M_1 \\ I\ M_1 \\ M_1\ M_2\ M_3 \\ M_3 \end{array}$$

$$P_{M_4} = \begin{matrix} M_4 \\ M_1\ M_2\ M_4 \\ I\ M_1\ M_2\ M_2\ M_3\ M_4 \\ I\ I\ M_2\ M_3\ M_3\ M_4 \\ I\ I\ M_1\ M_4 \\ I\ M_1\ M_1\ M_3 \\ M_1\ M_2\ M_3 \\ M_2\ M_3\ M_4 \\ M_4 \end{matrix}$$

Throughout this paper, all modules are finitely generated left modules. The (Jacobson) radical of $K\mathfrak{G}$ is denoted by $J(K\mathfrak{G})$. Let M be a $K\mathfrak{G}$-module. We denote by $L_i(M)$ the ith Loewy layer of M, that is

$$L_i(M) = J(K\mathfrak{G})^{i-1}M/J(K\mathfrak{G})^iM.$$

Let $\mathrm{soc}(M)$ denote the socle of M. We set $\mathrm{soc}_0(M) = 0$, $\mathrm{soc}_1(M) = \mathrm{soc}(M)$ and $\mathrm{soc}_i(M)/\mathrm{soc}_{i-1}(M) = \mathrm{soc}(M/\mathrm{soc}_{i-1}(M))$. Then the ith socle factor of M is denoted by $S_i(M)$, that is

$$S_i(M) = \mathrm{soc}_i(M)/\mathrm{soc}_{i-1}(M).$$

If the Loewy length of M is equal to m and

$$L_i(M) \simeq X_{i1} \oplus \dots \oplus X_{ir_i},$$

where X_{ij} is a simple module for all j, then as in [4], the Loewy series of M is denoted by

$$M = \begin{matrix} X_{11}\dots\dots X_{1r_1} \\ X_{21}\dots\dots\dots\dots X_{2r_2} \\ \vdots \\ X_{m1}\dots X_{mr_m}. \end{matrix}$$

Further the dual module of M is denoted by M^*. Given a simple $K\mathfrak{G}$-module X, we denote by $[M, X]$ the multiplicity of X as composition factor of M. If $\mathfrak{T}$ is a subgroup of $\mathfrak{G}$ then $M|_{\mathfrak{T}}$ is a $K\mathfrak{T}$-module obtained from M by restricting the domain of operators to $K\mathfrak{T}$. Given a $K\mathfrak{T}$-module N, we denote by $N^{\mathfrak{G}}$ the induced module $K\mathfrak{G} \otimes_{K\mathfrak{T}} N$. If A and B are K-subspaces of $K\mathfrak{G}$, then AB is the set of all finite sums of products of elements of A by those in B:

$$AB = \{\sum a_ib_i \mid a_i \in A,\ b_i \in B\}.$$

If Q is a subset of $\mathfrak{G}$, define $\hat{Q}$ in $K\mathfrak{G}$ by $\hat{Q} = \sum_{x \in Q} x$. Given $\alpha \in K\mathfrak{G}$ and

$g \in \mathfrak{G}$, we write a^g for gag^{-1}. Let V be a finite dimensional vector space over K. Then we denote by dim V, the K-dimension of V. Further, if V has a K-basis $\{v_1, \ldots, v_n\}$, then we write $V = \langle v_1, \ldots, v_n \rangle$.

In § 2, we determine the Loewy structure of the projective indecomposable $K\mathfrak{H}$-modules ($\mathfrak{H} = \langle \mathfrak{U}, \mathfrak{V} \rangle$), and in § 3, we determine the Loewy structure of P_I. P_{M_i} $(1 \leqq i \leqq 4)$ is isomorphic to the induced module of a certain projective indecomposable $K\mathfrak{H}$-module. Accordingly, in § 4, by making use of the Loewy structure of the projective indecomposable $K\mathfrak{H}$-modules obtained in § 2, we determine a part of the Loewy layers of P_{M_i}. Further, if e_i is a primitive idempotent in $K\mathfrak{G}$ corresponding to P_{M_i}, then as is well known, it holds that

$$\dim e_j J(K\mathfrak{G})^k e_i = [J(K\mathfrak{G})^k P_{M_i}, M_j].$$

So, in § 5, we determine the structure of powers of $J(K\mathfrak{G})$, and in §§ 6—9, calculating dim $e_j J(K\mathfrak{G})^k e_i$, we completely determine the Loewy structure of P_{M_i}.

Throughout this paper, we frequently use the next result, due to Landrock, and we cite this as Landrock's lemma.

Lemma ([4, Chap. I, Lemma 9.10]). *Let $\mathfrak{T}$ be an arbitrary finite group and let k be a splitting field for $\mathfrak{T}$. If M and N are simple $k\mathfrak{T}$-modules then for arbitrary s,*

$$[L_S(P_M), N] = [L_S(P_{N^*}), M^*],$$

where P_M, P_{N^*} are the projective indecomposable $k\mathfrak{T}$-modules for which $P_M/J(k\mathfrak{T})P_M \simeq M$, $P_{N^*}/J(k\mathfrak{T})P_{N^*} \simeq N^*$.

2. The Loewy structure of the projective indecomposable $K\mathfrak{H}$-modules. Since $\mathfrak{H}/\mathfrak{U} \simeq \mathfrak{V}$, each V_i may be regarded as a simple $K\mathfrak{V}$-module. Now we set $\varepsilon = \varepsilon_0 = \hat{\mathfrak{V}}$ and denote by ε_i the primitive idempotent in $K\mathfrak{V}$ corresponding to V_i for each i, $1 \leqq i \leqq 12$. Then $K\mathfrak{U}\varepsilon_i$ $(0 \leqq i \leqq 12)$ represent all types of the projective indecomposable $K\mathfrak{H}$-modules. To begin with we shall prove the following lemma which is useful for the caluculation in §§ 6—9.

Lemma 2.1. 1, *a and a^2 are the representatives of the conjugate classes of 3-elements in $\mathfrak{H}$, and for $n = 1, 2$,*

$$\varepsilon_i (a^n)^{v^k} \varepsilon_j = (\xi_i(v) \xi_j(v)^{-1})^k \varepsilon_i a^n \varepsilon_j, \quad 0 \leqq i, j, k \leqq 12.$$

Proof. The former half of the lemma is clear. Since

$$\varepsilon_i = \sum_{l=0}^{12} \xi_i(v)^{-l} v^l,$$

we have

$$\begin{aligned}\varepsilon_i(a^n)^{vk}\varepsilon_j &= \left(\sum_{l=0}^{12} \xi_i(v)^{-l} v^l\right)(a^n)^{vk}\varepsilon_j \\ &= \sum_{l=0}^{12} \xi_i(v)^{-l}(a^n)^{vk+l} v^l \varepsilon_j \\ &= \sum_{l=0}^{12} \xi_i(v)^{-l}(a^n)^{vk+l}\xi_j(v)^l \varepsilon_j \\ &= \sum_{l=0}^{12} (\xi_i(v)^{-1}\xi_j(v))^l (a^n)^{vk+l}\varepsilon_j.\end{aligned}$$

Setting $k+l=h$, we obtain

$$\begin{aligned}\varepsilon_i(a^n)^{vk}\varepsilon_j &= \sum_{h=0}^{12} (\xi_i(v)^{-1}\xi_j(v))^{h-k}(a^n)^{vh}\varepsilon_j \\ &= (\xi_i(v)\xi_j(v)^{-1})^k\left(\sum_{h=0}^{12} (\xi_i(v)^{-1}\xi_j(v))^h (a^n)^{vh}\varepsilon_j\right).\end{aligned}$$

Thus in particular,

$$\varepsilon_i a^n \varepsilon_j = \sum_{h=0}^{12} (\xi_i(v)^{-1}\xi_j(v))^h (a^n)^{vh}\varepsilon_j.$$

Hence for every k, we have

$$\varepsilon_i(a^n)^{vk}\varepsilon_j = (\xi_i(v)\xi_j(v)^{-1})^k \varepsilon_i a^n \varepsilon_j,$$

as required.

By the preceding lemma, we see that if $u\ (\in \mathfrak{U})$ is conjugate to a^n, then $\varepsilon_i u \varepsilon_j$ is a scalar multiple of $\varepsilon_i a^n \varepsilon_j$. Therefore, we see that

$$\varepsilon_i K\mathfrak{U}\varepsilon_j = \begin{cases} \langle \varepsilon_i,\ \varepsilon_i a \varepsilon_i,\ \varepsilon_i a^2 \varepsilon_i \rangle & \text{if } i = j, \\ \langle \varepsilon_i a \varepsilon_j,\ \varepsilon_i a^2 \varepsilon_j \rangle & \text{if } i \neq j. \end{cases}$$

Now we denote by $\tilde{P}_{V_i}$ the projective indecomposable $K\mathfrak{H}$-module for which $\tilde{P}_{V_i}/J(K\mathfrak{H})\tilde{P}_{V_i} \simeq V_i\ (0 \leqq i \leqq 12)$. Then by the above we have the following

Corollary 2.2. *The Cartan matrix of $K\mathfrak{H}$ is given by*

$$\begin{matrix}\tilde{P}_{V_0} \\ \tilde{P}_{V_1} \\ \vdots \\ \tilde{P}_{V_{12}}\end{matrix}\begin{pmatrix} 3 & 2 & \cdots\cdots & 2 \\ 2 & 3 & \cdots\cdots & 2 \\ \vdots & \vdots & & \vdots \\ 2 & 2 & \cdots\cdots & 3 \end{pmatrix}.$$

Now we shall prove the following

Proposition 2.3. *The Loewy and socle series of the projective indecomposable $K\mathfrak{H}$-modules are as follows:*

$$\widetilde{P}_{V_0} = \begin{array}{c} V_0 \\ V_1\ V_2\ V_3 \\ V_4\ V_5\ V_6\ V_7\ V_8\ V_9 \\ V_0\ V_7\ V_8\ V_9\ V_{10}\ V_{11}\ V_{12} \\ V_1\ V_2\ V_3\ V_{10}\ V_{11}\ V_{12} \\ V_4\ V_5\ V_6 \\ V_0 \end{array} \qquad \widetilde{P}_{V_1} = \begin{array}{c} V_1 \\ V_5\ V_6\ V_7 \\ V_0\ V_2\ V_8\ V_9\ V_{10}\ V_{11} \\ V_1\ V_2\ V_3\ V_4\ V_8\ V_{11}\ V_{12} \\ V_3\ V_4\ V_5\ V_6\ V_7\ V_{12} \\ V_0\ V_9\ V_{10} \\ V_1 \end{array}$$

$$\widetilde{P}_{V_2} = \begin{array}{c} V_2 \\ V_4\ V_6\ V_8 \\ V_0\ V_3\ V_7\ V_9\ V_{11}\ V_{12} \\ V_1\ V_2\ V_3\ V_5\ V_9\ V_{10}\ V_{12} \\ V_1\ V_4\ V_5\ V_6\ V_8\ V_{10} \\ V_0\ V_7\ V_{11} \\ V_2 \end{array} \qquad \widetilde{P}_{V_3} = \begin{array}{c} V_3 \\ V_4\ V_5\ V_9 \\ V_0\ V_1\ V_7\ V_8\ V_{10}\ V_{12} \\ V_1\ V_2\ V_3\ V_6\ V_7\ V_{10}\ V_{11} \\ V_2\ V_4\ V_5\ V_6\ V_9\ V_{11} \\ V_0\ V_8\ V_{12} \\ V_3 \end{array}$$

$$\widetilde{P}_{V_4} = \begin{array}{c} V_4 \\ V_0\ V_7\ V_{12} \\ V_1\ V_2\ V_3\ V_6\ V_9\ V_{10} \\ V_1\ V_4\ V_5\ V_6\ V_8\ V_9\ V_{11} \\ V_0\ V_5\ V_7\ V_8\ V_{11}\ V_{12} \\ V_2\ V_3\ V_{10} \\ V_4 \end{array} \qquad \widetilde{P}_{V_5} = \begin{array}{c} V_5 \\ V_0\ V_8\ V_{10} \\ V_1\ V_2\ V_3\ V_4\ V_7\ V_{11} \\ V_2\ V_4\ V_5\ V_6\ V_7\ V_9\ V_{12} \\ V_0\ V_6\ V_8\ V_9\ V_{10}\ V_{12} \\ V_1\ V_3\ V_{11} \\ V_5 \end{array}$$

$$\widetilde{P}_{V_6} = \begin{array}{c} V_6 \\ V_0\ V_9\ V_{11} \\ V_1\ V_2\ V_3\ V_5\ V_8\ V_{12} \\ V_3\ V_4\ V_5\ V_6\ V_7\ V_8\ V_{10} \\ V_0\ V_4\ V_7\ V_9\ V_{10}\ V_{11} \\ V_1\ V_2\ V_{12} \\ V_6 \end{array} \qquad \widetilde{P}_{V_7} = \begin{array}{c} V_7 \\ V_2\ V_9\ V_{10} \\ V_1\ V_4\ V_6\ V_8\ V_{11}\ V_{12} \\ V_0\ V_3\ V_5\ V_6\ V_7\ V_{11}\ V_{12} \\ V_0\ V_2\ V_3\ V_5\ V_9\ V_{10} \\ V_1\ V_4\ V_8 \\ V_7 \end{array}$$

$$\widetilde{P}_{V_8} = \begin{array}{c} V_8 \\ V_3\ V_7\ V_{11} \\ V_2\ V_4\ V_5\ V_9\ V_{10}\ V_{12} \\ V_0\ V_1\ V_4\ V_6\ V_8\ V_{10}\ V_{12} \\ V_0\ V_1\ V_3\ V_6\ V_7\ V_{11} \\ V_2\ V_5\ V_9 \\ V_8 \end{array} \qquad \widetilde{P}_{V_9} = \begin{array}{c} V_9 \\ V_1\ V_8\ V_{12} \\ V_3\ V_5\ V_6\ V_7\ V_{10}\ V_{11} \\ V_0\ V_2\ V_4\ V_5\ V_9\ V_{10}\ V_{11} \\ V_0\ V_1\ V_2\ V_4\ V_8\ V_{12} \\ V_3\ V_6\ V_7 \\ V_9 \end{array}$$

$$\tilde{P}_{V_{10}} = \begin{matrix} V_{10} \\ V_1\ V_4\ V_{11} \\ V_0\ V_2\ V_5\ V_6\ V_7\ V_{12} \\ V_0\ V_2\ V_3\ V_6\ V_8\ V_9\ V_{10} \\ V_1\ V_3\ V_4\ V_8\ V_9\ V_{11} \\ V_5\ V_7\ V_{12} \\ V_{10} \end{matrix} \qquad \tilde{P}_{V_{11}} = \begin{matrix} V_{11} \\ V_2\ V_5\ V_{12} \\ V_0\ V_3\ V_4\ V_6\ V_8\ V_{10} \\ V_0\ V_1\ V_3\ V_4\ V_7\ V_9\ V_{11} \\ V_1\ V_2\ V_5\ V_7\ V_9\ V_{12} \\ V_6\ V_8\ V_{10} \\ V_{11} \end{matrix}$$

$$\tilde{P}_{V_{12}} = \begin{matrix} V_{12} \\ V_3\ V_6\ V_{10} \\ V_0\ V_1\ V_4\ V_5\ V_9\ V_{11} \\ V_0\ V_1\ V_2\ V_5\ V_7\ V_8\ V_{12} \\ V_2\ V_3\ V_6\ V_7\ V_8\ V_{10} \\ V_4\ V_9\ V_{11} \\ V_{12} \end{matrix}$$

Proof. We may, and shall assume that $\tilde{P}_{V_0} = K\mathfrak{U}\varepsilon$. Then we have $J(K\mathfrak{H})^i\tilde{P}_{V_0} = J(K\mathfrak{U})^i\varepsilon$ for every i, because $J(K\mathfrak{H}) = J(K\mathfrak{U})K\mathfrak{H}$. View $K\mathfrak{U}$ as a $K\mathfrak{H}$-module via conjugation of $\mathfrak{H}$ on $\mathfrak{U}$. Then, by the above, we have a $K\mathfrak{H}$-isomorphism

$$J(K\mathfrak{H})^i\tilde{P}_{V_0}/J(K\mathfrak{H})^{i+1}\tilde{P}_{V_0} = J(K\mathfrak{U})^i\varepsilon/J(K\mathfrak{U})^{i+1}\varepsilon \simeq J(K\mathfrak{U})^i/J(K\mathfrak{U})^{i+1},$$

for every i. Now we set $M = J(K\mathfrak{U})/J(K\mathfrak{U})^2$. Then the following elements form a K-basis of M:

$$(a-1)+J(K\mathfrak{U})^2,\ (b-1)+J(K\mathfrak{U})^2,\ (c-1)+J(K\mathfrak{U})^2.$$

Operating v on these elements, we obtain

$$\begin{aligned} (a-1)^v+J(K\mathfrak{U})^2 &= (c-1)+J(K\mathfrak{U})^2, \\ (b-1)^v+J(K\mathfrak{U})^2 &= (a^2c-1)+J(K\mathfrak{U})^2, \\ (c-1)^v+J(K\mathfrak{U})^2 &= (a^2b^2c-1)+J(K\mathfrak{U})^2. \end{aligned}$$

Hence by the congruences:

$$\begin{aligned} a^2c-1 &\equiv -(a-1)+(c-1) && \bmod J(K\mathfrak{U})^2, \\ a^2b^2c-1 &\equiv -(a-1)-(b-1)+(c-1) && \bmod J(K\mathfrak{U})^2, \end{aligned}$$

we see that M affords the matrix representation

$$T:\ v \to T(v) = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The characteristic polynomial of the matrix $T(v)$ is X^3-X^2-X-1. Therefore we see that

$$(2.4) \qquad L_2(\tilde{P}_{V_0}) \simeq M \simeq V_1 \oplus V_2 \oplus V_3.$$

From this we can choose a K-basis $\{m_1, m_2, m_3\}$ of M such that $m_i^v = \zeta^{3^{i-1}} m_i$. For each i, $1 \leqq i \leqq 3$, m_i can be expressed in the form

$$m_i = \alpha_i + J(K\mathfrak{U})^2, \qquad \alpha_i \in J(K\mathfrak{U}).$$

We set $N = J(K\mathfrak{U})^2/J(K\mathfrak{U})^3$. Then N is spanned by

$$\alpha_i\alpha_j + J(K\mathfrak{U})^3, \qquad 1 \leqq i \leqq j \leqq 3.$$

But dim $N = 6$. Hence these six elements form a K-basis of N. Operating v on these elements, we obtain

$$\begin{aligned}
(\alpha_1^2)^v + J(K\mathfrak{U})^3 &= \zeta^2(\alpha_1^2 + J(K\mathfrak{U})^3),\\
(\alpha_2^2)^v + J(K\mathfrak{U})^3 &= \zeta^6(\alpha_2^2 + J(K\mathfrak{U})^3),\\
(\alpha_3^2)^v + J(K\mathfrak{U})^3 &= \zeta^5(\alpha_3^2 + J(K\mathfrak{U})^3),\\
(\alpha_1\alpha_2)^v + J(K\mathfrak{U})^3 &= \zeta^4(\alpha_1\alpha_2 + J(K\mathfrak{U})^3),\\
(\alpha_1\alpha_3)^v + J(K\mathfrak{U})^3 &= \zeta^{10}(\alpha_1\alpha_3 + J(K\mathfrak{U})^3),\\
(\alpha_2\alpha_3)^v + J(K\mathfrak{U})^3 &= \zeta^{12}(\alpha_2\alpha_3 + J(K\mathfrak{U})^3).
\end{aligned}$$

This shows that

$$(2.5) \qquad L_3(\tilde{P}_{V_0}) \simeq N \simeq V_4 \oplus V_5 \oplus V_6 \oplus V_7 \oplus V_8 \oplus V_9.$$

Now, it is easy to see that $J(K\mathfrak{U})^7 = 0$, and so the Loewy length of $\tilde{P}_{V_0}$ is 7. Further by [3, Corollary], the Loewy and socle series of $\tilde{P}_{V_0}$ coincide. Hence, noting that $\tilde{P}_{V_0}$ is self-dual, we conclude, from Corollary 2.2 and the above facts (2.4), (2.5), that $\tilde{P}_{V_0}$ has the Loewy and socle series given in the proposition. It is easy to see that

$$\tilde{P}_{V_i} \simeq \tilde{P}_{V_0} \otimes_K V_i,\ L_j(\tilde{P}_{V_i}) \simeq L_j(\tilde{P}_{V_0}) \otimes_K V_i, \qquad 1 \leqq i \leqq 12,\ 1 \leqq j \leqq 7.$$

From this, we see immediately that $\tilde{P}_{V_i}$ has the Loewy and socle series given in the proposition. Thus the proposition is proved

We can easily see that $P_I \simeq \tilde{P}_{V_0}^{\mathfrak{G}}$, $P_{M_1} \simeq \tilde{P}_{V_1}^{\mathfrak{G}}$, $P_{M_2} \simeq \tilde{P}_{V_4}^{\mathfrak{G}}$, $P_{M_3} \simeq \tilde{P}_{V_7}^{\mathfrak{G}}$ and $P_{M_4} \simeq \tilde{P}_{V_{10}}^{\mathfrak{G}}$. Therefore the next corollary follows at once from Corollary 2.2.

Corollary 2.6. *The Cartan matrix of $K\mathfrak{G}$ is given by*

$$\begin{matrix} P_I \\ P_{M_1} \\ P_{M_2} \\ P_{M_3} \\ P_{M_4} \end{matrix} \begin{bmatrix} 9 & 6 & 6 & 6 & 6 \\ 6 & 7 & 6 & 6 & 6 \\ 6 & 6 & 7 & 6 & 6 \\ 6 & 6 & 6 & 7 & 6 \\ 6 & 6 & 6 & 6 & 7 \end{bmatrix}.$$

3. The Loewy structure of P_I. In this section, we shall determine the Loewy series of P_I. We set $\mathfrak{S} = \mathfrak{V}\mathfrak{W} = \langle \mathfrak{V}, \mathfrak{W} \rangle$, and we let $\hat{I}$ be the trivial simple $K\mathfrak{S}$-module and $\hat{P}_{\hat{I}}$ the projective indecomposable $K\mathfrak{S}$-module for which $\hat{P}_{\hat{I}}/J(K\mathfrak{S})P_{\hat{I}} \simeq \hat{I}$. Then it is easy to see that $P_I \simeq \hat{P}_{\hat{I}}^{\mathfrak{G}}$. Hence to determine the Loewy series of P_I, it suffices to determine that of $\hat{P}_{\hat{I}}^{\mathfrak{G}}$. We now set

$$X = (J(K\mathfrak{S})^2 \hat{P}_{\hat{I}})^{\mathfrak{G}},\ Y = (J(K\mathfrak{S})\hat{P}_{\hat{I}})^{\mathfrak{G}},\ Z = \hat{P}_{\hat{I}}^{\mathfrak{G}}.$$

Then identifying $\hat{P}_{\hat{I}}$ with $K\mathfrak{W}\varepsilon$, we may set

$$X = K\mathfrak{U}\varepsilon\hat{\mathfrak{W}},\ Y = K\mathfrak{U}\varepsilon J(K\mathfrak{W}),\ Z = K\mathfrak{U}\varepsilon K\mathfrak{W}.$$

Throughout this section, we set $J = J(K\mathfrak{G})$, $J_{\mathfrak{U}} = J(K\mathfrak{U})$ and $J_{\mathfrak{W}} = J(K\mathfrak{W})$. At first, we have at once the following, because $\mathfrak{G}/\mathfrak{U}$ is a Frobenius group (see the proof of [6, Proposition 3]).

Lemma 3.1. $J = K\mathfrak{G}J_{\mathfrak{U}} + \varepsilon J_{\mathfrak{W}}$.

Now we determine the Loewy series of X.

Lemma 3.2.

$$X = \begin{matrix} I \\ M_1 \\ M_2\ M_3 \\ I\ M_3\ M_4 \\ M_1\ M_4 \\ M_2 \\ I \end{matrix}.$$

Proof. Since $X = K\mathfrak{U}\varepsilon\hat{\mathfrak{W}}$, we have

$$\varepsilon J_{\mathfrak{W}} X = \varepsilon J_{\mathfrak{W}} K\mathfrak{U}\varepsilon\hat{\mathfrak{W}} = J_{\mathfrak{W}}\varepsilon K\mathfrak{U}\varepsilon\hat{\mathfrak{W}}.$$

Now let C_{a^i} $(i = 1, 2)$ be the conjugate class in $\mathfrak{H}$ containing a^i. Then $\varepsilon K\mathfrak{U}\varepsilon = \langle \varepsilon, \hat{C}_a\varepsilon, \hat{C}_{a^2}\varepsilon \rangle$, and so we can see that each element of $\varepsilon K\mathfrak{U}\varepsilon$ commutes with s. Therefore

$$\varepsilon J_{\mathfrak{W}}X = \varepsilon K\mathfrak{U}\varepsilon J_{\mathfrak{W}}\hat{\mathfrak{W}} = 0.$$

This together with Lemma 3.1 implies that $J^iX = J^i_{\mathfrak{U}}\varepsilon\hat{\mathfrak{W}}$ for every i. Hence we have

$$J^iX|_{\mathfrak{H}} \simeq J^i_{\mathfrak{U}}\varepsilon \simeq J(K\mathfrak{H})^i\tilde{P}_{\nu_0}.$$

Thus we conclude that

$$(J^iX/J^{i+1}X)|_{\mathfrak{H}} \simeq J(K\mathfrak{H})^i\tilde{P}_{\nu_0}/J(K\mathfrak{H})^{i+1}\tilde{P}_{\nu_0}.$$

Therefore the result follows from Proposition 2.3.

Next, we determine the Loewy structure of Y.

Lemma 3.3. (1) $L_1(Y) \simeq L_1(X)$.
(2) $L_i(Y) \simeq L_i(X) \oplus L_{i-1}(X)$ *for* $2 \leqq i \leqq 7$.
(3) $L_8(Y) = L_7(X)$.

Proof. At first we shall show, by induction, that

$$J^iY = J^i_{\mathfrak{U}}\varepsilon J_{\mathfrak{W}} + J^{i-1}X, \qquad \text{where } J^0 = K\mathfrak{G}. \tag{3.4}$$

By Lemma 3.1, we have

$$\begin{aligned} JY &= J_{\mathfrak{U}}\varepsilon J_{\mathfrak{W}} + \varepsilon J_{\mathfrak{W}}K\mathfrak{U}\varepsilon J_{\mathfrak{W}} = J_{\mathfrak{U}}\varepsilon J_{\mathfrak{W}} + J_{\mathfrak{W}}(\varepsilon K\mathfrak{U}\varepsilon)J_{\mathfrak{W}} \\ &= J_{\mathfrak{U}}\varepsilon J_{\mathfrak{W}} + (\varepsilon K\mathfrak{U}\varepsilon)J^2_{\mathfrak{W}} = J_{\mathfrak{U}}\varepsilon J_{\mathfrak{W}} + \varepsilon K\mathfrak{U}\varepsilon\hat{\mathfrak{W}}. \end{aligned}$$

Hence $JY \subset J_{\mathfrak{U}}\varepsilon J_{\mathfrak{W}} + X$. On the other hand, for any $u \in \mathfrak{U}$, we have

$$\begin{aligned} u\varepsilon\hat{\mathfrak{W}} &= (1-\varepsilon)u\varepsilon\hat{\mathfrak{W}} + \varepsilon u\varepsilon\hat{\mathfrak{W}} = (u - \textstyle\sum_{v\in\mathfrak{B}} u^v)\varepsilon\hat{\mathfrak{W}} + \varepsilon u\varepsilon\hat{\mathfrak{W}} \\ &\in J_{\mathfrak{U}}\varepsilon J_{\mathfrak{W}} + \varepsilon K\mathfrak{U}\varepsilon\hat{\mathfrak{W}} = JY. \end{aligned}$$

Hence $X = K\mathfrak{U}\varepsilon\hat{\mathfrak{W}} \subset JY$. Thus (3.4) is proved for $i = 1$. Next, assume that (3.4) holds for some i. Then

$$\begin{aligned} J^{i+1}Y &= J^{i+1}_{\mathfrak{U}}\varepsilon J_{\mathfrak{W}} + \varepsilon J_{\mathfrak{W}}J^i_{\mathfrak{U}}\varepsilon J_{\mathfrak{W}} + J^iX \\ &= J^{i+1}_{\mathfrak{U}}\varepsilon J_{\mathfrak{W}} + J_{\mathfrak{W}}(\varepsilon J^i_{\mathfrak{U}}\varepsilon)J_{\mathfrak{W}} + J^iX \\ &= J^{i+1}_{\mathfrak{U}}\varepsilon J_{\mathfrak{W}} + (\varepsilon J^i_{\mathfrak{U}}\varepsilon)J^2_{\mathfrak{W}} + J^iX. \end{aligned}$$

Hence, observing that

$$\varepsilon J^i_{\mathfrak{U}}\varepsilon J^2_{\mathfrak{W}} = \varepsilon J^i_{\mathfrak{U}}\varepsilon\hat{\mathfrak{W}} \subset J^i_{\mathfrak{U}}\varepsilon\hat{\mathfrak{W}} = J^iX,$$

we have

$$J^{i+1}Y = J_{\mathfrak{U}}^{i+1}\varepsilon J_{\mathfrak{W}} + J^{i}X.$$

Thus (3.4) holds for every i.

Because of (3.4), we have

$$\begin{aligned} L_1(Y) &= Y/JY = K\mathfrak{U}\varepsilon J_{\mathfrak{W}}/(J_{\mathfrak{U}}\varepsilon J_{\mathfrak{W}} + X) \\ &\simeq (K\mathfrak{U}\varepsilon J_{\mathfrak{W}}/K\mathfrak{U}\varepsilon\hat{\mathfrak{W}})/((J_{\mathfrak{U}}\varepsilon J_{\mathfrak{W}} + K\mathfrak{U}\varepsilon\hat{\mathfrak{W}})/K\mathfrak{U}\varepsilon\hat{\mathfrak{W}}) \\ &\simeq K\mathfrak{U}\varepsilon\hat{\mathfrak{W}}/J_{\mathfrak{U}}\varepsilon\hat{\mathfrak{W}} \simeq K\hat{\mathfrak{G}} \simeq L_1(X). \end{aligned}$$

Thus we obtain (1).

From (3.4), we have the following inclusions:

$$J^{i}Y \subset J^{i}Y + J^{i-2}X \subset J^{i-1}Y, \qquad 2 \leqq i \leqq 7.$$

To prove (2), it suffices then to show that

$$J^{i-1}Y/(J^{i}Y + J^{i-2}X) \simeq L_i(X), \tag{3.5}$$

$$(J^{i}Y + J^{i-2}X)/J^{i}Y \simeq L_{i-1}(X). \tag{3.6}$$

In fact,

$$\begin{aligned} J^{i-1}Y/(J^{i}Y + J^{i-2}X) &= (J_{\mathfrak{U}}^{i-1}\varepsilon J_{\mathfrak{W}} + J^{i-2}X)/(J_{\mathfrak{U}}^{i}\varepsilon J_{\mathfrak{W}} + J^{i-2}X) \\ &\simeq J_{\mathfrak{U}}^{i-1}\varepsilon J_{\mathfrak{W}}/(J_{\mathfrak{U}}^{i-1}\varepsilon J_{\mathfrak{W}} \cap (J_{\mathfrak{U}}^{i}\varepsilon J_{\mathfrak{W}} + J^{i-2}X)) \\ &= J_{\mathfrak{U}}^{i-1}\varepsilon J_{\mathfrak{W}}/(J_{\mathfrak{U}}^{i}\varepsilon J_{\mathfrak{W}} + J_{\mathfrak{U}}^{i-1}\varepsilon\hat{\mathfrak{W}}) \\ &\simeq (J_{\mathfrak{U}}^{i-1}\varepsilon J_{\mathfrak{W}}/J_{\mathfrak{U}}^{i-1}\varepsilon\hat{\mathfrak{W}})/((J_{\mathfrak{U}}^{i}\varepsilon J_{\mathfrak{W}} + J_{\mathfrak{U}}^{i-1}\varepsilon\hat{\mathfrak{W}})/J_{\mathfrak{U}}^{i-1}\varepsilon\hat{\mathfrak{W}}) \\ &\simeq J_{\mathfrak{U}}^{i-1}\varepsilon\hat{\mathfrak{W}}/J_{\mathfrak{U}}^{i}\varepsilon\hat{\mathfrak{W}} \\ &= J^{i-1}X/J^{i}X = L_i(X), \end{aligned}$$

and

$$\begin{aligned} (J^{i}Y + J^{i-2}X)/J^{i}Y &\simeq J^{i-2}X/(J^{i}Y \cap J^{i-2}X) \\ &= J^{i-2}X/J^{i-1}X = L_{i-1}(X). \end{aligned}$$

Thus we obtain (3.5) and (3.6), and complete the proof of (2).

(3) is clear because $J^{7}Y = J^{6}X = L_7(X)$ by (3.4).

Finally, we determine the Loewy structure of Z.

Lemma 3.7. (1) $L_1(Z) \simeq L_1(X)$.
(2) $L_i(Z) \simeq L_i(X) \oplus L_{i-1}(Y)$ *for* $2 \leqq i \leqq 7$.
(3) $L_8(Z) = L_7(Y)$.
(4) $L_9(Z) = L_8(Y)$.

Proof. First we shall prove, by induction, that

(3.8) $$J^iZ = J_{\mathfrak{U}}^i\varepsilon K\mathfrak{W}+J^{i-1}Y, \qquad \text{where } J^0 = K\mathfrak{G}.$$

By Lemma 3.1, we have

$$\begin{aligned} JZ &= J_{\mathfrak{U}}\varepsilon K\mathfrak{W}+\varepsilon J_{\mathfrak{W}}K\mathfrak{U}\varepsilon K\mathfrak{W} = J_{\mathfrak{U}}\varepsilon K\mathfrak{W}+J_{\mathfrak{W}}(\varepsilon K\mathfrak{U}\varepsilon)K\mathfrak{W} \\ &= J_{\mathfrak{U}}\varepsilon K\mathfrak{W}+(\varepsilon K\mathfrak{U}\varepsilon)J_{\mathfrak{W}} \subset J_{\mathfrak{U}}\varepsilon K\mathfrak{W}+Y. \end{aligned}$$

On the other hand, given $u \in \mathfrak{U}$ and $w \in J_{\mathfrak{W}}$, we have

$$\begin{aligned} u\varepsilon w &= (1-\varepsilon)u\varepsilon w+\varepsilon u\varepsilon w = (u-\textstyle\sum_{v\in\mathfrak{V}} u^v)\varepsilon w+\varepsilon u\varepsilon w \\ &\in J_{\mathfrak{U}}\varepsilon K\mathfrak{W}+\varepsilon K\mathfrak{U}\varepsilon J_{\mathfrak{W}} = JZ. \end{aligned}$$

Hence $Y = K\mathfrak{U}\varepsilon J_{\mathfrak{W}} \subset JZ$, and so (3.8) is proved for $i = 1$. Next, assume that (3.8) holds for some i. Then

$$\begin{aligned} J^{i+1}Z &= J_{\mathfrak{U}}^{i+1}\varepsilon K\mathfrak{W}+\varepsilon J_{\mathfrak{W}}J_{\mathfrak{U}}^i\varepsilon K\mathfrak{W}+J^iY \\ &= J_{\mathfrak{U}}^{i+1}\varepsilon K\mathfrak{W}+J_{\mathfrak{W}}(\varepsilon J_{\mathfrak{U}}^i\varepsilon)K\mathfrak{W}+J^iY \\ &= J_{\mathfrak{U}}^{i+1}\varepsilon K\mathfrak{W}+(\varepsilon J_{\mathfrak{U}}^i\varepsilon)J_{\mathfrak{W}}+J^iY. \end{aligned}$$

Hence, noting that

$$\varepsilon J_{\mathfrak{U}}^i\varepsilon J_{\mathfrak{W}} \subset J_{\mathfrak{U}}^i\varepsilon J_{\mathfrak{W}} \subset J^iY,$$

we have

$$J^{i+1}Z = J_{\mathfrak{U}}^{i+1}\varepsilon K\mathfrak{W}+J^iY.$$

Thus (3.8) holds for every i.

By (3.8), we have

$$\begin{aligned} L_1(Z) &= Z/JZ = K\mathfrak{U}\varepsilon K\mathfrak{W}/(J_{\mathfrak{U}}\varepsilon K\mathfrak{W}+K\mathfrak{U}\varepsilon J_{\mathfrak{W}}) \\ &\simeq (K\mathfrak{U}\varepsilon K\mathfrak{W}/K\mathfrak{U}\varepsilon J_{\mathfrak{W}})/((J_{\mathfrak{U}}\varepsilon K\mathfrak{W}+K\mathfrak{U}\varepsilon J_{\mathfrak{W}})/K\mathfrak{U}\varepsilon J_{\mathfrak{W}}) \\ &\simeq K\mathfrak{U}\varepsilon\hat{\mathfrak{W}}/J_{\mathfrak{U}}\varepsilon\hat{\mathfrak{W}} \simeq K\hat{\mathfrak{G}} \simeq L_1(X), \end{aligned}$$

proving (1).

Further, from (3.8), we have the following inclusions:

$$J^iZ \subset J^iZ+J^{i-2}Y \subset J^{i-1}Z, \qquad 2 \leqq i \leqq 7.$$

Hence, in order to prove (2), it suffices to show that

(3.9) $$J^{i-1}Z/(J^iZ+J^{i-2}Y) \simeq L_i(X),$$

(3.10) $$(J^iZ+J^{i-2}Y)/J^iZ \simeq L_{i-1}(Y).$$

Indeed we have

$$\begin{aligned} J^{i-1}Z/(J^iZ+J^{i-2}Y) &= (J_{\mathfrak{U}}^{i-1}\varepsilon K\mathfrak{W}+J^{i-2}Y)/(J_{\mathfrak{U}}^{i}\varepsilon K\mathfrak{W}+J^{i-2}Y) \\ &\simeq J_{\mathfrak{U}}^{i-1}\varepsilon K\mathfrak{W}/(J_{\mathfrak{U}}^{i-1}\varepsilon K\mathfrak{W} \cap (J_{\mathfrak{U}}^{i}\varepsilon K\mathfrak{W}+J^{i-2}Y)) \\ &= J_{\mathfrak{U}}^{i-1}\varepsilon K\mathfrak{W}/(J_{\mathfrak{U}}^{i}\varepsilon K\mathfrak{W}+J_{\mathfrak{U}}^{i-1}\varepsilon J_{\mathfrak{W}}) \\ &\simeq (J_{\mathfrak{U}}^{i-1}\varepsilon K\mathfrak{W}/J_{\mathfrak{U}}^{i-1}\varepsilon J_{\mathfrak{W}})/((J_{\mathfrak{U}}^{i}\varepsilon K\mathfrak{W}+J_{\mathfrak{U}}^{i-1}\varepsilon J_{\mathfrak{W}})/J_{\mathfrak{U}}^{i-1}\varepsilon J_{\mathfrak{W}}) \\ &\simeq J_{\mathfrak{U}}^{i-1}\varepsilon\hat{\mathfrak{W}}/J_{\mathfrak{U}}^{i}\varepsilon\hat{\mathfrak{W}} = L_i(X), \end{aligned}$$

and

$$\begin{aligned} (J^iZ+J^{i-2}Y)/J^iZ &\simeq J^{i-2}Y/(J^iZ \cap J^{i-2}Y) \\ &= J^{i-2}Y/J^{i-1}Y = L_{i-1}(Y). \end{aligned}$$

Thus we have proved (3.9) and (3.10), and complete the proof of (2).

Because of (3.8), we have $J^7Z = J^6Y$ and $J^8Z = J^7Y$. Hence we have

$$\begin{aligned} L_8(Z) &= J^7Z/J^8Z = J^6Y/J^7Y = L_7(Y), \\ L_9(Z) &= J^7Y = L_8(Y). \end{aligned}$$

Thus (3) and (4) are proved.

Since $Z \simeq P_I$, combining the preceding lemma with Lemmas 3.2 and 3.3, we obtain

Proposition 3.11. *P_I has the Loewy series given in Theorem.*

4. A partial proof of Theorem. In this section, by making use of Proposition 2.3, we determine the upper part of the Loewy series of each P_{M_i}. From now on, we set $J = J(K\mathfrak{G})$ and $\tilde{J} = J(K\mathfrak{H})$.

At first we prove the following

Lemma 4.1. (1) $L_2(P_{M_1}) \simeq M_2 \oplus M_2 \oplus M_3$.

(2) $L_3(P_{M_1}) \simeq I \oplus M_1 \oplus M_3 \oplus M_3 \oplus M_4 \oplus M_4$.

(3) $L_2(P_{M_2}) \simeq I \oplus M_3 \oplus M_4$.

(4) $L_2(P_{M_3}) \simeq M_1 \oplus M_3 \oplus M_4$.

(5) $L_3(P_{M_3}) \simeq M_1 \oplus M_2 \oplus M_2 \oplus M_3 \oplus M_4 \oplus M_4$.

(6) $L_4(P_{M_3}) \simeq I \oplus M_1 \oplus M_2 \oplus M_2 \oplus M_3 \oplus M_4 \oplus M_4$.

(7) $L_2(P_{M_4}) \simeq M_1 \oplus M_2 \oplus M_4$.

(8) $L_3(P_{M_4}) \simeq I \oplus M_1 \oplus M_2 \oplus M_2 \oplus M_3 \oplus M_4$.

Proof. (1) Since it holds that

$$P_{M_1}/JP_{M_1} \simeq \tilde{P}_{V_1}^{\mathfrak{G}}/\tilde{J}\tilde{P}_{V_1}^{\mathfrak{G}} \simeq \tilde{P}_{V_1}^{\mathfrak{G}}/(\tilde{J}\tilde{P}_{V_1})^{\mathfrak{G}} \simeq (\tilde{P}_{V_1}/\tilde{J}\tilde{P}_{V_1})^{\mathfrak{G}} \simeq V_1^{\mathfrak{G}} = M_1,$$

we obtain $JP_{M_1} = \tilde{J}P_{M_1}$. Therefore we have

$$\begin{aligned} JP_{M_1}/\tilde{J}^2P_{M_1} &= \tilde{J}P_{M_1}/\tilde{J}^2P_{M_1} \simeq \tilde{J}\tilde{P}^{(\sigma)}_{V_1}/\tilde{J}^2\tilde{P}^{(\sigma)}_{V_1} \simeq (\tilde{J}\tilde{P}_{V_1})^{(\sigma)}/(\tilde{J}^2\tilde{P}_{V_1})^{(\sigma)} \\ &\simeq (\tilde{J}\tilde{P}_{V_1}/\tilde{J}^2\tilde{P}_{V_1})^{(\sigma)} \simeq (V_5 \oplus V_6 \oplus V_7)^{(\sigma)} \\ &= M_2 \oplus M_2 \oplus M_3. \end{aligned}$$

Hence $JP_{M_1}/\tilde{J}^2P_{M_1}$ is completely reducible, and so noting that $\tilde{J}^2P_{M_1} \subset J^2P_{M_1}$, we obtain

$$\tilde{J}^2P_{M_1} = J^2P_{M_1} \text{ and } L_2(P_{M_1}) \simeq M_2 \oplus M_2 \oplus M_3,$$

proving (1).

(2) Since $J^2P_{M_1} = \tilde{J}^2P_{M_1}$, we have

$$\begin{aligned} J^2P_{M_1}/\tilde{J}^3P_{M_1} &= \tilde{J}^2P_{M_1}/\tilde{J}^3P_{M_1} \simeq \tilde{J}^2\tilde{P}^{(\sigma)}_{V_1}/\tilde{J}^3\tilde{P}^{(\sigma)}_{V_1} \\ &\simeq (\tilde{J}^2\tilde{P}_{V_1})^{(\sigma)}/(\tilde{J}^3\tilde{P}_{V_1})^{(\sigma)} \simeq (\tilde{J}^2\tilde{P}_{V_1}/\tilde{J}^3\tilde{P}_{V_1})^{(\sigma)} \\ &\simeq (V_0 \oplus V_2 \oplus V_8 \oplus V_9 \oplus V_{10} \oplus V_{11})^{(\sigma)} \\ &= \begin{array}{l} I\ M_1\ M_3\ M_3\ M_4\ M_4 \\ I \\ I \end{array} \end{aligned}$$

This implies that

$$L_3(P_{M_1}) \simeq I \oplus M_1 \oplus M_3 \oplus M_3 \oplus M_4 \oplus M_4,$$

because $\tilde{J}^3P_{M_1} \subset J^3P_{M_1}$, proving (2).

(3) through (8) are obtained in the same manner as in the proof of (1) and (2), and hence we omit the proof.

Further, concerning the Loewy layers of P_{M_1}, we have the following

Lemma 4.2. (1) $L_4(P_{M_1}) \supset I \oplus M_1 \oplus M_2 \oplus M_3 \oplus M_4 \oplus M_4$.
(2) $L_5(P_{M_1}) \supset I \oplus M_2$.
(3) $L_6(P_{M_1}) \supset M_1$.

Proof. (1) In view of Proposition 2.3, we see that $\tilde{J}^2\tilde{P}_{V_1}/\tilde{J}^4\tilde{P}_{V_1}$ has a factor module

$$X = \begin{array}{l} V_0\ V_2\ V_8\ V_9\ V_{10}\ V_{11} \\ \quad V_4\ V_8\ V_{11}\ V_{12} \end{array}.$$

Since $S_2(\tilde{P}_{V_i}) \simeq L_6(\tilde{P}_{V_i})$ for $i = 4, 8, 11, 12$, from Proposition 2.3, it follows that $[S_2(\tilde{P}_{V_i}), V_0] = 0$. Hence we conclude that $[S_1(X), V_0] = 1$, and so we have

$$X = V_0 \oplus \begin{matrix} V_2\ V_8\ V_9\ V_{10}\ V_{11} \\ V_4\ V_8\ V_{11}\ V_{12} \end{matrix}.$$

Hence we obtain

$$X^{(\mathfrak{G})} = \begin{matrix} I \\ I \\ I \end{matrix} \oplus \begin{matrix} M_1\ M_3\ M_3\ M_4\ M_4 \\ M_2\ M_3\ M_4\ M_4 \end{matrix}.$$

Since $X^{(\mathfrak{G})}$ is a homomorphic image of $(\tilde{J}^2\tilde{P}_{V_1})^{(\mathfrak{G})}$, recalling that $(\tilde{J}^2\tilde{P}_{V_1})^{(\mathfrak{G})} \simeq J^2P_{M_1}$, we have

(4.3) $$L_4(P_{M_1}) \supsetneq L_2(X^{(\mathfrak{G})}) \simeq I \oplus M_2 \oplus M_3 \oplus M_4 \oplus M_4.$$

Further $\tilde{J}^2\tilde{P}_{V_1}/\tilde{J}^4\tilde{P}_{V_1}$ has another factor module

$$Y = \begin{matrix} V_0\ V_2\ V_8\ V_9\ V_{10}\ V_{11} \\ V_1\ V_2\ V_3 \end{matrix}.$$

It is easy to see that the mapping

$$\tilde{J}^2\tilde{P}_{V_1}/\tilde{J}^4\tilde{P}_{V_1} \to X \oplus Y$$

given by

$$\alpha \to (\alpha + (V_4 \oplus V_8 \oplus V_{11} \oplus V_{12}),\ \alpha + (V_1 \oplus V_2 \oplus V_3))\ (\alpha \in \tilde{J}^2\tilde{P}_{V_1}/\tilde{J}^4\tilde{P}_{V_1}),$$

is a $K\mathfrak{H}$-monomorphism. Therefore from the fact that $[S_2(\tilde{J}^2\tilde{P}_{V_1}/\tilde{J}^4\tilde{P}_{V_1}),\ V_0] = 1$, it follows that $[S_2(X \oplus Y),\ V_0] \neq 0$. Hence, recalling that $[S_2(X),\ V_0] = 0$, we have $[S_2(Y),\ V_0] = 1$. Thus we see that the Loewy series of $Y^{(\mathfrak{G})}$ is one of the following:

(a)	(b)	(c)
$I\ M_1\ M_3\ M_3\ M_4\ M_4$	$I\ M_1\ M_3\ M_3\ M_4\ M_4$	$I\ M_1\ M_3\ M_3\ M_4\ M_4$
I	I	$I\ M_1$
I	$I\ M_1$	I
$M_1\ M_1\ M_1$	$M_1\ M_1$	$M_1\ M_1$

(d)	(e)	(f)
$I\ M_1\ M_3\ M_3\ M_4\ M_4$	$I\ M_1\ M_3\ M_3\ M_4\ M_4$	$I\ M_1\ M_3\ M_3\ M_4\ M_4$
I	$I\ M_1$	$I\ M_1\ M_1$
$I\ M_1\ M_1$	$I\ M_1$	I
M_1	M_1	M_1

If (a) (resp. (b), (c), (d)) were the Loewy series of $Y^{(\mathfrak{G})}$, then we could choose a module

$$\begin{array}{c} I \\ M_1\ M_1\ M_1 \end{array} \left(\text{resp.}\ \begin{array}{c} I \\ I\ M_1 \\ M_1\ M_1 \end{array},\ \begin{array}{c} I \\ M_1\ M_1 \end{array},\ \begin{array}{c} I \\ I\ M_1\ M_1 \\ M_1 \end{array} \right)$$

among the submodules of $Y^{(\sigma)}$. Each of these modules is a homomorphic image of P_I because its head is isomorphic to I. But by Proposition 3.11, we see that P_I does not have such a homomorphic image. This contradiction shows that the Loewy series of $Y^{(\sigma)}$ is (e) or (f), so that we have $[L_2(Y^{(\sigma)}), M_1] \neq 0$. Since $Y^{(\sigma)}$ is a homomorphic image of $(\tilde{J}^2\tilde{P}_{V_1})^{(\sigma)}$ and $(\tilde{J}^2\tilde{P}_{V_1})^{(\sigma)} \simeq J^2P_{M_1}$, the above yields

(4.4) $$[L_4(P_{M_1}), M_1] \neq 0.$$

Thus (1) follows from (4.3) and (4.4).

(2) From the proof of (1), it follows that $[L_3(X^{(\sigma)}), I] = 1$, and so $[L_5(P_{M_1}), I] \neq 0$. Since $L_1(J\hat{I}^{(\sigma)}) \simeq M_1$ (Lemma 3.2), we see that $J\hat{I}^{(\sigma)}$ is a homomorphic image of P_{M_1}, and so $L_5(P_{M_1}) \supset L_5(J\hat{I}^{(\sigma)}) \simeq M_2$. Thus we have $L_5(P_{M_1}) \supset I \oplus M_2$, proving (2).

(3) In the proof of (1), we have shown that $[L_4(Y^{(\sigma)}), M_1] = 1$. Hence, clearly $[L_6(P_{M_1}), M_1] \neq 0$, and (3) is proved.

Now, by making use of an argument similar to the one used in the proof of the preceding lemma, we shall prove the following lemmas.

Lemma 4.5. (1) $L_3(P_{M_2}) \supset I \oplus M_1 \oplus M_2 \oplus M_3 \oplus M_4$.

(2) $L_5(P_{M_2}) \supset M_1$.

Proof. In view of Proposition 2.3, we see that $\tilde{J}\tilde{P}_{V_4}/\tilde{J}^3\tilde{P}_{V_4}$ has a factor module

$$X = \begin{array}{c} V_0\ V_7\ V_{12} \\ V_6\ V_9\ V_{10} \end{array}.$$

Noting that $[S_2(\tilde{P}_{V_i}), V_0] = 0$ for $i = 6, 9, 10$, we have

$$X = V_0 \oplus \begin{array}{c} V_7\ V_{12} \\ V_6\ V_9\ V_{12} \end{array}, \text{ and so } X^{(\sigma)} = \begin{array}{c} I \\ I \\ I \end{array} \oplus \begin{array}{c} M_3\ M_4 \\ M_2\ M_3\ M_4 \end{array}.$$

Since $(\tilde{J}\tilde{P}_{V_4})^{(\sigma)} \simeq JP_{M_2}$, the above implies that

(4.6) $$L_3(P_{M_2}) \supset L_2(X^{(\sigma)}) \simeq I \oplus M_2 \oplus M_3 \oplus M_4.$$

$\tilde{J}\tilde{P}_{V_4}/\tilde{J}^3\tilde{P}_{V_4}$ has another factor module

$$Y = \frac{V_0\ V_7\ V_{12}}{V_1\ V_2\ V_3}.$$

As in the proof of Lemma 4.2, we see that there exists a $K\mathfrak{H}$-monomorphism

$$\tilde{J}\tilde{P}_{V_4}/\tilde{J}^3\tilde{P}_{V_4} \to X \oplus Y.$$

Hence $[S_2(Y),\ V_0] = 1$ because $[S_2(X),\ V_0] = 0$. Therefore the Loewy series of $Y^{\mathfrak{G}}$ is one of the following:

(a)	(b)	(c)	(d)	(e)	(f)
$I\ M_3\ M_4$	$I\ M_3\ M_4$	$I\ M_3\ M_4$	$I\ M_3\ M_4$	$I\ M_3\ M_4$	$I\ M_3\ M_4$
I	I	$I\ M_1$	I	$I\ M_1$	$I\ M_1\ M_1$
I	$I\ M_1$	I	$I\ M_1\ M_1$	$I\ M_1$	I
$M_1\ M_1\ M_1$	$M_1\ M_1$	$M_1\ M_1$	M_1	M_1	M_1

But in view of Proposition 3.11, we can see that the Loewy series of $Y^{\mathfrak{G}}$ must be (e) or (f). Hence we obtain

(4.7) $$L_3(P_{M_2}) \supset L_2(Y^{\mathfrak{G}}) \supset M_1.$$

Thus (1) follows from (4.6) and (4.7). Further we have $L_5(P_{M_2}) \supset L_4(Y^{\mathfrak{G}}) \simeq M_1$, proving (2).

Lemma 4.8. (1) $L_5(P_{M_3}) \supset I \oplus I \oplus M_2 \oplus M_3 \oplus M_4$.
(2) $L_7(P_{M_3}) \supset M_1$.

Proof. By Proposition 2.3, $\tilde{J}^3\tilde{P}_{V_7}/\tilde{J}^5\tilde{P}_{V_7}$ has a factor module

$$X = \frac{V_0\ V_3\ V_5\ V_6\ V_7\ V_{11}\ V_{12}}{V_0\ V_5\ V_9\ V_{10}}.$$

Since $[S_2(\tilde{P}_{V_i}),\ V_0] = 0$ for $i = 0, 5, 9, 10$, we have

$$X = V_0 \oplus \frac{V_3\ V_5\ V_6\ V_7\ V_{11}\ V_{12}}{V_0\ V_5\ V_9\ V_{10}}$$

and so

$$X^{\mathfrak{G}} = \begin{matrix} I \\ I \\ I \end{matrix} \oplus \begin{matrix} M_1\ M_2\ M_2\ M_3\ M_4\ M_4 \\ I\ M_2\ M_3\ M_4 \\ I \\ I \end{matrix}.$$

Hence noting that $(\tilde{J}^3\tilde{P}_{V_7})^{\mathfrak{G}} \simeq J^3P_{M_3}$, we obtain

$$L_5(P_{M_3}) \supsetneq L_2(X^{\mathfrak{G}}) \simeq I \oplus I \oplus M_2 \oplus M_3 \oplus M_4,$$

proving (1).

(2) By Proposition 2.3, $\tilde{J}^3\tilde{P}_{V_7}/\tilde{J}^5\tilde{P}_{V_7}$ has another factor module

$$Y = \begin{matrix} V_0\ V_3\ V_5\ V_6\ V_7\ V_{11}\ V_{12} \\ V_2\ V_3 \end{matrix}.$$

Since there exists a $K\mathfrak{H}$-monomorphism

$$\tilde{J}^3\tilde{P}_{V_7}/\tilde{J}^5\tilde{P}_{V_7} \to X \oplus Y,$$

we see that $[S_2(Y), V_0] = 1$. Therefore the Loewy series of $Y^{\mathfrak{G}}$ is one of the following:

(a)	(b)	(c)
$I\,M_1\,M_2\,M_2\,M_3\,M_4\,M_4$	$I\,M_1\,M_2\,M_2\,M_3\,M_4\,M_4$	$I\,M_1\,M_2\,M_2\,M_3\,M_4\,M_4$
I	I	$I\,M_1$
I	$I\,M_1$	I
$M_1\,M_1$	M_1	M_1

Hence $L_7(P_{M_3}) \supsetneq L_4(Y^{\mathfrak{G}}) \supsetneq M_1$, proving (2). (Note: By Proposition 3.11, we see that (a) is not the Loewy series of $Y^{\mathfrak{G}}$. But we do not need this fact for our proof.)

Lemma 4.9. (1) $L_4(P_{M_4}) \supsetneq I \oplus I \oplus M_2 \oplus M_3 \oplus M_3 \oplus M_4$.
(2) $L_6(P_{M_4}) \supsetneq M_1$.

Proof. By Proposition 2.3, $\tilde{J}^2\tilde{P}_{V_{10}}/\tilde{J}^4\tilde{P}_{V_{10}}$ has factor modules

$$X = \begin{matrix} V_0\ V_2\ V_5\ V_6\ V_7\ V_{12} \\ V_0\ V_6\ V_8\ V_9\ V_{10} \end{matrix} \text{ and } Y = \begin{matrix} V_0\ V_2\ V_5\ V_6\ V_7\ V_{12} \\ V_2\ V_3 \end{matrix}.$$

From the fact that $[S_2(\tilde{P}_{V_i}), V_0] = 0$ $(i = 0, 6, 8, 9, 10)$, it follows that

$$X = V_0 \oplus \begin{matrix} V_2\ V_5\ V_6\ V_7\ V_{12} \\ V_0\ V_6\ V_8\ V_9\ V_{10} \end{matrix},$$

and so we obtain

$$X^{\mathfrak{G}} = \begin{matrix} I \\ I \\ I \end{matrix} \oplus \begin{matrix} M_1\,M_2\,M_2\,M_3\,M_4 \\ I\,M_2\,M_3\,M_3\,M_4 \\ I \\ I \end{matrix}.$$

Further from the existence of a $K\mathfrak{H}$-monomorphism

$$\tilde{J}^2\tilde{P}_{V_{10}}/\tilde{J}^4\tilde{P}_{V_{10}} \to X \oplus Y,$$

it follows that $[S_2(Y), V_0] = 1$. Hence the Loewy series of $Y^{\mathfrak{G}}$ is one of the following:

(a)	(b)	(c)
$I\ M_1\ M_2\ M_2\ M_3\ M_4$	$I\ M_1\ M_2\ M_2\ M_3\ M_4$	$I\ M_1\ M_2\ M_2\ M_3\ M_4$
I	I	$I\ M_1$
I	$I\ M_1$	I
$M_1\ M_1$	M_1	M_1

Hence, observing that $(\tilde{J}^2\tilde{P}_{V_{10}})^{\mathfrak{G}} \simeq J^2P_{M_4}$, we obtain

$$L_4(P_{M_4}) \supset L_2(X^{\mathfrak{G}}) \simeq I \oplus I \oplus M_2 \oplus M_3 \oplus M_3 \oplus M_4,$$
$$L_6(P_{M_4}) \supset L_4(Y^{\mathfrak{G}}) \supset M_1.$$

Thus the result follows. (Note: By Proposotion 3.11, we see that (a) is not the Loewy series of $Y^{\mathfrak{G}}$. But we do not need this fact for our proof.)

Since $M_2 \simeq M_1^*$ and $M_4 \simeq M_3^*$, Landrock's lemma (see § 1) together with Proposition 3.11 and Lemmas 4.1, 4.2, 4.5, 4.8, 4.9 implies the following

Corollary 4.10. (1) *$L_5(P_{M_1}) \supset I \oplus M_2 \oplus M_4$, $L_6(P_{M_1}) \supset I \oplus M_1$ and $L_i(P_{M_1}) \supset I$ for $i = 7, 8$.*

(2) *$L_4(P_{M_2}) \supset I \oplus M_2 \oplus M_4$, $L_5(P_{M_2}) \supset I \oplus M_1$, $L_6(P_{M_2}) \supset I \oplus M_2 \oplus M_3$ and $L_7(P_{M_2}) \supset I \oplus M_4$.*

(3) *$L_6(P_{M_3}) \supset I \oplus I$ and $L_7(P_{M_3}) \supset I \oplus M_1$.*

(4) *$L_5(P_{M_4}) \supset I \oplus I \oplus M_4$ and $L_6(P_{M_4}) \supset I \oplus M_1$.*

5. The structure of powers of the radical. As in § 2, we set $\varepsilon = \hat{\mathfrak{B}}$ and denote by ε_i the primitive idempotent in $K\mathfrak{B}$ corresponding to the simple $K\mathfrak{B}$-module V_i for every i, $1 \leqq i \leqq 12$. Then $K\mathfrak{G}\varepsilon$, $K\mathfrak{G}\varepsilon_1$, $K\mathfrak{G}\varepsilon_4$, $K\mathfrak{G}\varepsilon_7$ and $K\mathfrak{G}\varepsilon_{10}$ are the projective indecomposable $K\mathfrak{G}$-modules for which $K\mathfrak{G}\varepsilon/J(K\mathfrak{G})\varepsilon \simeq I$, $K\mathfrak{G}\varepsilon_1/J(K\mathfrak{G})\varepsilon_1 \simeq M_1$, $K\mathfrak{G}\varepsilon_4/J(K\mathfrak{G})\varepsilon_4 \simeq M_2$, $K\mathfrak{G}\varepsilon_7/J(K\mathfrak{G})\varepsilon_7 \simeq M_3$ and $K\mathfrak{G}\varepsilon_{10}/J(K\mathfrak{G})\varepsilon_{10} \simeq M_4$ respectively. So in the remainder of this paper, we set $e_1 = \varepsilon_1$, $e_2 = \varepsilon_4$, $e_3 = \varepsilon_7$ and $e_4 = \varepsilon_{10}$. Then there holds the following:

$$\varepsilon_2 = e_1^{S^2},\ \varepsilon_3 = e_1^S,\ \varepsilon_5 = e_2^{S^2},\ \varepsilon_6 = e_2^S,$$
$$\varepsilon_8 = e_3^{S^2},\ \varepsilon_9 = e_3^S,\ \varepsilon_{11} = e_4^{S^2},\ \varepsilon_{12} = e_4^S.$$

In the subsequent sections, by making use of the fact that

$$\dim e_j J(K\mathfrak{G})^k e_i - \dim e_j J(K\mathfrak{G})^{k+1} e_i = [L_{k+1}(P_{M_i}),\ M_j],$$

we shall determine the remaining part of the Loewy series of P_{M_i}. As preliminary to the proof, in this section, we give the structure of powers of $J(K\mathfrak{G})$, and state a few remarks which are helpful for the calculation of $\dim e_j J(K\mathfrak{G})^k e_i$. From now on, we set $J = J(K\mathfrak{G})$, $\mathfrak{M} = J(K\mathfrak{U})K\mathfrak{G}$ and $\mathfrak{N} = \varepsilon J(K\mathfrak{W})$. Then we have $J = \mathfrak{M}+\mathfrak{N}$ by Lemma 3.1. To begin with we shall prove the following

Lemma 5.1. (1) $\varepsilon J(K\mathfrak{U})\varepsilon = \varepsilon J(K\mathfrak{U})^3\varepsilon = \langle \varepsilon(a-1)\varepsilon,\ \varepsilon(a^2-1)\varepsilon\rangle$ *and* $\varepsilon J(K\mathfrak{U})^4\varepsilon = \varepsilon J(K\mathfrak{U})^6\varepsilon = K\hat{\mathfrak{U}}\varepsilon$.

(2) $\mathfrak{N}\mathfrak{M}\mathfrak{N} = \mathfrak{N}\mathfrak{M}^3\mathfrak{N} = \langle \varepsilon(a-1)\varepsilon,\ \varepsilon(a^2-1)\varepsilon\rangle\hat{\mathfrak{W}}$ and $\mathfrak{N}\mathfrak{M}^4\mathfrak{N} = \mathfrak{N}\mathfrak{M}^6\mathfrak{N} = K\hat{\mathfrak{G}}$.

(3) $\mathfrak{N}^2\mathfrak{M}\mathfrak{N} = \mathfrak{N}\mathfrak{M}\mathfrak{N}^2 = 0$.

(4) $\mathfrak{N}\mathfrak{M}\mathfrak{N}\mathfrak{M}\mathfrak{N} = 0$.

Proof. Since $\dim \varepsilon J(K\mathfrak{U})^i\varepsilon$ is the multiplicity of V_0 as composition factor of $J(K\mathfrak{H})^i P_{V_0}$, (1) follows immediately from Proposition 2.3. Recalling that each element of $\varepsilon K\mathfrak{U}\varepsilon$ commutes with s, by (1), we have

$$\begin{aligned}\mathfrak{N}\mathfrak{M}\mathfrak{N} &= \varepsilon J(K\mathfrak{W})J(K\mathfrak{U})\varepsilon J(K\mathfrak{W}) = J(K\mathfrak{W})(\varepsilon J(K\mathfrak{U})\varepsilon)J(K\mathfrak{W})\\ &= (\varepsilon J(K\mathfrak{U})\varepsilon)J(K\mathfrak{W})^2 = (\varepsilon J(K\mathfrak{U})^3\varepsilon)\hat{\mathfrak{W}}\\ &= \langle \varepsilon(a-1)\varepsilon,\ \varepsilon(a^2-1)\varepsilon\rangle\hat{\mathfrak{W}},\\ \mathfrak{N}\mathfrak{M}^3\mathfrak{N} &= \varepsilon J(K\mathfrak{W})J(K\mathfrak{U})^3\varepsilon J(K\mathfrak{W}) = J(K\mathfrak{W})(\varepsilon J(K\mathfrak{U})^3\varepsilon)J(K\mathfrak{W})\\ &= (\varepsilon J(K\mathfrak{U})^3\varepsilon)J(K\mathfrak{W})^2 = \langle \varepsilon(a-1)\varepsilon,\ \varepsilon(a^2-1)\varepsilon\rangle\hat{\mathfrak{W}}.\end{aligned}$$

Hence, it follows that

$$\mathfrak{N}\mathfrak{M}\mathfrak{N} = \mathfrak{N}\mathfrak{M}^3\mathfrak{N} = \langle \varepsilon(a-1)\varepsilon,\ \varepsilon(a^2-1)\varepsilon\rangle\hat{\mathfrak{W}}.$$

Further, we have

$$\begin{aligned}\mathfrak{N}\mathfrak{M}^4\mathfrak{N} &= \varepsilon J(K\mathfrak{W})J(K\mathfrak{U})^4\varepsilon J(K\mathfrak{W}) = J(K\mathfrak{W})(\varepsilon J(K\mathfrak{U})^4\varepsilon)J(K\mathfrak{W})\\ &= (\varepsilon J(K\mathfrak{U})^4\varepsilon)J(K\mathfrak{W})^2 = (\varepsilon J(K\mathfrak{U})^6\varepsilon)\hat{\mathfrak{W}},\\ \mathfrak{N}\mathfrak{M}^6\mathfrak{N} &= \varepsilon J(K\mathfrak{W})J(K\mathfrak{U})^6\varepsilon J(K\mathfrak{W}) = J(K\mathfrak{W})(\varepsilon J(K\mathfrak{U})^6\varepsilon)J(K\mathfrak{W})\\ &= (\varepsilon J(K\mathfrak{U})^6\varepsilon)J(K\mathfrak{W})^2 = (\varepsilon J(K\mathfrak{U})^6\varepsilon)\hat{\mathfrak{W}}.\end{aligned}$$

Since $J(K\mathfrak{U})^6 = K\hat{\mathfrak{U}}$, the above implies that

$$\mathfrak{N}\mathfrak{M}^4\mathfrak{N} = \mathfrak{N}\mathfrak{M}^6\mathfrak{N} = K\hat{\mathfrak{G}}.$$

Thus (2) is proved. Further, (3) follows immediately from (2) because

$$J(K\mathfrak{W})\varepsilon(a^i-1)\varepsilon\hat{\mathfrak{W}} = \varepsilon(a^i-1)\varepsilon\hat{\mathfrak{W}}J(K\mathfrak{W}) = 0 \qquad \text{for } i = 1, 2.$$

Because of (2) and (3), we have

$$\begin{aligned}\mathfrak{N}\mathfrak{M}\mathfrak{N}\mathfrak{M}\mathfrak{N} &= \langle\varepsilon(a-1)\varepsilon,\ \varepsilon(a^2-1)\varepsilon\rangle\hat{\mathfrak{W}}\mathfrak{M}\mathfrak{N} \\ &= \langle\varepsilon(a-1)\varepsilon,\ \varepsilon(a^2-1)\varepsilon\rangle\varepsilon\hat{\mathfrak{W}}\mathfrak{M}\mathfrak{N} \\ &= \langle\varepsilon(a-1)\varepsilon,\ \varepsilon(a^2-1)\varepsilon\rangle\mathfrak{N}^2\mathfrak{M}\mathfrak{N} = 0,\end{aligned}$$

proving (4). Thus we complete the proof of the lemma.

Using the preceding lemma, we can easily obtain the following

Lemma 5.2. (1) $J^2 = \mathfrak{M}^2+\mathfrak{M}\mathfrak{N}+\mathfrak{N}\mathfrak{M}+\mathfrak{N}^2$.

(2) $J^3 = \mathfrak{M}^3+\mathfrak{M}^2\mathfrak{N}+\mathfrak{M}\mathfrak{N}\mathfrak{M}+\mathfrak{M}\mathfrak{N}^2+\mathfrak{N}\mathfrak{M}^2+\mathfrak{N}^2\mathfrak{M}$.

(3) $J^4 = \mathfrak{M}^4+\mathfrak{M}^3\mathfrak{N}+\mathfrak{M}^2\mathfrak{N}\mathfrak{M}+\mathfrak{M}^2\mathfrak{N}^2+\mathfrak{M}\mathfrak{N}\mathfrak{M}^2+\mathfrak{M}\mathfrak{N}^2\mathfrak{M}+\mathfrak{N}\mathfrak{M}^3$
$+\mathfrak{N}^2\mathfrak{M}^2$.

(4) $J^5 = \mathfrak{M}^5+\mathfrak{M}^4\mathfrak{N}+\mathfrak{M}^3\mathfrak{N}\mathfrak{M}+\mathfrak{M}^3\mathfrak{N}^2+\mathfrak{M}^2\mathfrak{N}\mathfrak{M}^2+\mathfrak{M}^2\mathfrak{N}^2\mathfrak{M}+\mathfrak{M}\mathfrak{N}\mathfrak{M}^3$
$+\mathfrak{M}\mathfrak{N}^2\mathfrak{M}^2+\mathfrak{N}\mathfrak{M}^4+\mathfrak{N}\mathfrak{M}^3\mathfrak{N}+\mathfrak{N}^2\mathfrak{M}^3$.

(5) $J^6 = \mathfrak{M}^6+\mathfrak{M}^5\mathfrak{N}+\mathfrak{M}^4\mathfrak{N}\mathfrak{M}+\mathfrak{M}^4\mathfrak{N}^2+\mathfrak{M}^3\mathfrak{N}\mathfrak{M}^2+\mathfrak{M}^3\mathfrak{N}^2\mathfrak{M}+\mathfrak{M}^2\mathfrak{N}\mathfrak{M}^3$
$+\mathfrak{M}^2\mathfrak{N}^2\mathfrak{M}^2+\mathfrak{M}\mathfrak{N}\mathfrak{M}^4+\mathfrak{M}\mathfrak{N}^2\mathfrak{M}^3+\mathfrak{N}\mathfrak{M}^5+\mathfrak{N}^2\mathfrak{M}^4$.

(6) $J^7 = \mathfrak{M}^6\mathfrak{N}+\mathfrak{M}^5\mathfrak{N}\mathfrak{M}+\mathfrak{M}^5\mathfrak{N}^2+\mathfrak{M}^4\mathfrak{N}\mathfrak{M}^2+\mathfrak{M}^4\mathfrak{N}^2\mathfrak{M}+\mathfrak{M}^3\mathfrak{N}\mathfrak{M}^3$
$+\mathfrak{M}^3\mathfrak{N}^2\mathfrak{M}^2+\mathfrak{M}^2\mathfrak{N}\mathfrak{M}^4+\mathfrak{M}^2\mathfrak{N}^2\mathfrak{M}^3+\mathfrak{M}\mathfrak{N}\mathfrak{M}^5+\mathfrak{M}\mathfrak{N}^2\mathfrak{M}^4+\mathfrak{N}^2\mathfrak{M}^5$.

(7) $J^8 = \mathfrak{M}^5\mathfrak{N}^2\mathfrak{M}+\mathfrak{M}^4\mathfrak{N}^2\mathfrak{M}^2+\mathfrak{M}^3\mathfrak{N}^2\mathfrak{M}^3+\mathfrak{M}^2\mathfrak{N}^2\mathfrak{M}^4+\mathfrak{M}\mathfrak{N}^2\mathfrak{M}^5$.

(8) $J^9 = 0$.

We close this section with two lemmas which are helpful for the calculation in the remainder of this paper.

Lemma 5.3. *For $n = 1, 2$, it holds that*

$$e_k a^n \varepsilon = e_k\left(\textstyle\sum_{l=0}^{12} \xi_{3k-2}(\nu)^{-l}(a^n)^{\nu^l}\right), \qquad 1 \leqq k \leqq 4.$$

Proof. Since $e_k = \varepsilon_{3k-2}$, we have

$$\begin{aligned}e_k a^n \varepsilon &= e_k a^n\left(\textstyle\sum_{l=0}^{12} \nu^l\right) = \textstyle\sum_{l=0}^{12} e_k \nu^l (a^n)^{\nu^{-l}} \\ &= \textstyle\sum_{l=0}^{12} \xi_{3k-2}(\nu)^l e_k (a^n)^{\nu^{-l}} \\ &= e_k\left(\textstyle\sum_{l=0}^{12} \xi_{3k-2}(\nu)^l (a^n)^{\nu^{-l}}\right) \\ &= e_k\left(\textstyle\sum_{l=0}^{12} \xi_{3k-2}(\nu)^{-l} (a^n)^{\nu^l}\right).\end{aligned}$$

Thus the lemma is proved.

Lemma 5.4. *Let k be a positive integer. Then for every i, $1 \leqq i \leqq 4$, there holds that*

(1) $\dim e_i J(K\mathfrak{U})^k \varepsilon \leqq 2$ *and* $\dim \varepsilon J(K\mathfrak{U})^k e_i \leqq 2$.

(2) *If* $\dim e_i J(K\mathfrak{U})^k \varepsilon = 2$ *then* $e_i J(K\mathfrak{U})^k \varepsilon = \langle e_i a\varepsilon, e_i a^2 \varepsilon \rangle$.

(3) *If* $\dim \varepsilon J(K\mathfrak{U})^k e_i = 2$ *then* $\varepsilon J(K\mathfrak{U})^k e_i = \langle \varepsilon a e_i, \varepsilon a^2 e_i \rangle$.

Proof. (1) is a direct consequence of Corollary 2.2. If $\dim e_i J(K\mathfrak{U})^k \varepsilon = 2$, then $e_i J(K\mathfrak{U})^k \varepsilon = e_i K\mathfrak{U}\varepsilon$, and so (2) follows from Lemma 2.1. (3) is also obtained similarly.

We use the last lemma without any citation in our subsequent study.

6. The Loewy structure of P_{M_1}. In this section, we shall determine the Loewy structure of P_{M_1}. The following lemmas are obtained directly from Proposition 2.3.

Lemma 6.1. (1) $e_1 J(K\mathfrak{U})^2 \varepsilon = e_1 J(K\mathfrak{U})^4 \varepsilon$, $e_1 J(K\mathfrak{U})^5 \varepsilon = 0$.

(2) $e_2 J(K\mathfrak{U}) \varepsilon = e_2 J(K\mathfrak{U})^2 \varepsilon$, $e_2 J(K\mathfrak{U})^3 \varepsilon = e_2 J(K\mathfrak{U})^5 \varepsilon$.

(3) $e_3 J(K\mathfrak{U}) \varepsilon = e_3 J(K\mathfrak{U})^2 \varepsilon$, $e_3 J(K\mathfrak{U})^4 \varepsilon = 0$.

(4) $e_4 J(K\mathfrak{U}) \varepsilon = e_4 J(K\mathfrak{U})^3 \varepsilon$, $e_4 J(K\mathfrak{U})^5 \varepsilon = 0$.

Lemma 6.2. $\varepsilon J(K\mathfrak{U}) e_1 = \varepsilon J(K\mathfrak{U})^2 e_1$, $\varepsilon J(K\mathfrak{U})^3 e_1 = \varepsilon J(K\mathfrak{U})^5 e_1$.

Using Lemmas 5.2, 6.1 and 6.2, we prove the following

Lemma 6.3. (1) $e_1 J^4 e_1 = e_1 \mathfrak{M}^4 e_1 + e_1 \mathfrak{M}\mathfrak{N}\mathfrak{M}^2 e_1$.

(2) $e_1 J^5 e_1 = e_1 \mathfrak{M}^5 e_1 + e_1 \mathfrak{M}\mathfrak{N}^2 \mathfrak{M}^2 e_1$.

(3) $e_2 J^5 e_1 = e_2 \mathfrak{M}^5 e_1 + e_2 \mathfrak{M}^2 \mathfrak{N}\mathfrak{M}^2 e_1$.

(4) $e_2 J^6 e_1 = e_2 \mathfrak{M}^2 \mathfrak{N}^2 \mathfrak{M}^2 e_1$.

(5) $e_3 J^6 e_1 = e_3 \mathfrak{M}^3 \mathfrak{N}\mathfrak{M}^2 e_1 + e_3 \mathfrak{M}^2 \mathfrak{N}^2 \mathfrak{M}^2 e_1$.

(6) $e_3 J^7 e_1 = e_3 \mathfrak{M}^3 \mathfrak{N}^2 \mathfrak{M}^2 e_1$.

(7) $e_4 J^7 e_1 = e_4 \mathfrak{M}^3 \mathfrak{N}^2 \mathfrak{M}^2 e_1$.

Proof. (1) Since $e_1 \varepsilon = \varepsilon e_1 = 0$, we have $e_1 \mathfrak{N} = \mathfrak{N} e_1 = 0$. Hence, from Lemma 5.2, it follows that

$$e_1 J^4 e_1 = e_1 \mathfrak{M}^4 e_1 + e_1 \mathfrak{M}^2 \mathfrak{N}\mathfrak{M} e_1 + e_1 \mathfrak{M}\mathfrak{N}\mathfrak{M}^2 e_1.$$

Therefore, observing that

$$\begin{aligned} e_1\mathfrak{M}^2\mathfrak{N}\mathfrak{M}e_1 &= e_1J(K\mathfrak{U})^2\varepsilon J(K\mathfrak{W})J(K\mathfrak{U})e_1 \\ &= (e_1J(K\mathfrak{U})^2\varepsilon)J(K\mathfrak{W})(\varepsilon J(K\mathfrak{U})e_1) \\ &= (e_1J(K\mathfrak{U})^4\varepsilon)J(K\mathfrak{W})(\varepsilon J(K\mathfrak{U})^2e_1) \\ &= e_1J(K\mathfrak{U})^4\varepsilon J(K\mathfrak{W})J(K\mathfrak{U})^2e_1 \\ &= e_1\mathfrak{M}^4\mathfrak{N}\mathfrak{M}^2e_1 \subset e_1\mathfrak{M}^4e_1, \end{aligned}$$

we obtain $e_1J^4e_1 = e_1\mathfrak{M}^4e_1 + e_1\mathfrak{M}\mathfrak{N}\mathfrak{M}^2e_1$, proving (1).

(2), (3), (6) and (7) are also obtained similarly, and we omit the poof. Further, for $i = 2, 3$, we have

$$\begin{aligned} e_i\mathfrak{M}^6e_1 &= e_i\hat{\mathfrak{U}}K\mathfrak{G}e_1 = \hat{\mathfrak{U}}e_iK\mathfrak{W}e_1 \\ &= \hat{\mathfrak{U}}(Ke_ie_1 + Ke_ie_1^s s + Ke_ie_1^{s^2}s^2) = 0. \end{aligned}$$

Noting this fact, we also obtain (4) and (5) from Lemmas 5.2, 6.1 and 6.2.

Lemma 6.4. $[L_5(P_{M1}), M_1] = 2$.

Proof. To our end, it suffices to prove that $\dim e_1J^4e_1 = 4$ and $\dim e_1J^5e_1 = 2$. At first we prove $\dim e_1J^5e_1 = 2$. By Lemma 6.3,

$$e_1J^5e_1 = e_1\mathfrak{M}^5e_1 + e_1\mathfrak{M}\mathfrak{N}^2\mathfrak{M}^2e_1.$$

Since $e_1J(K\mathfrak{U})^5e_1^{s^i} = 0$ for $i = 1, 2$ (Proposition 2.3), we have

$$\begin{aligned} e_1\mathfrak{M}^5e_1 &= e_1J(K\mathfrak{U})^5K\mathfrak{G}e_1 = e_1J(K\mathfrak{U})^5K\mathfrak{W}e_1 \\ &= e_1J(K\mathfrak{U})^5e_1 + e_1J(K\mathfrak{U})^5e_1^s s + e_1J(K\mathfrak{U})^5e_1^{s^2}s^2 \\ &= e_1J(K\mathfrak{U})^5e_1. \end{aligned}$$

Thus we obtain $e_1\mathfrak{M}^5e_1 = K\hat{\mathfrak{U}}e_1$, because $\dim e_1J(K\mathfrak{U})^5e_1 = 1$ by Proposition 2.3. Further, from the equality

$$\begin{aligned} e_1\mathfrak{M}\mathfrak{N}^2\mathfrak{M}^2e_1 &= e_1J(K\mathfrak{U})\varepsilon\hat{\mathfrak{W}}J(K\mathfrak{U})^2e_1 \\ &= (e_1J(K\mathfrak{U})\varepsilon)\hat{\mathfrak{W}}(\varepsilon J(K\mathfrak{U})^2e_1) \\ &= \langle e_1a\varepsilon,\ e_1a^2\varepsilon\rangle\hat{\mathfrak{W}}\langle\varepsilon ae_1,\ \varepsilon a^2e_1\rangle, \end{aligned}$$

it follows that the vector space $e_1\mathfrak{M}\mathfrak{N}^2\mathfrak{M}^2e_1$ over K is spanned by the elements:

$$\begin{aligned} &x_1 = e_1a\varepsilon\hat{\mathfrak{W}}\varepsilon ae_1,\ x_2 = e_1a\varepsilon\hat{\mathfrak{W}}\varepsilon a^2e_1,\ x_3 = e_1a^2\varepsilon\hat{\mathfrak{W}}\varepsilon ae_1, \\ &x_4 = e_1a^2\varepsilon\hat{\mathfrak{W}}\varepsilon a^2e_1. \end{aligned}$$

Therefore $e_1J^5e_1$ is spanned by the elements $\hat{\mathfrak{U}}e_1$, x_1, x_2, x_3, and x_4. Since $\zeta^3 = \zeta^2+\zeta+1$, calculating by taking advantage of Lemmas 2.1 and 5.3, we have

$$(6.5)\qquad \begin{cases} e_1 a\varepsilon a e_1 = -e_1 a^2 \varepsilon a^2 e_1 = -e_1 a e_1 + e_1 a^2 e_1, \\ e_1 a \varepsilon a^2 e_1 = e_1 - e_1 a^2 e_1, \\ e_1 a^2 \varepsilon a e_1 = e_1 - e_1 a e_1, \\ e_1 a \varepsilon a e_1^s = e_1 a \varepsilon a^2 e_1^s = -e_1 a^2 \varepsilon a e_1^s = -e_1 a^2 \varepsilon a^2 e_1^s \\ \qquad = e_1 a e_1^s - e_1 a^2 e_1^s, \\ e_1 a \varepsilon a e_1^{s^2} = e_1 a \varepsilon a^2 e_1^{s^2} = -e_1 a^2 \varepsilon a e_1^{s^2} = -e_1 a^2 \varepsilon a^2 e_1^{s^2} \\ \qquad = e_1 a e_1^{s^2} - e_1 a^2 e_1^{s^2}. \end{cases}$$

Hence, noting that

$$e_1 a^i \varepsilon \hat{\mathfrak{W}} \varepsilon a^j e_1 = e_1 a^i \varepsilon \hat{\mathfrak{W}} a^j e_1 = \textstyle\sum_{k=0}^{2} e_1 a^i \varepsilon a^j e_1^{s^k} s^k, \qquad 1 \leqq i,\ j \leqq 2,$$

and

$$e_1 + e_1 a e_1 + e_1 a^2 e_1 = (1 + \hat{C}_a + \hat{C}_{a^2}) e_1 = \hat{\mathfrak{U}} e_1,$$

we obtain $e_1 J^5 e_1 = \langle \hat{\mathfrak{U}} e_1, x_1 \rangle$, and so $\dim e_1 J^5 e_1 = 2$. Next, we prove $\dim e_1 J^4 e_1 = 4$. By Lemma 6.3,

$$e_1 J^4 e_1 = e_1 \mathfrak{M}^4 e_1 + e_1 \mathfrak{M} \mathfrak{N} \mathfrak{M}^2 e_1.$$

We now claim that

$$e_1 \mathfrak{M}^4 e_1 = \langle \hat{\mathfrak{U}} e_1,\ (e_1 a e_1^{s^2} + e_1 a^2 e_1^{s^2}) s^2 \rangle.$$

Indeed,

$$\begin{aligned} e_1 \mathfrak{M}^4 e_1 &= e_1 J (K\mathfrak{U})^4 e_1 K \mathfrak{W} e_1 \\ &= e_1 J (K\mathfrak{U})^4 e_1 + e_1 J (K\mathfrak{U})^4 e_1^s s + e_1 J (K\mathfrak{U})^4 e_1^{s^2} s^2 \\ &= e_1 J (K\mathfrak{U})^4 e_1 + e_1 J (K\mathfrak{U})^4 e_1^{s^2} s^2, \end{aligned}$$

because $e_1 J (K\mathfrak{U})^4 e_1^s = 0$ by Proposition 2.3. Further as $\dim e_1 J (K\mathfrak{U})^4 e_1 = 1$ (Proposition 2.3), we have $e_1 J (K\mathfrak{U})^4 e_1 = K \hat{\mathfrak{U}} e_1$. Calculating with the aid of Lemma 2.1, we see that

$$e_1 (a-1)^2 (b-1)^2 e_1^{s^2} = (-\zeta^2 + \zeta + 1)(e_1 a e_1^{s^2} + e_1 a^2 e_1^{s^2})$$

is a non zero element of $e_1 J (K\mathfrak{U})^4 e_1^{s^2}$. Hence, noting that $\dim e_1 J (K\mathfrak{U})^4 e_1^{s^2} = 1$ (Proposition 2.3), we obtain

$$e_1 J (K\mathfrak{U})^4 e_1^{s^2} s^2 = \langle (e_1 a e_1^{s^2} + e_1 a^2 e_1^{s^2}) s^2 \rangle.$$

Thus we conclude

$$e_1 \mathfrak{M}^4 e_1 = \langle \hat{\mathfrak{U}} e_1,\ (e_1 a e_1^{s^2} + e_1 a^2 e_1^{s^2}) s^2 \rangle.$$

Now observing that

$$\begin{aligned} e_1J^4e_1 &\supset e_1\mathfrak{M}\mathfrak{N}\mathfrak{M}^2e_1 \\ &= (e_1J(K\mathfrak{U})\varepsilon)J(K\mathfrak{W})(\varepsilon J(K\mathfrak{U})^2e_1) \\ &= \langle e_1a\varepsilon,\ e_1a^2\varepsilon\rangle J(K\mathfrak{W})\langle \varepsilon ae_1,\ \varepsilon a^2e_1\rangle, \end{aligned}$$

we see that $e_1J^4e_1$ contains the following two elements:

$$\begin{aligned} y_1 &= e_1a\varepsilon(1-s)\varepsilon ae_1 = e_1a\varepsilon ae_1 - e_1a\varepsilon ae_1^s s, \\ y_2 &= e_1a\varepsilon(1-s^2)\varepsilon ae_1 = e_1a\varepsilon ae_1 - e_1a\varepsilon ae_1^{s^2}s^2. \end{aligned}$$

From (6.5), it follows that $\hat{\mathfrak{U}}e_1$, $(e_1ae_1^{s^2}+e_1a^2e_1^{s^2})s^2$, y_1 and y_2 are linearly independent, and so we have $\dim e_1J^4e_1 \geqq 4$. Since $L_1(P_{M_1}) \simeq M_1$, we have $[P_{M_1}/J^4P_{M_1},\ M_1] \geqq 3$ by Lemmas 4.1 and 4.2. Therefore, by Corollary 2.6, we obtain $\dim e_1J^4e_1 = 4$, and the lemma is established.

Landrock's lemma together with the preceding lemma implies the following

Corollary 6.6. $[L_5(P_{M_2}),\ M_2] = 2$.

Lemma 6.7. $[L_6(P_{M_1}),\ M_2] = [L_7(P_{M_1}),\ M_2] = 1$.

Proof. At first we prove that $[L_6(P_{M_1}),\ M_2] = 1$. To prove this it suffices to show that $\dim e_2J^5e_1 = 2$ and $\dim e_2J^6e_1 = 1$. By Lemma 6.3, we have

$$\begin{aligned} e_2J^6e_1 = e_2\mathfrak{M}^2\mathfrak{N}^2\mathfrak{M}^2e_1 &= (e_2J(K\mathfrak{U})^2\varepsilon)\hat{\mathfrak{W}}(\varepsilon J(K\mathfrak{U})^2e_1) \\ &= \langle e_2a\varepsilon,\ e_2a^2\varepsilon\rangle\hat{\mathfrak{W}}\langle\varepsilon ae_1,\ \varepsilon a^2e_1\rangle. \end{aligned}$$

Calculating by taking advantage of Lemmas 2.1 and 5.3, we have

$$\text{(6.8)}\qquad \begin{cases} e_2a^i\varepsilon a^je_1 = -e_2ae_1 - e_2a^2e_1, \\ e_2a^i\varepsilon a^je_1^s = e_2ae_1^s + e_2a^2e_1^s, \\ e_2a^i\varepsilon a^je_1^{s^2} = e_2ae_1^{s^2} + e_2a^2e_1^{s^2}. \end{cases} \qquad (1 \leqq i,\ j \leqq 2)$$

Hence from the equality

$$e_2a^i\varepsilon\hat{\mathfrak{W}}\varepsilon a^je_1 = e_2a^i\varepsilon\hat{\mathfrak{W}}a^je_1 = \textstyle\sum_{k=0}^{2} e_2a^i\varepsilon a^je_1^{s^k}s^k,$$

we obtain $e_2J^6e_1 = \langle e_2a\varepsilon\hat{\mathfrak{W}}ae_1\rangle$, and so $\dim e_2J^6e_1 = 1$. Now we prove that $\dim e_2J^5e_1 = 2$. By Lemma 6.3,

$$e_2J^5e_1 = e_2\mathfrak{M}^5e_1 + e_2\mathfrak{M}^2\mathfrak{N}\mathfrak{M}^2e_1.$$

Noting that $e_2J(K\mathfrak{U})^5e_1^{s^i} = 0$ for $i = 0,\ 1,\ 2$ (Proposition 2.3), we have

$$e_2\mathfrak{M}^5e_1 = e_2J(K\mathfrak{U})^5e_1 + e_2J(K\mathfrak{U})^5e_1^s s + e_2J(K\mathfrak{U})^5e_1^{s^2}s^2 = 0.$$

Hence it follows that

$$\begin{aligned} e_2J^5e_1 &= e_2\mathfrak{M}^2\mathfrak{N}\mathfrak{M}^2e_1 \\ &= (e_2J(K\mathfrak{U})^2\varepsilon)J(K\mathfrak{W})(\varepsilon J(K\mathfrak{U})^2e_1) \\ &= \langle e_2a\varepsilon,\ e_2a^2\varepsilon\rangle J(K\mathfrak{W})\langle \varepsilon ae_1,\ \varepsilon a^2e_1\rangle. \end{aligned}$$

Since

$$e_2a^i\varepsilon(1-s^k)\varepsilon a^je_1 = e_2a^i\varepsilon a^je_1 - e_2a^i\varepsilon a^je_1^{s^k}s^k, \qquad 1 \leqq i,\ j,\ k \leqq 2,$$

from (6.8), it follows that

$$e_2J^5e_1 = \langle e_2a\varepsilon(1-s)\varepsilon ae_1,\ e_2a\varepsilon(1-s^2)\varepsilon ae_1\rangle,$$

and so dim $e_2J^5e_1 = 2$. Thus we obtain $[L_6(P_{M_1}),\ M_2] = 1$. Next, recalling that $L_8(P_{M_1}) \supsetneq I$ (Corollary 4.10) and $J^9 = 0$, we see that the Loewy length of P_{M_1} is 9. Hence we have

$$\begin{aligned} L_8(P_{M_1}) &\subsetneq S_2(P_{M_1}) \\ &\simeq L_2(P_{M_2})^* \qquad ([4,\ \text{Chap. I, Lemma 8.4}]) \\ &\simeq I \oplus M_3 \oplus M_4 \qquad (\text{Lemma 4.1}). \end{aligned}$$

Since $L_9(P_{M_1}) \simeq M_1$, the above implies that $[J^7P_{M_1},\ M_2] = 0$. Thus we obtain $[L_7(P_{M_1}),\ M_2] = 1$ from the fact that dim $e_2J^6e_1 = 1$, and the lemma is proved.

Lemma 6.9. $[L_7(P_{M_1}),\ M_3] = [L_8(P_{M_1}),\ M_3] = 1$.

Proof. Since the Loewy length of P_{M_1} is 9, and $L_9(P_{M_1}) \simeq M_1$, in order to prove the lemma, it suffices to show that dim $e_3J^6e_1 = 2$ and dim $e_3J^7e_1 = 1$. At first we prove that dim $e_3J^6e_1 = 2$. By Lemma 6.3,

$$e_3J^6e_1 = e_3\mathfrak{M}^3\mathfrak{N}\mathfrak{M}^2e_1 + e_3\mathfrak{M}^2\mathfrak{N}^2\mathfrak{M}^2e_1.$$

Noting that dim $e_3J(K\mathfrak{U})^3\varepsilon = 1$ (Proposition 2.3) and

$$\begin{aligned} e_3J(K\mathfrak{U})^3\varepsilon &\ni e_3(a-1)^2(b-1)\varepsilon \\ &= (\zeta^2+1)(e_3a\varepsilon - e_3a^2\varepsilon) \neq 0, \end{aligned}$$

we obtain

$$(6.10) \qquad e_3J(K\mathfrak{U})^3\varepsilon = \langle e_3(a-a^2)\varepsilon\rangle.$$

Hence we have

$$\begin{aligned} e_3\mathfrak{M}^3\mathfrak{N}\mathfrak{M}^2e_1 &= (e_3J(K\mathfrak{U})^3\varepsilon)J(K\mathfrak{W})(\varepsilon J(K\mathfrak{U})^2e_1) \\ &= \langle e_3(a-a^2)\varepsilon\rangle J(K\mathfrak{W})\langle \varepsilon ae_1,\ \varepsilon a^2e_1\rangle. \end{aligned}$$

This implies that $e_3\mathfrak{M}^3\mathfrak{N}\mathfrak{M}^2e_1$ contains an element

$$e_3(a-a^2)\varepsilon(1-s^2)\varepsilon ae_1.$$

Since $e_3J(K\mathfrak{U})^5e_1 = 0$ (Proposition 2.3), we have

$$\begin{aligned}(e_3(a-a^2)\varepsilon)(\varepsilon ae_1) &\in (e_3J(K\mathfrak{U})^3\varepsilon)(\varepsilon J(K\mathfrak{U})^2e_1)\\ &\subset e_3J(K\mathfrak{U})^5e_1 = 0.\end{aligned}$$

From this it follows that

$$\begin{aligned}e_3(a-a^2)\varepsilon(1-s^2)\varepsilon ae_1 &= -e_3(a-a^2)\varepsilon s^2\varepsilon ae_1\\ &= (-e_3a\varepsilon ae_1^{s^2}+e_3a^2\varepsilon ae_1^{s^2})s^2.\end{aligned}$$

Further calculating with the aid of Lemmas 2.1 and 5.3, we obtain

$$e_3a\varepsilon ae_1^{s^2} = -e_3a^2\varepsilon ae_1^{s^2} = e_3ae_1^{s^2}-e_3a^2e_1^{s^2} \neq 0. \tag{6.11}$$

Accordingly we have

$$e_3(a-a^2)\varepsilon(1-s^2)\varepsilon ae_1 = e_3a\varepsilon ae_1^{s^2}s^2,$$

and so

$$e_3J^6e_1 \ni e_3a\varepsilon ae_1^{s^2}s^2. \tag{6.12}$$

Since

$$\begin{aligned}e_3\mathfrak{M}^2\mathfrak{N}^2\mathfrak{M}^2e_1 &= (e_3J(K\mathfrak{U})^2\varepsilon)\hat{\mathfrak{W}}(\varepsilon J(K\mathfrak{U})^2e_1)\\ &= \langle e_3a\varepsilon,\ e_3a^2\varepsilon\rangle\hat{\mathfrak{W}}\langle\varepsilon ae_1,\ \varepsilon a^2e_1\rangle,\end{aligned}$$

we see that $e_3a\varepsilon\hat{\mathfrak{W}}\varepsilon ae_1$ is an element of $e_3\mathfrak{M}^2\mathfrak{N}^2\mathfrak{M}^2e_1$. Noting that $e_3J(K\mathfrak{U})^4e_1^s = 0$ (Proposition 2.3), we have

$$e_3a\varepsilon\hat{\mathfrak{W}}\varepsilon ae_1 = e_3a\varepsilon\hat{\mathfrak{W}}ae_1 = e_3a\varepsilon ae_1+e_3a\varepsilon ae_1^{s^2}s^2.$$

We see, by calculation, that

$$e_3a\varepsilon ae_1 = -(e_3ae_1+e_3a^2e_1) \neq 0.$$

Hence, from the above and (6.12), it follows that

$$e_3J^6e_1 \supset \langle e_3a\varepsilon ae_1,\ e_3a\varepsilon ae_1^{s^2}s^2\rangle.$$

On the other hand, from Lemmas 4.1 and 4.2, it follows that

$$[P_{M_1}/J^4P_{M_1},\ M_3] \geqq 4.$$

Hence we conclude that dim $e_3J^6e_1 = 2$ by Corollary 2.6. Next we prove dim $e_3J^7e_1 = 1$. By Lemma 6.3 and (6.10), we have

$$\begin{aligned} e_3J^7e_1 &= e_3\mathfrak{M}^3\mathfrak{N}^2\mathfrak{M}^2e_1 \\ &= (e_3J(K\mathfrak{U})^3\varepsilon)\hat{\mathfrak{W}}(\varepsilon J(K\mathfrak{U})^2e_1) \\ &= \langle e_3(a-a^2)\varepsilon\rangle\hat{\mathfrak{W}}\langle\varepsilon ae_1,\ \varepsilon a^2e_1\rangle. \end{aligned}$$

Hence $e_3(a-a^2)\varepsilon\hat{\mathfrak{W}}\varepsilon ae_1$ is an element of $e_3J^7e_1$. Since $e_3J(K\mathfrak{U})^5e_1^{s^i} = 0$ for $i = 0, 1$ (Proposition 2.3), by (6.11), we have

$$\begin{aligned} e_3(a-a^2)\varepsilon\hat{\mathfrak{W}}\varepsilon ae_1 &= (e_3a\varepsilon ae_1^{s^2} - e_3a^2\varepsilon ae_1^{s^2})s^2 \\ &= -e_3a\varepsilon ae_1^{s^2}s^2 \neq 0. \end{aligned}$$

Hence $e_3J^7e_1 \neq 0$. On the other hand, we have

$$L_8(P_{M_1}) \subsetneq S_2(P_{M_1}) \simeq L_2(P_{M_2})^* \simeq I \oplus M_3 \oplus M_4.$$

This shows that dim $e_3J^7e_1 \leqq 1$, because $L_9(P_{M_1}) \simeq M_1$. Thus we conclude that dim $e_3J^7e_1 = 1$, and the result follows.

Landrock's lemma together with the preceding lemma implies the following

Corollary 6.13. $[L_7(P_{M_4}), M_2] = [L_8(P_{M_4}), M_2] = 1$.

Lemma 6.14. $[L_8(P_{M_1}), M_4] = 1$.

Proof. By Lemma 6.3, we have

$$\begin{aligned} e_4J^7e_1 = e_4\mathfrak{M}^3\mathfrak{N}^2\mathfrak{M}^2e_1 &= (e_4J(K\mathfrak{U})^3\varepsilon)\hat{\mathfrak{W}}(\varepsilon J(K\mathfrak{U})^2e_1) \\ &= \langle e_4a\varepsilon,\ e_4a^2\varepsilon\rangle\hat{\mathfrak{W}}\langle\varepsilon ae_1,\ \varepsilon a^2e_1\rangle. \end{aligned}$$

Hence $e_4a\varepsilon\hat{\mathfrak{W}}\varepsilon ae_1$ is an element of $e_4J^7e_1$. By Proposition 2.3, we have for $k = 1, 2$,

$$\begin{aligned} e_4a\varepsilon ae_1^{s^k} &= (e_4a\varepsilon)(\varepsilon ae_1^{s^k}) \in (e_4J(K\mathfrak{U})^3\varepsilon)(\varepsilon J(K\mathfrak{U})^2e_1^{s^k}) \\ &\subset e_4J(K\mathfrak{U})^5e_1^{s^k} = 0. \end{aligned}$$

Hence

$$e_4a\varepsilon\hat{\mathfrak{W}}\varepsilon ae_1 = e_4a\varepsilon\hat{\mathfrak{W}}ae_1 = e_4a\varepsilon ae_1.$$

Further, by calculation, we have

$$e_4a\varepsilon ae_1 = e_4ae_1 - e_4a^2e_1 \neq 0.$$

Thus it follows that $e_4J^7e_1 \neq 0$. Therefore, from the fact that

$$L_8(P_{M_1}) \subsetneq I \oplus M_3 \oplus M_4,$$

we obtain dim $e_4J^7e_1 = 1$, because $L_9(P_{M_1}) \simeq M_1$. Thus we conclude that $[L_8(P_{M_1}), M_4] = 1$.

We have the following by Landrock's lemma and the preceding lemma.

Corollary 6.15. $[L_8(P_{M_3}), M_2] = 1$.

Combining Corollary 2.6, Lemmas 4.1, 4.2, Corollary 4.10 and Lemmas 6.4, 6.7, 6.9, 6.14, we obtain the following

Proposition 6.16. *P_{M_1} has the Loewy series given in Theorem.*

7. The Loewy structure of P_{M_2}. In the remainder of this paper, by making use of an argument similar to the one used in the preceding section, we shall determine the Loewy structure of P_{M_2}, P_{M_3} and P_{M_4}, and consequently we complete the proof of Theorem. To begin with, in this section, we determine the Loewy structure of P_{M_2}. The following is a direct consequence of Proposition 2.3.

Lemma 7.1. $\varepsilon J(K\mathfrak{U})^2e_2 = \varepsilon J(K\mathfrak{U})^4e_2$ and $\varepsilon J(K\mathfrak{U})^5e_2 = 0$.

By Lemmas 5.2, 6.1 and 7.1, we have the following

Lemma 7.2. (1) $e_1J^3e_2 = e_1\mathfrak{M}^3e_2 + e_1\mathfrak{M}\mathfrak{N}\mathfrak{M}e_2$.
(2) $e_1J^4e_2 = e_1\mathfrak{M}^4e_2 + e_1\mathfrak{M}\mathfrak{N}^2\mathfrak{M}e_2$.
(3) $e_1J^6e_2 = e_1\mathfrak{M}^4\mathfrak{N}\mathfrak{M}e_2 + e_1\mathfrak{M}\mathfrak{N}\mathfrak{M}^4e_2$.
(4) $e_1J^7e_2 = e_1\mathfrak{M}^4\mathfrak{N}^2\mathfrak{M}e_2 + e_1\mathfrak{M}\mathfrak{N}^2\mathfrak{M}^4e_2$.
(5) $e_3J^4e_2 = e_3\mathfrak{M}^4e_2 + e_3\mathfrak{M}^2\mathfrak{N}\mathfrak{M}e_2$.
(6) $e_3J^5e_2 = e_3\mathfrak{M}^3\mathfrak{N}\mathfrak{M}e_2 + e_3\mathfrak{M}^2\mathfrak{N}^2\mathfrak{M}e_2$.
(7) $e_3J^6e_2 = e_3\mathfrak{M}^3\mathfrak{N}^2\mathfrak{M}e_2$.
(8) $e_4J^5e_2 = e_4\mathfrak{M}^5e_2 + e_4\mathfrak{M}^3\mathfrak{N}\mathfrak{M}e_2$.
(9) $e_4J^6e_2 = e_4\mathfrak{M}^4\mathfrak{N}\mathfrak{M}e_2 + e_4\mathfrak{M}^3\mathfrak{N}^2\mathfrak{M}e_2$.
(10) $e_4J^7e_2 = e_4\mathfrak{M}^4\mathfrak{N}^2\mathfrak{M}e_2$.

Calculating by taking advantage of Lemmas 2.1 and 5.3, we obtain the following

Lemma 7.3. (1) $e_1a\varepsilon ae_2 = e_1a^2\varepsilon a^2e_2 = -e_1a\varepsilon a^2e_2 = -e_1a^2\varepsilon ae_2 = e_1ae_2 + e_1a^2e_2$.
(2) $e_1a\varepsilon ae_2^S = e_1ae_2^S$, $e_1a\varepsilon a^2e_2^S = e_1a^2\varepsilon ae_2^S = e_1ae_2^S + e_1a^2e_2^S$, $e_1a^2\varepsilon a^2a_2^S$

$=e_1a^2e_2^S$.

(3) $e_1a\varepsilon ae_2^{S^2} = e_1ae_2^{S^2}$, $e_1a\varepsilon a^2e_2^{S^2} = e_1a^2\varepsilon ae_2^{S^2} = e_1ae_2^{S^2}+e_1a^2e_2^{S^2}$, $e_1a^2\varepsilon a^2e_2^{S^2} = e_1a^2e_2^{S^2}$.

(4) $e_3a\varepsilon ae_2 = e_3a^2\varepsilon a^2e_2 = -e_3a\varepsilon a^2e_2 = -e_3a^2\varepsilon ae_2 = -(e_3ae_2 + e_3a^2e_2)$.

(5) $e_3a\varepsilon ae_2^S = -e_3a\varepsilon a^2e_2^S = e_3a^2e_2^S$, $e_3a^2\varepsilon ae_2^S = e_3a^2\varepsilon a^2e_2^S = e_3ae_2^S$.

(6) $e_3a\varepsilon ae_2^{S^2} = e_3a^2\varepsilon ae_2^{S^2} = -e_3a\varepsilon a^2e_2^{S^2} = -e_3a^2\varepsilon a^2e_2^{S^2} = -e_3ae_2^{S^2} + e_3a^2e_2^{S^2}$.

(7) $e_4a\varepsilon ae_2 = e_4a^2\varepsilon ae_2 = -e_4a\varepsilon a^2e_2 = -e_4a^2\varepsilon a^2e_2 = -e_4ae_2 + e_4a^2e_2$.

(8) $e_4a\varepsilon ae_2^S = e_4a^2\varepsilon a^2e_2^S = -e_4a\varepsilon a^2e_2^S = -e_4a^2\varepsilon ae_2^S = e_4ae_2^S + e_4a^2e_2^S$.

(9) $e_4a\varepsilon ae_2^{S^2} = e_4a^2\varepsilon a^2e_2^{S^2} = -e_4a\varepsilon a^2e_2^{S^2} = -e_4a^2\varepsilon ae_2^{S^2} = -(e_4ae_2^{S^2} + e_4a^2e_2^{S^2})$.

At first, we shall prove the following

Lemma 7.4. (1) $[L_4(P_{M_2}), M_1] = 2$.
(2) $[L_7(P_{M_2}), M_1] = 1$.
(3) $[L_8(P_{M_2}), M_1] = 1$.

Proof. (1) To our end, it suffices to prove that $\dim e_1J^3e_2 = 5$ and $\dim e_1J^4e_2 = 3$. By Lemma 7.2,

$$e_1J^3e_2 = e_1\mathfrak{M}^3e_2+e_1\mathfrak{M}\mathfrak{N}\mathfrak{M}e_2.$$

We set

$$x_i = e_1(a-1)^2(b-1)e_2^{S^i} \in e_1J(K\mathfrak{U})^3e_2^{S^i}, \qquad 0 \leqq i \leqq 2.$$

Then by Lemma 2.1, we have

$$\begin{aligned} x_0 &= e_1(a-1)^2(b-1)e_2 = (\zeta^2+1)(e_1ae_2-e_1a^2e_2) \neq 0, \\ x_1 &= e_1(a-1)^2(b-1)e_2^S = (\zeta^2-\zeta+1)(e_1ae_2^S-e_1a^2e_2^S) \neq 0, \\ x_2 &= e_1(a-1)^2(b-1)e_2^{S^2} = (-\zeta+1)(e_1ae_2^{S^2}-e_1ae_2^{S^2}) \neq 0. \end{aligned}$$

Since $\dim e_1J(K\mathfrak{U})^3e_2^{S^i} = 1$ for $i = 0, 1, 2$, from the above, it follows that

$$e_1J^3e_2 \supset e_1\mathfrak{M}^3e_2 = \langle x_0, x_1, x_2\rangle.$$

Further, since

$$\begin{aligned} e_1J^3e_2 \supset e_1\mathfrak{M}\mathfrak{N}\mathfrak{M}e_2 &= (e_1J(K\mathfrak{U})\varepsilon)J(K\mathfrak{W})(\varepsilon J(K\mathfrak{U})e_2) \\ &= \langle e_1a\varepsilon, e_1a^2\varepsilon\rangle J(K\mathfrak{W})\langle \varepsilon ae_2, \varepsilon a^2e_2\rangle, \end{aligned}$$

we see that $e_1J^3e_2$ contains the elements:

$$y_1 = e_1a\varepsilon(1-s)\varepsilon ae_2 = e_1a\varepsilon ae_2 - e_1a\varepsilon ae_2^s s,$$
$$y_2 = e_1a\varepsilon(1-s)\varepsilon a^2e_2 = e_1a\varepsilon a^2e_2 - e_1a\varepsilon a^2e_2^s s.$$

Now by Lemma 7.3, we see at once that x_0, x_1, x_2, y_1 and y_2 are linearly independent, and so dim $e_1J^3e_2 \geqq 5$. Therefore we obtain dim $e_1J^3e_2 = 5$ by Lemma 4.5 and Corollary 2.6. Next, by Lemma 7.2,

$$e_1J^4e_2 = e_1\mathfrak{M}^4e_2 + e_1\mathfrak{M}\mathfrak{N}^2\mathfrak{M}e_2,$$

and by Proposition 2.3,

$$\begin{aligned} e_1\mathfrak{M}^4e_2 &= e_1J(K\mathfrak{U})^4e_2^s s + e_1J(K\mathfrak{U})^4e_2^{s^2}s^2 \\ &= e_1J(K\mathfrak{U})^3e_2^s s + e_1J(K\mathfrak{U})^3e_2^{s^2}s^2 \\ &= \langle x_1, x_2\rangle. \end{aligned}$$

Further, observing that

$$\begin{aligned} e_1\mathfrak{M}\mathfrak{N}^2\mathfrak{M}e_2 &= (e_1J(K\mathfrak{U})\varepsilon)\hat{\mathfrak{W}}(\varepsilon J(K\mathfrak{U})e_2) \\ &= \langle e_1a\varepsilon,\ e_1a^2\varepsilon\rangle\hat{\mathfrak{W}}\langle\varepsilon ae_2,\ \varepsilon a^2e_2\rangle \end{aligned}$$

and

$$e_1a^i\varepsilon\hat{\mathfrak{W}}\varepsilon a^je_2 = e_1a^i\varepsilon\hat{\mathfrak{W}}a^je_2 = \textstyle\sum_{k=0}^2 e_1a^i\varepsilon a^je_2^{s^k}s^k, \qquad 1 \leqq i,\ j \leqq 2,$$

we obtain

$$e_1J^4e_2 = \langle x_1,\ x_2,\ e_1a\varepsilon\hat{\mathfrak{W}}ae_2\rangle$$

by Lemma 7.3. Hence dim $e_1J^4e_2 = 3$. Thus (1) is proved.

(2) and (3) It suffices to prove that dim $e_1J^6e_2 = 2$ and dim $e_1J^7e_2 = 1$, because $L_9(P_{M_2}) \simeq M_2$. At first, by Lemma 7.2,

$$e_1J^6e_2 = e_1\mathfrak{M}^4\mathfrak{N}\mathfrak{M}e_2 + e_1\mathfrak{M}\mathfrak{N}\mathfrak{M}^4e_2.$$

Since dim $e_1J(K\mathfrak{U})^4\varepsilon = 1$ and

$$\begin{aligned} e_1J(K\mathfrak{U})^4\varepsilon &\ni e_1(a-1)^2(b-1)^2\varepsilon \\ &= (\zeta^2-\zeta+1)(e_1a\varepsilon + e_1a^2\varepsilon) \neq 0, \end{aligned}$$

we have

$$e_1J(K\mathfrak{U})^4\varepsilon = \langle e_1(a+a^2)\varepsilon\rangle. \tag{7.5}$$

Therefore $e_1J^6e_2$ contains the elements:

$$\begin{aligned} y_1 &= e_1(a+a^2)\varepsilon(1-s)\varepsilon a e_2 \\ &= e_1(a+a^2)\varepsilon a e_2 - e_1(a+a^2)\varepsilon a e_2^s s, \\ y_2 &= e_1(a+a^2)\varepsilon(1-s^2)\varepsilon a e_2 \\ &= e_1(a+a^2)\varepsilon a e_2 - e_1(a+a^2)\varepsilon a e_2^{s^2} s^2, \end{aligned}$$

because

$$\begin{aligned} e_1 J^6 e_2 &\supset e_1 \mathfrak{M}^4 \mathfrak{N} \mathfrak{M} e_2 \\ &= (e_1 J(K\mathfrak{U})^4 \varepsilon) J(K\mathfrak{W})(\varepsilon J(K\mathfrak{U}) e_2) \\ &= \langle e_1(a+a^2)\varepsilon\rangle J(K\mathfrak{W}) \langle \varepsilon a e_2,\ \varepsilon a^2 e_2\rangle. \end{aligned}$$

By Lemma 7.3, we see that y_1 and y_2 are linearly independent, and so it follows that $\dim e_1 J^6 e_2 = 2$, because $\dim e_1 J^6 e_2 \leqq 2$ by Corollary 2.6, Lemma 4.5, and (1).· Next, we prove that $\dim e_1 J^7 e_2 = 1$. Since $\dim \varepsilon J(K\mathfrak{U})^4 e_2 = 1$ and

$$\begin{aligned} \varepsilon J(K\mathfrak{U})^4 e_2 &\ni \varepsilon(a-1)^2(b-1)^2 e_2 \\ &= (\zeta^2-\zeta+1)(\varepsilon a e_2 + \varepsilon a^2 e_2) \neq 0, \end{aligned}$$

we have

$$\varepsilon J(K\mathfrak{U})^4 e_2 = \langle \varepsilon(a+a^2) e_2\rangle. \tag{7.6}$$

Hence, from (7.5) and (7.6), it follows that

$$\begin{aligned} e_1 J^7 e_2 &= e_1 \mathfrak{M}^4 \mathfrak{N}^2 \mathfrak{M} e_2 + e_1 \mathfrak{M} \mathfrak{N}^2 \mathfrak{M}^4 e_2 \\ &= (e_1 J(K\mathfrak{U})^4 \varepsilon)\hat{\mathfrak{W}}(\varepsilon J(K\mathfrak{U}) e_2) + (e_1 J(K\mathfrak{U})\varepsilon)\hat{\mathfrak{W}}(\varepsilon J(K\mathfrak{U})^4 e_2) \\ &= \langle e_1(a+a^2)\varepsilon\rangle \hat{\mathfrak{W}} \langle \varepsilon a e_2,\ \varepsilon a^2 e_2\rangle \\ &\quad + \langle e_1 a\varepsilon,\ e_1 a^2 \varepsilon\rangle \hat{\mathfrak{W}} \langle \varepsilon(a+a^2) e_2\rangle. \end{aligned}$$

Therefore $e_1 J^7 e_2$ is spanned by the following four elements:

$$\begin{aligned} e_1(a+a^2)\varepsilon\hat{\mathfrak{W}}\varepsilon a^i e_2 &= \textstyle\sum_{k=0}^2 e_1(a+a^2)\varepsilon a^i e_2^{s^k} s^k \ (i=1,\ 2), \\ e_1 a^i \varepsilon\hat{\mathfrak{W}}\varepsilon(a+a^2) e_2 &= \textstyle\sum_{k=0}^2 e_1 a^i \varepsilon(a+a^2) e_2^{s^k} s^k \ (i=1,\ 2). \end{aligned}$$

But Lemma 7.3 asserts that

$$\begin{aligned} e_1(a+a^2)\varepsilon\hat{\mathfrak{W}}\varepsilon a e_2 &= e_1 a\varepsilon\hat{\mathfrak{W}}\varepsilon(a+a^2) e_2 \\ &= -e_1(a+a^2)\varepsilon\hat{\mathfrak{W}}\varepsilon a^2 e_2 \\ &= -e_1 a^2 \varepsilon\hat{\mathfrak{W}}\varepsilon(a+a^2) e_2. \end{aligned}$$

Thus we obtain $\dim e_1 J^7 e_2 = 1$, and so (2) and (3) are proved.

Lemma 7.7. (1) $[L_5(P_{M_2}),\ M_3] = 1$.

(2) $[L_6(P_{M_2}),\ M_3] = 2.$
(3) $[L_7(P_{M_2}),\ M_3] = 1.$

Proof. To prove the lemma, it suffices to show that $\dim e_3J^4e_2 = 4$, $\dim e_3J^5e_2 = 3$ and $\dim e_3J^6e_2 = 1$, because

$$L_8(P_{M_2}) \subsetneq S_2(P_{M_2}) \simeq L_2(P_{M_1})^* \simeq M_1 \oplus M_1 \oplus M_4$$

and $L_9(P_{M_2}) \simeq M_2$. At first, we prove $\dim e_3J^4e_2 = 4$. By Lemma 7.2,

$$e_3J^4e_2 = e_3\mathfrak{M}^4e_2 + e_3\mathfrak{M}^2\mathfrak{N}\mathfrak{M}e_2.$$

Since $e_3J(K\mathfrak{U})^4e_2^{s^2} = 0$, it holds that

$$e_3\mathfrak{M}^4e_2 = e_3J(K\mathfrak{U})^4e_2 + e_3J(K\mathfrak{U})^4e_2^s s.$$

Hence, noting that

$$\begin{aligned} e_3J(K\mathfrak{U})^4e_2 \ni x_1 &= e_3(a-1)^2(b-1)^2e_2 \\ &= (-\zeta+1)(e_3ae_2 + e_3a^2e_2) \neq 0, \\ e_3J(K\mathfrak{U})^4e_2^s \ni x_2 &= e_3(a-1)^2(b-1)^2e_2^s \\ &= (-\zeta^2+\zeta+1)(e_3ae_2^s + e_3a^2e_2^s) \neq 0, \end{aligned}$$

we obtain

$$e_3J^4e_2 \supset e_3\mathfrak{M}^4e_2 = \langle x_1,\ x_2\rangle,$$

because $\dim e_3J(K\mathfrak{U})^4e_2 = \dim e_3J(K\mathfrak{U})^4e_2^s = 1$. Further, from the equality:

$$\begin{aligned} e_3\mathfrak{M}^2\mathfrak{N}\mathfrak{M}e_2 &= (e_2J(K\mathfrak{U})^2\varepsilon)J(K\mathfrak{W})(\varepsilon J(K\mathfrak{U})e_2) \\ &= \langle e_3a\varepsilon,\ e_3a^2\varepsilon\rangle J(K\mathfrak{W})\langle \varepsilon ae_2,\ \varepsilon a^2e_2\rangle, \end{aligned}$$

we see that $e_3\mathfrak{M}^2\mathfrak{N}\mathfrak{M}e_2$ contains the elements:

$$\begin{aligned} y_1 &= e_3a\varepsilon(1-s)\varepsilon ae_2 = e_3a\varepsilon ae_2 - e_3a\varepsilon ae_2^s s, \\ y_2 &= e_3a\varepsilon(1-s^2)\varepsilon ae_2 = e_3a\varepsilon ae_2 - e_3a\varepsilon ae_2^{s^2}s^2. \end{aligned}$$

By Lemma 7.3, we see immediately that x_1, x_2, y_1 and y_2 are linearly independent. Thus it follows that $\dim e_3J^4e_2 = 4$ from Corollary 2.6, because $[P_{M_2}/J^3P_{M_2},\ M_3] \geqq 2$ by Lemmas 4.1 and 4.5. Next, by (6.10), we have

$$\begin{aligned} e_3J^5e_2 &= e_3\mathfrak{M}^3\mathfrak{N}\mathfrak{M}e_2 + e_3\mathfrak{M}^2\mathfrak{N}^2\mathfrak{M}e_2 \\ &= (e_3J(K\mathfrak{U})^3\varepsilon)J(K\mathfrak{W})(\varepsilon J(K\mathfrak{U})e_2) + (e_3J(K\mathfrak{U})^2\varepsilon)\hat{\mathfrak{W}}(\varepsilon J(K\mathfrak{U})e_2) \\ &= \langle e_3(a-a^2)\varepsilon\rangle J(K\mathfrak{W})\langle \varepsilon ae_2,\ \varepsilon a^2e_2\rangle \\ &\quad + \langle e_3a\varepsilon,\ e_3a^2\varepsilon\rangle \hat{\mathfrak{W}}\langle \varepsilon ae_2,\ \varepsilon a^2e_2\rangle. \end{aligned}$$

Hence, from Lemma 7.3 and the equalities:

$$\begin{aligned} e_3(a-a^2)\varepsilon(1-s^l)\varepsilon a^j e_2 &= e_3(a-a^2)(1-s^l)\varepsilon a^j e_2 \\ &= e_3(a-a^2)\varepsilon a^j e_2 - e_3(a-a^2)\varepsilon a^j e_2^{s^l} s^l, \\ e_3 a^i \varepsilon \hat{\mathfrak{W}} \varepsilon a^j e_2 &= e_3 a^i \varepsilon \hat{\mathfrak{W}} a^j e_2 = \textstyle\sum_{k=0}^{2} e_3 a^i \varepsilon a^j e_2^{s^k} s^k, \end{aligned}$$

where $1 \leqq i, j, l \leqq 2$, it follows that $e_3J^5e_2$ has a K-basis consisting of the elements:

$$e_3(a-a^2)(1-s)\varepsilon a e_2,\ e_3(a-a^2)(1-s^2)\varepsilon a e_2,\ e_3 a\varepsilon\hat{\mathfrak{W}} a e_2.$$

Thus we obtain dim $e_3J^5e_2 = 3$. In final, since

$$\begin{aligned} e_3J^6e_2 &= e_3\mathfrak{M}^3\mathfrak{N}^2\mathfrak{M}e_2 \\ &= (e_3J(K\mathfrak{U})^3\varepsilon)\hat{\mathfrak{W}}(\varepsilon J(K\mathfrak{U})e_2) \\ &= \langle e_3(a-a^2)\varepsilon\rangle\hat{\mathfrak{W}}\langle \varepsilon a e_2,\ \varepsilon a^2 e_2\rangle \end{aligned}$$

and

$$e_3(a-a^2)\varepsilon\hat{\mathfrak{W}}\varepsilon a e_2 = -e_3(a-a^2)\varepsilon\hat{\mathfrak{W}}\varepsilon a^2 e_2 \text{ (Lemma 7.3)},$$

we obtain

$$e_3J^6e_2 = \langle e_3(a-a^2)\varepsilon\hat{\mathfrak{W}} a e_2\rangle,$$

proving dim $e_3J^6e_2 = 1$. Thus we complete the proof of the lemma.

By Landrock's lemma and the preceding lemma, we have the following

Corollary 7.8. (1) $[L_5(P_{M_4}), M_1] = 1$.
(2) $[L_6(P_{M_4}), M_1] = 2$.
(3) $[L_7(P_{M_4}), M_1] = 1$.

Lemma 7.9. $[L_6(P_{M_2}), M_4] = [L_8(P_{M_2}), M_4] = 1$.

Proof. We prove that dim $e_4J^5e_2 = 3$, dim $e_4J^6e_2 = 2$ and dim $e_4J^7e_2 = 1$. Then the result follows. By Lemma 7.2,

$$e_4J^5e_2 = e_4\mathfrak{M}^5e_2 + e_4\mathfrak{M}^3\mathfrak{N}\mathfrak{M}e_2.$$

Since

$$\begin{aligned} e_4\mathfrak{M}^3\mathfrak{N}\mathfrak{M}e_2 &= (e_4J(K\mathfrak{U})^3\varepsilon)J(K\mathfrak{W})(\varepsilon J(K\mathfrak{U})e_2) \\ &= \langle e_4 a\varepsilon,\ e_4 a^2\varepsilon\rangle J(K\mathfrak{W})\langle\varepsilon a e_2,\ \varepsilon a^2 e_2\rangle, \end{aligned}$$

we see that $e_4\mathfrak{M}^3\mathfrak{N}\mathfrak{M}e_2$ contains the elements:

$$e_4a\varepsilon(1-s)\varepsilon ae_2 = e_4a\varepsilon ae_2 - e_4a\varepsilon ae_2^s s,$$
$$e_4a^2\varepsilon(1-s)\varepsilon ae_2 = e_4a^2\varepsilon ae_2 - e_4a^2\varepsilon ae_2^s s,$$
$$e_4a\varepsilon(1-s^2)\varepsilon ae_2 = e_4a\varepsilon ae_2 - e_4a\varepsilon ae_2^{s^2} s^2.$$

Further, by Lemma 7.3, we see that these elements are linearly independent, and so, by Corollary 2.6, Lemmas 4.1, 4.5 and Corollary 4.10, we obtain dim $e_4J^5e_2 = 3$. Next, we have

$$\begin{aligned} e_4J^6e_2 &= e_4\mathfrak{M}^4\mathfrak{N}\mathfrak{M}e_2 + e_4\mathfrak{M}^3\mathfrak{N}^2\mathfrak{M}e_2 \\ &= (e_4J(K\mathfrak{U})^4\varepsilon)J(K\mathfrak{W})(\varepsilon J(K\mathfrak{U})e_2) + (e_4J(K\mathfrak{U})^3\varepsilon)\hat{\mathfrak{W}}(\varepsilon J(K\mathfrak{U})e_2). \end{aligned}$$

Since dim $e_4J(K\mathfrak{U})^4\varepsilon = 1$ and

$$\begin{aligned} e_4J(K\mathfrak{U})^4\varepsilon &\ni e_4(a-1)^2(b-1)^2\varepsilon \\ &= (-\zeta^2+\zeta+1)(e_4a\varepsilon + e_4a^2\varepsilon) \neq 0, \end{aligned}$$

we have $e_4J(K\mathfrak{U})^4\varepsilon = \langle e_4(a+a^2)\varepsilon\rangle$. Noting that $e_4J(K\mathfrak{U})^5e_2^{sk} = 0$ for $k = 1, 2$, we obtain

$$e_4(a+a^2)\varepsilon(1-s^k)\varepsilon a^je_2 = e_4(a+a^2)\varepsilon a^je_2, \qquad j = 1, 2.$$

Hence, by Lemma 7.3, we obtain

$$e_4\mathfrak{M}^4\mathfrak{N}\mathfrak{M}e_2 = \langle e_4(a+a^2)\varepsilon ae_2\rangle. \tag{7.10}$$

Further, from the equality:

$$e_4a^i\varepsilon\hat{\mathfrak{W}}\varepsilon a^je_2 = e_4a^i\varepsilon\hat{\mathfrak{W}}a^je_2 = \textstyle\sum_{k=0}^2 e_4a^i\varepsilon a^je_2^{sk}s^k, \qquad 1 \leqq i,\ j \leqq 2$$

and Lemma 7.3, it follows that

$$\begin{aligned} e_4\mathfrak{M}^3\mathfrak{N}^2\mathfrak{M}e_2 &= (e_4J(K\mathfrak{U})^3\varepsilon)\hat{\mathfrak{W}}(\varepsilon J(K\mathfrak{U})e_2) \\ &= \langle e_4a\varepsilon,\ e_4a^2\varepsilon\rangle\hat{\mathfrak{W}}\langle\varepsilon ae_2,\ \varepsilon a^2e_2\rangle \\ &= \langle e_4a\varepsilon\hat{\mathfrak{W}}ae_2,\ e_4a^2\varepsilon\hat{\mathfrak{W}}a^2e_2\rangle. \end{aligned} \tag{7.11}$$

Thus, by Lemma 7.3, (7.10) and (7.11), we obtain

$$e_4J^6e_2 = e_4\mathfrak{M}^4\mathfrak{N}\mathfrak{M}e_2 + e_4\mathfrak{M}^3\mathfrak{N}^2\mathfrak{M}e_2 = \langle e_4a\varepsilon\hat{\mathfrak{W}}ae_2,\ e_4a^2\varepsilon\hat{\mathfrak{W}}a^2e_2\rangle,$$

proving dim $e_4J^6e_2 = 2$. Finally, from the equalities:

$$\begin{aligned} e_4J^7e_2 = e_4\mathfrak{M}^4\mathfrak{N}^2\mathfrak{M}e_2 &= (e_4J(K\mathfrak{U})^4\varepsilon)\hat{\mathfrak{W}}(\varepsilon J(K\mathfrak{U})e_2) \\ &= \langle e_4(a+a^2)\varepsilon\rangle\hat{\mathfrak{W}}\langle\varepsilon ae_2,\ \varepsilon a^2e_2\rangle \end{aligned}$$

and

$$e_4(a+a^2)\varepsilon\hat{\mathfrak{W}}\varepsilon ae_2 = e_4(a+a^2)\varepsilon ae_2 = e_4ae_2 - e_4a^2e_2,$$
$$e_4(a+a^2)\varepsilon\hat{\mathfrak{W}}\varepsilon a^2e_2 = e_4(a+a^2)\varepsilon a^2e_2 = -(e_4ae_2 - e_4a^2e_2),$$

we obtain dim $e_4J^7e_2 = 1$. Thus the lemma is proved.

Landrock's lemma together with the preceding lemma implies the following

Corollary 7.12. $[L_6(P_{M_3}), M_1] = [L_8(P_{M_3}), M_1] = 1$.

Combining Corollary 2.6, Lemmas 4.1, 4.5, Corollaries 4.10, 6.6 and Lemmas 7.4, 7.7, 7.9 we obtain the following

Proposition 7.13. *P_{M_2} has the Loewy series given in Theorem.*

8. The Loewy structure of P_{M_3}. Here, we determine the Loewy structure of P_{M_3}. For this we need only prove the following

Lemma 8.1. $[L_8(P_{M_3}), M_3] = 1$.

Proof. Because $L_9(P_{M_3}) \simeq M_3$, it suffices to prove that dim $e_3J^7e_3 = 2$. Since $\varepsilon J(K\mathfrak{U})e_3 = \varepsilon J(K\mathfrak{U})^3e_3$ and $\varepsilon J(K\mathfrak{U})^5e_3 = 0$, by Lemmas 5.2 and 6.1, we obtain

$$\begin{aligned} e_3J^8e_3 &= e_3\mathfrak{M}^3\mathfrak{N}^2\mathfrak{M}^3e_3 + e_3\mathfrak{M}^2\mathfrak{N}^2\mathfrak{M}^4e_3 \\ &\subset e_3\mathfrak{M}^6e_3 = e_3J(K\mathfrak{U})^6K\mathfrak{W}e_3 \\ &= \hat{\mathfrak{U}}e_3K\mathfrak{W}e_3 = K\hat{\mathfrak{U}}e_3. \end{aligned}$$

Hence $e_3J^8e_3 = K\hat{\mathfrak{U}}e_3$, because dim $e_3J^8e_3 = 1$. Thus we see that $e_3J^7e_3$ contains $\hat{\mathfrak{U}}e_3$. Further, we have

$$\begin{aligned} e_3J^7e_3 &\supset e_3\mathfrak{M}^2\mathfrak{N}^2\mathfrak{M}^3e_3 = (e_3J(K\mathfrak{U})^2\varepsilon)\hat{\mathfrak{W}}(\varepsilon J(K\mathfrak{U})^3e_3) \\ &= \langle e_3a\varepsilon,\ e_3a^2\varepsilon\rangle\hat{\mathfrak{W}}\langle\varepsilon ae_3,\ \varepsilon a^2e_3\rangle, \end{aligned}$$

and so $e_3J^7e_3$ contains an element $x = (e_3a\varepsilon)\hat{\mathfrak{W}}(\varepsilon ae_3)$. Noting that $e_3J(K\mathfrak{U})^5e_3^{s^2} = 0$, we have

$$x = e_3a\varepsilon ae_3 + e_3a\varepsilon ae_3^s s.$$

Calculating with the aid of Lemmas 2.1 and 5.3, we find

$$e_3a\varepsilon ae_3^s = -e_3ae_3^s + e_3a^2e_3^s \neq 0.$$

Further,

$$\begin{aligned} e_3a\varepsilon ae_3 &\in e_3J(K\mathfrak{U})^2\varepsilon J(K\mathfrak{U})^3e_3 \subset e_3J(K\mathfrak{U})^5e_3 \\ &= e_3J(K\mathfrak{U})^6e_3 = K\hat{\mathfrak{U}}e_3. \end{aligned}$$

Therefore we obtain

$$e_3 J^7 e_3 \supset \langle \hat{\mathfrak{U}} e_3,\ x \rangle.$$

Hence dim $e_3 J^7 e_3 \geqq 2$. On the other hand, it holds that

$$L_8(P_{M_3}) \subsetneq S_2(P_{M_3}) \simeq L_2(P_{M_4})^* \simeq M_1 \oplus M_2 \oplus M_3,$$
$$L_9(P_{M_3}) \simeq M_3.$$

Hence dim $e_3 J^7 e_3 \leqq 2$, and so dim $e_3 J^7 e_3 = 2$. Thus the lemma is proved.

Landrock's lemma together with the preceding lemma implies the following

Corollary 8.2. $[L_8(P_{M_4}),\ M_4] = 1$.

By Corollary 2.6, Lemmas 4.1, 4.8, Corollaries 4.10, 6.15, 7.12 and Lemma 8.1, we obtain the following

Proposition 8.3. *P_{M_3} has the Loewy series given in Theorem.*

9. The Loewy structure of P_{M_4}. Finally, in this section, we determine the Loewy structure of P_{M_4}, and complete the proof of Theorem. The following is a direct consequence of Proposition 2.3.

Lemma 9.1. $\varepsilon J(K\mathfrak{U})e_4 = \varepsilon J(K\mathfrak{U})^2 e_4$ *and* $\varepsilon J(K\mathfrak{U})^4 e_4 = 0$.

This together with Lemmas 5.2 and 6.1 yields the following

Lemma 9.2. (1) $e_3 J^5 e_4 = e_3 \mathfrak{M}^5 e_4 + e_3 \mathfrak{M}^2 \mathfrak{N} \mathfrak{M}^2 e_4$.
(2) $e_3 J^6 e_4 = e_3 \mathfrak{M}^3 \mathfrak{N} \mathfrak{M}^2 e_4 + e_3 \mathfrak{M}^2 \mathfrak{N} \mathfrak{M}^3 e_4 + e_3 \mathfrak{M}^2 \mathfrak{N}^2 \mathfrak{M}^2 e_4$.
(3) $e_3 J^7 e_4 = e_3 \mathfrak{M}^3 \mathfrak{N}^2 \mathfrak{M}^2 e_4 + e_3 \mathfrak{M}^2 \mathfrak{N}^2 \mathfrak{M}^3 e_4$.

Calculating with the aid of Lemmas 2.1 and 5.3, we have the following

Lemma 9.3. (1) $e_3 a \varepsilon a e_4 = -e_3 a^2 \varepsilon a^2 e_4 = e_3 a e_4 - e_3 a^2 e_4$, $e_3 a \varepsilon a^2 e_4 = e_3 a^2 \varepsilon a e_4 = 0$.
(2) $e_3 a \varepsilon a e_4^s = e_3 a \varepsilon a^2 e_4^s = e_3 a^2 \varepsilon a e_4^s = e_3 a^2 \varepsilon a^2 e_4^s = -(e_3 a e_4^s + e_3 a^2 e_4^s)$.
(3) $e_3 a \varepsilon a e_4^{s^2} = e_3 a \varepsilon a^2 e_4^{s^2} = e_3 a^2 \varepsilon a e_4^{s^2} = e_3 a^2 \varepsilon a^2 e_4^{s^2} = -(e_3 a e_4^{s^2} + e_3 a^2 e_4^{s^2})$.

Now we prove the following

Lemma 9.4. $[L_6(P_{M_4}), M_3] = [L_7(P_{M_4}), M_3] = [L_8(P_{M_4}), M_3] = 1$.

Proof. To prove the lemma, it suffices to show that $\dim e_3J^5e_4 = 3$, $\dim e_3J^6e_4 = 2$ and $\dim e_3J^7e_4 = 1$, because $L_9(P_{M_4}) \simeq M_4$. By Lemma 9.2, we have

$$\begin{aligned} e_3J^5e_4 &\supset e_3\mathfrak{M}^2\mathfrak{N}\mathfrak{M}^2e_4 \\ &= (e_3J(K\mathfrak{U})^2\varepsilon)J(K\mathfrak{W})(\varepsilon J(K\mathfrak{U})^2e_4) \\ &= \langle e_3a\varepsilon,\ e_3a^2\varepsilon\rangle J(K\mathfrak{W})\langle \varepsilon ae_4,\ \varepsilon a^2e_4\rangle, \end{aligned}$$

and so $e_3J^5e_4$ contains the elements:

$$\begin{aligned} x_1 &= e_3a\varepsilon(1-s)\varepsilon ae_4 = e_3a\varepsilon ae_4 - e_3a\varepsilon ae_4^s s, \\ x_2 &= e_3a^2\varepsilon(1-s)\varepsilon a^2e_4 = e_3a^2\varepsilon a^2e_4 - e_3a^2\varepsilon a^2e_4^s s, \\ x_3 &= e_3a\varepsilon(1-s^2)\varepsilon ae_4 = e_3a\varepsilon ae_4 - e_3a\varepsilon ae_4^{s^2}s^2. \end{aligned}$$

By Lemma 9.3, we see that these three elements are linearly independent. On the other hand, by Lemmas 4.1 and 4.9,

$$[P_{M_4}/J^5P_{M_4},\ M_3] \geqq 3.$$

Thus we obtain $\dim e_3J^5e_4 = 3$ by Corollary 2.6. Next, by Lemma 9.2, we have

$$\begin{aligned} e_3J^6e_4 &= e_3\mathfrak{M}^3\mathfrak{N}\mathfrak{M}^2e_4 + e_3\mathfrak{M}^2\mathfrak{N}\mathfrak{M}^3e_4 + e_3\mathfrak{M}^2\mathfrak{N}^2\mathfrak{M}^2e_4 \\ &= (e_3J(K\mathfrak{U})^3\varepsilon)J(K\mathfrak{W})(\varepsilon J(K\mathfrak{U})^2e_4) \\ &\quad + (e_3J(K\mathfrak{U})^2\varepsilon)J(K\mathfrak{W})(\varepsilon J(K\mathfrak{U})^3e_4) \\ &\quad + (e_3J(K\mathfrak{U})^2\varepsilon)\hat{\mathfrak{W}}(\varepsilon J(K\mathfrak{U})^2e_4). \end{aligned}$$

Further, since $\dim \varepsilon J(K\mathfrak{U})^3e_4 = 1$ and

$$\varepsilon J(K\mathfrak{U})^3e_4 \ni \varepsilon(a-1)^2(b-1)e_4 = (\zeta^2+1)(\varepsilon ae_4 - \varepsilon a^2e_4) \neq 0,$$

it follows that

$$\varepsilon J(K\mathfrak{U})^3e_4 = \langle \varepsilon(a-a^2)e_4\rangle. \tag{9.5}$$

Hence, by (6.10) and (9.5), we have

$$\begin{aligned} e_3J^6e_4 = &\langle e_3(a-a^2)\varepsilon\rangle J(K\mathfrak{W})\langle \varepsilon ae_4,\ \varepsilon a^2e_4\rangle \\ &+ \langle e_3a\varepsilon,\ e_3a^2\varepsilon\rangle J(K\mathfrak{W})\langle \varepsilon(a-a^2)e_4\rangle \\ &+ \langle e_3a\varepsilon,\ e_3a^2\varepsilon\rangle \hat{\mathfrak{W}}\langle \varepsilon ae_4,\ \varepsilon a^2e_4\rangle. \end{aligned}$$

By Lemma 9.3, for every i, k, $1 \leqq i$, $k \leqq 2$, it holds that

$$e_3(a-a^2)\varepsilon(1-s^k)\varepsilon a^i e_4 = e_3 a^i \varepsilon(1-s^k)\varepsilon(a-a^2)e_4$$
$$= e_3 a e_4 - e_3 a^2 e_4.$$

Therefore noting that

$$e_3 a^i \varepsilon \hat{\mathfrak{W}} \varepsilon a^j e_4 = e_3 a^i \varepsilon \hat{\mathfrak{W}} a^j e_4 = \sum_{k=0}^{2} e_3 a^i \varepsilon a^j e_4^{s^k} s^k, \quad 1 \leqq i, j \leqq 2,$$

from Lemma 9.3, we obtain

$$e_3 J^6 e_4 = \langle e_3 a \varepsilon(1-s)\varepsilon(a-a^2)e_4,\ e_3 a \varepsilon \hat{\mathfrak{W}} a e_4 \rangle,$$

and so dim $e_3 J^6 e_4 = 2$. Finally, by Lemma 9.2, (6.10) and (9.5), we have

$$e_3 J^7 e_4 = e_3 \mathfrak{M}^3 \mathfrak{N}^2 \mathfrak{M}^2 e_4 + e_3 \mathfrak{M}^2 \mathfrak{N}^2 \mathfrak{M}^3 e_4$$
$$= \langle e_3(a-a^2)\varepsilon \rangle \hat{\mathfrak{W}} \langle \varepsilon a e_4,\ \varepsilon a^2 e_4 \rangle$$
$$+ \langle e_3 a \varepsilon,\ e_3 a^2 \varepsilon \rangle \hat{\mathfrak{W}} \langle \varepsilon(a-a^2)e_4 \rangle.$$

Since $e_3 J(K\mathfrak{U})^5 e_4^{s^k} = 0$ for $k = 1, 2$, there holds that

$$(e_3(a-a^2)\varepsilon)\hat{\mathfrak{W}}(\varepsilon a^i e_4) = e_3(a-a^2)\varepsilon a^i e_4, \qquad i = 1, 2,$$
$$(e_3 a^i \varepsilon)\hat{\mathfrak{W}}(\varepsilon(a-a^2)e_4) = e_3 a^i \varepsilon(a-a^2)e_4, \qquad i = 1, 2.$$

Further Lemma 9.3 yields the following:

$$e_3(a-a^2)\varepsilon a^i e_4 = e_3 a^i \varepsilon(a-a^2)e_4 = e_3 a e_4 - e_3 a^2 e_4.$$

Hence we see at once that dim $e_3 J^7 e_4 = 1$. Thus we complete the proof of the lemma.

Combining Corollary 2.6, Lemmas 4.1, 4.9, Corollaries 4.10, 6.13, 7.8, 8.2 and Lemma 9.4, we obtain the following

Proposition 9.6. *P_{M_4} has the Loewy series given in Theorem.*

Thus we complete the proof of Theorem.

REFERENCES

[1] B. HUPPERT and N. BLACKBURN : Finite Groups II, Springer-Verlag, 1982.

[2] S. KOSHITANI : On the Loewy series of the group algebra of a finite p-solvable group with p-length > 1, Comm. Alg. 13 (1985), 2175–2198.

[3] P. LANDROCK : Some remarks on Loewy lengths of projective modules, In Representation Theory II : Lecture Notes in Math. 832, Springer-Verlag, 1980, 369–381.

[4] P. LANDROCK : Finite Group Algebras and their Modules, London Math. Soc. Lecture Note Series 84, Cambridge Univ. Press, 1983.

[5] R. LIDL and H. NIEDERREITER : Finite Fields, Addison-Wesley, 1983.

[6] K. MOTOSE : On the nilpotency index of the radical of a group algebra III, J. London Math. Soc. 25 (1982), 39–42.
[7] K. MOTOSE and Y. NINOMIYA : On the nilpotency index of the radical of a group algebra, Hokkaido Math. J. 4 (1975), 261–264.
[8] D. A. R. WALLACE : Lower bounds for the radical of the group algebra of a finite p-soluble group, Proc. Edinburgh Math. Soc. 16 (1968/69) 127–134.

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