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FAMILIES OF GEODESICS WHICH DISTINGUISH FLAT TORI

NOBUHIRO INNAMI

0. Introduction. The flat tori play distinguished roles in the study of manifolds with nonpositive sectional curvature and of nonnegative sectional (or Ricci) curvature. Namely, the Riemannian metrics on a torus having one of these curvature conditions are flat. And, the behavior of geodesics on these manifolds are very different. This suggests to us that some behavior of geodesics distinguishes flat tori from any other Riemannian manifold. In the present paper we study the conditions (0.1)−(0.4) on surfaces.

Throughout the paper let M be a complete Riemannian manifold which is homeomorphic to a plane and N a complete Riemannian manifold with dimension 2. We denote by $N = M/D$ a quotient manifold, where D is the group of isometries acting freely on M which is properly discontinuous. All geodesics are always parametrized by arc-length, unless otherwise stated. Let SN be the unit tangent bundle of N and $\pi : SN \to N$ the projection. For any $v \in SN$ let $\gamma_v : (-\infty, \infty) \to N$ be the geodesic with $\dot{\gamma}_v(0) = v$, but the interval may be restricted if stated.

(0.1) For any $v \in SN$ there exists a section $\tau_v : N \to SN$, $\tau_v(\pi(v)) = v$, such that $f_v^t = \pi \circ g^t \circ \tau_v$ is a flow on N, where $g^t : SN \to SN$ is the geodesic flow of N.

(0.2) Each non-trivial homotopy class of closed curves in N contains a family of geodesics which cover N simply.

(0.3) The set of all tangent vectors of geodesics which are dense in N is dense in SN.

(0.4) For any non-contractible closed curve K and for any point p in N there exists at most two geodesic ray emanating from p which does not intersect K.

The flat tori T^2 satisfy these conditions. We shall show that the converse of (0.1), (0.2) and (0.4) are true in Section 1, 3 and 5, resp.. In Section 2 we provide the properties of axial isometries on M which is used in Section 3, 4, 5. We discuss the condition (0.1) and (0.2) in Section 4 and obtain some results concerning (0.3). The results are sometimes obtained by applying a theorem of E. Hopf ([9]) that the tori T^2 without conjugate points are flat. And, the study of Busemann-Pedersen ([6]) concerning axial isometries would be very useful to the investigation here. In the paper they have studied G-spaces (see [4]) which are much more general than complete Riemannian manifolds. Hence, if the readers are interested in the theory of G-spaces, all conclusions should be understood that the universal covering G-surfaces are straight and satisfy the parallel axiom(for the definitions of "straight" and "parallel axiom" see [4]).

1. Surfaces having flows whose orbits are geodesics. The important property of surfaces is that a curve can separate the space locally and sometimes globally. This is the principal tools of the investigation here. Let A be an open domain in N. We say that A is covered *simply* by a family Γ of geodesics if the geodesic in Γ through p uniquely exists locally for each point $p \in A$ and is maximal in A, i.e., any extension does not stay in A. In the definition, "locally" means "globally" also. If A is simply connected, then we can give all geodesics of the family an orientation so that the tangent vectors $v(x)$ of all geodesics depend continuously on their foot points x in A. And, each geodesic of the family decomposes A into two components.

Lemma 1.1. *Let A be a simply connected open domain in N which is covered simply by a family Γ of geodesics. Then, all subarcs of all geodesics in Γ are the unique geodesic in A connecting their endpoints. In particular, all geodesics of Γ are minimizing in A and the vectors $v(x)$ depend differentiably on the points $x \in A$.*

Proof. Let $v(x)$ be the tangent vector at $x \in A$ of the geodesic of Γ determined by x. We take the signed angle of unit vectors at x from $v(x)$ for any $x \in A$. Suppose there is a geodesic $\alpha : [0, L] \to N$ joining p and q in A both of which are in a geodesic γ of Γ such that α is not a subarc of γ. If $\alpha(L_1)$, $0 < L_1 \leq L$, is the first intersection point of α and γ, then the signs of $\alpha(L_1)$ and $\alpha(0)$ are different. Hence, the angle of $\alpha(L_0)$ is either zero or π for some L_0, $0 < L_0 \leq L_1$. This implies that α or the reversed geodesic of α is a subarc of a geodesic of Γ because of the simplicity of the family of geodesics, a contradiction.

The existence of a family of geodesics which entirely covers N simply would be controlled by the topological structure of N.

Lemma 1.2. *Suppose N is compact. If N is covered simply by a family Γ of geodesics, then N is topologically either the torus or one-sided torus.*

Proof. It suffices to prove that the statement is true if N is orientable. Let $N = M/D$, where M is the universal covering space of N. The lift $\widetilde{\Gamma}$ of Γ to M also covers M simply. Define an orientation of $\widetilde{\Gamma}$ and denote by $v(x)$ the tangent vector at x of the geodesic of $\widetilde{\Gamma}$ passing through x. Since N is orientable, the vector field v is invariant under D. This implies that v induces the vector field on N. Since it has no singular points, it follows from the Hopf-Lefschetz fixed point theorem that N is topologically the torus.

It should be noted that N is orientable if N has two families of geodesics which cover N simply in such a way that the tangent vectors at each point of N of the geodesics are linearly independent.

Theorem 1.3. *Suppose N is compact. If the geodesic frow on N satisfies the property*(0.1), *then N is a flat torus.*

Proof. For any unit vector $v \in SN$ the section τ_v induces a family Γ of geodesics which cover N simply with $\gamma_v \in \Gamma$. By Lemma 1.2, N is topologically either the torus or one-sided torus. To apply the theorem of E. Hopf we must prove that N has no conjugate points. Since the lift of γ_v to the universal covering space M of N is minimizing, there is no conjugate points along γ_v. Thus, N is flat. This completes the proof.

2. Axial isometries and the structure of surfaces. Let φ be an isometry of M. The function $d_\varphi : M \to \boldsymbol{R}$ given by $d_\varphi(p) = d(p, \varphi p)$ for any $p \in M$ is called the *displacement function* of φ. We say that φ is *axial* if there is a minimizing geodesic $\gamma : (-\infty, \infty) \to M$ such that $\varphi\gamma(t) = \gamma(t+a)$ for any $t \in (-\infty, \infty)$ and some constant $a > 0$. The γ is called an *axis* of φ. We can find important results on axial isometries in a paper of Busemann-Pedersen([6], and also see [5] pp. 64-67).

(2.1) If γ is an axis of φ, then $a = \min d_\varphi$ and γ is also an axis of φ^n, $n \geq 1$.

(2.2) Let φ be an isometry which preserves an orientation of M. If $\gamma : (-\infty, \infty) \to M$ is an axis of φ^n, $n > 1$, then γ is also an axis of φ.

(2.3) Let φ be an isometry which preserves an orientation of M. If $0 < d_\varphi(p) = \min d_\varphi$ for some $p \in M$, then the geodesic $\gamma : (-\infty, \infty) \to M$, with $\gamma(0) = p$ and $\gamma(d_\varphi(p)) = \varphi p$, is minimizing and hence an axis of φ.

We see in a book of Cheeger-Ebin([7] p. 156) that

(2.4) If $N = M/D$ is compact, where D is the group of isometries, then all d_φ, $\varphi \in D$, assume their minimums on M.

Lemma 2.1. *Let M be oriented and let φ be an axial isometry of M. If the displacement function d_φ of φ is bounded, then φ preserves the orientation of M.*

Proof. Let $\gamma : (-\infty, \infty) \to M$ be an axis of φ. Suppose φ dose not preserve the orientation of M. Then, it follows that

$$d_\varphi(p) \geq 2d(p, \gamma(-\infty, \infty))$$

for any $p \in M$, since all minimizing geodesics joining p and φp intersect $\gamma(-\infty, \infty)$. This contradicts that d_φ is bounded on M.

Now we characterize the torus by a property of displacement functions. The idea of the proof is seen in a paper of Eberlein-O'Neill ([8], Proposition 6.8).

Proposition 2.2. *Suppose $N = M/D$ is compact and oriented. If the displacement function of an isometry in $D-\{1\}$ is bounded on M, then all the displacement functions of D are bounded on M and D is abelian. In particular, N is topologically the torus.*

Proof. Suppose d_φ, $\varphi \in D-\{1\}$, is bounded on M. Let $1 \neq \Psi \in D$ be an arbitrary isometry. Then, Ψ has an axis $\gamma : (-\infty, \infty) \to M$, i.e., $\Psi\gamma(t) = \gamma(t+a)$ for all $t \in (-\infty, \infty)$, where $a = \min d_\Psi > 0$. If we put $p = \gamma(0)$ and $\gamma_1 = \varphi\gamma$, then

$$d(p, \Psi^{-n}\varphi\Psi^{n}\varphi^{-1}(\varphi p)) = d(\Psi^{n}p, \varphi\Psi^{n}p) \leq C$$

for all integers n and some constant $C > 0$, since d_φ is bounded on M. Thus, the set $\{\Psi^{-n}\varphi\Psi^{-1}(\varphi p) ;\ n \in \boldsymbol{Z}\}$ is bounded. Since D acts freely and is properly discontinuous, there exist $m > n \in \boldsymbol{Z}$ such that

$$\Psi^{-n}\varphi\Psi^{n}\varphi^{-1} = \Psi^{-m}\varphi\Psi^{m}\varphi^{-1}.$$

Therefore,

$$\Psi^{m-n} = \varphi\Psi^{m-n}\varphi^{-1}.$$

It follows from(2.1) that γ_1 is also an axis of Ψ^{m-n}. And, Ψ has an axis γ_1 because of(2.2). Thus,

$$\Psi\varphi p = \gamma_1(a) = \varphi\gamma(a) = \varphi\Psi p.$$

This implies that $\Psi\varphi = \varphi\Psi$, since D acts freely on M.

Next we prove that d_Ψ is bounded on M. Let $\alpha : [0, L] \to M$ be a minimizing geodesic joining p and φp, where $L = d(p, \varphi p)$. Let A be the compact domain bounded by $\gamma[0, a] \cup \Psi\alpha[0, L] \cup \gamma_1[0, a] \cup \alpha[0, L]$. If $D_1 \subset D$ is the abelian subgroup generated by φ and Ψ, then $D_1 A = M$. Therefore,

$$d_\Psi \leq \max d_\Psi(A),$$

since $d_\varphi(q) = d_\Psi(x)$ for any $q \in M$ and a point $x \in A$ such that $\sigma q = x$ for some $\sigma \in D_1$.

If we apply the first part of the argument above to any isometry of D, then we conclude that D is abelian.

3. Surfaces having families of closed geodesics. In the present section we shall prove the converse of (0.2). We first prove

Theorem 3.1. *Suppose $N = M/D$ is compact. If all the displacement functions of D are constant on M, then N is a flat torus.*

Proof. By Lemma 2.1, all $\varphi \in D$ preserve an orientation of M, so N is orientable. It follows from Proposition 2.2 that N is topologically the torus. Hence, to apply the theorem of E. Hopf, we must prove only that N or M has no conjugate points. We first of all should remark that if x and y in M are over the same point in N, then there is the unique geodesic joining x and y which is a subarc of an axis of some isometry of D. This is shown from Lemma 2.1, (2.3), and Lemma 1.1.

Assume that φ and Ψ generate D. Let z be an arbitrary point of M. We define a coordinate on M as follows. Let $\alpha : (-\infty, \infty) \to M$ and $\beta : (-\infty, \infty) \to M$ be axes of φ and Ψ with $\alpha(0) = z = \beta(0)$ and with speed $c = \min d_\varphi$, $d = \min d_\Psi$, resp.. For any point x in M the axis of Ψ(and φ) through x intersects $\alpha(-\infty, \infty)$(and $\beta(-\infty, \infty)$) at exactly one point, say $\alpha(u)$(and $\beta(v)$, resp.). We give the x a coordinate (u, v). In the coordinate, all curves $u = const.$ and $v = const.$ are minimizing geodesics. And, the u- and v-coordinates of other geodesics are strictly monotone for their parameters. Thus, we represent as a curve a geodesic by $v = f(u)$ for any $u \in (-\infty, \infty)$, where f is strictly monotone. Since all $v = const.$ are parallel to each other in the sense of Busemann ([5] p. 65 (2)), $|f(u)| \to \infty$ as $|u| \to \infty$ ([4] p. 216). Hence, as seen in [4] p. 216, we can define the "slope" for any geodesic $v = f(u)$ by $\lim_{u\to\pm\infty} \frac{f(u)}{u}$. For the slope, we have from the same argument as Busemann's (see [4] pp. 216–219).

Assertion. For any $\mu \in (-\infty, \infty)$ there passes the unique minimizing geodesic through each point p of M with slope μ:

Although he proved the assertion under the hypothesis that M is straight, i.e., in the Riemannian case it is equivalent to having no conjugate points, his proof is valid in our situation by virtue of the above remark.

Let S_zM be the set of all unit vectors at z and A the set of all vectors $w \in S_zM$ such that the geodesic $\gamma_w : (-\infty, \infty) \to M$ with $\dot{\gamma}_w(0) = w$ is minimizing. The set A is closed in S_zM and contains the initial vectors of geodesics from $z = (0, 0)$ to (m, n) for all $(m, n) \in \mathbf{Z} \times \mathbf{Z}$. We want to prove that $A = S_zM$. Suppose for an indirect proof that $S_zM - A \neq \phi$. Let w_0 and w_1 be the boundary vectors of a component of $S_zM - A$. Then, we may assume that the geodesics with initial vectors w_0 and w_1 have representations of the forms

$$\gamma_{w_0} : v = f(u), \ \gamma_{w_1} : v = g(u), \text{ with } g(u) > f(u),$$

for $u > 0$. Then for any integer $n > 0$

$$0 < g(u) - f(u) < 1,$$

since, otherwise the geodesic S_n joining $(n, f(n))$ and $(n, g(n))$ contains a point of the form (n, m) with integer m, and this contradicts the choice of w_0 and w_1. However, if so, then

$$\lim_{u \to \pm\infty} \frac{f(u)}{u} = \lim_{u \to \pm\infty} \frac{g(u)}{u},$$

contradicting Assertion. Thus, $S_zM = A$. Therefore, by the theorem of Hopf, N is flat. This completes the proof.

Busemann could prove that $S_zM = A$ in the course of showing Assertion, but it was not stated because his interest was different from ours. We express Theorem 3.1 in terms of closed geodesics.

Corollary 3.2. *Suppose $N = M/D$ is compact. If each non-trivial homotopy class of closed curves in N contains a family of closed geodesics which cover N simply, then N is a flat torus.*

Proof. Let H be a non-trivial homotopy class of closed curves in N and $\varphi \in D$ correspond to H. Let Γ be a family of closed geodesics in H which cover N simply. The lift $\widetilde{\Gamma}$ of Γ to M covers M simply also. It follows from Lemma 1.1 that the geodesic connecting p and φp uniquely exists and is a subarc of the geodesics in $\widetilde{\Gamma}$ through p for any point $p \in M$. Thus, if a point p in M is over p_0 in N, then $d_\varphi(p)$ is the length of the closed geodesic in Γ through p_0. Since, because of the first variation formula, all closed geodesics in Γ have the same length, d_φ is constant on M. Corollary 3.2 follows from Theorem 3.1.

4. Example and discussion. In the present section we show that the assumptions of Theorem 1.3 and Theorem 3.1 is best in some sense. The examples and discussion here are due to Bliss([3]), Busemann-Pedersen([6]), Busemann([5]) and Innami([10]). After studying examples we investigate the condition (0.3).

Let $f: \boldsymbol{R} \to \boldsymbol{R}$ be a positive periodic function with period 1. Define a Riemannian metric on $\boldsymbol{R}^2$ by $ds^2 = dx^2 + f(x)^2 dy^2$. Then, the affine translations $\varphi: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ given by $\varphi(x, y) = (x+1, y)$ and $\Psi_t: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ given by $\Psi_t(x, y) = (x, y+t)$ for each $t \in \boldsymbol{R}$ are isometries. Hence, if D is the group of isometries generated by φ and Ψ_1, then $(\boldsymbol{R}^2, ds^2)/D = T^2$ is a torus. And, if $f(x_0) = \min f$, then (x_0, y) is a pole for any $y \in (-\infty, \infty)$. Let D_1 be the isometry group generated by $\{\varphi\} \cup \{\Psi_t : t \in (-\infty, \infty)\}$. Then, d_η assumes its minimum at all poles for any $\eta \in D_1$. Hence, for any unit vector v except for $\dfrac{\partial}{f\partial y}$ at any pole we construct a section τ_v from T^2 to ST^2 such that $f_v^t = \pi \circ g^t \circ \tau_v$ is a flow on T^2. If we choose the function f such that the length of the minimum set of f is sufficiently greater than the complement, we have

(4.1) For any $\varepsilon > 0$ there is a non-flat Riemannian metric G on T^2 such that $\lambda > \mu \geq (1-\varepsilon)\lambda$, where λ is the volume of ST^2 and μ is the volume of the set of all $v \in ST^2$ such that f_v^t is a flow on T^2.

On the assumption of Theorem 3.1 and Corollary 3.2 we have

(4.2) For any $\eta \in D$ except for $D_2 = \{\Psi_1^n \,;\, n \in \boldsymbol{Z}\}$ the displacement function d_η is constant on $(\boldsymbol{R}^2, ds^2)$.

Each non-trivial homotopy class of closed curves in T^2 except for ones corresponding to D_2 contains a family of closed geodesics which cover T^2 simply.

Let $f(x_0) = \min f$ and $f(x_1) > \min f$, $x_0 < x_1 < x_0 + 1$. Let $\alpha: (-\infty, \infty) \to (\boldsymbol{R}^2, ds^2)$ be a geodesic with $\alpha(t) = (x(t), y(t))$ and $\alpha(0) = (x_1, y_1)$. It follows from a theorem of Clairaut that $f(x(t))\cos\theta(t) = f(x(0))\cos\theta(0)$

for all $t \in (-\infty, \infty)$, where $\theta(t)$ is the angle of $\dot{\alpha}(t)$ and $\frac{\partial}{\partial y}$ at $\alpha(t)$. Hence, if $f(x_1)\cos\theta(0) > \min f$, and if $x_0 < x_2 \leq x_1 \leq x_3 < x_0+1$ with $f(x_2) = f(x_3) = f(x_1)\cos\theta(0)$, then α lies in the strip bounded by the lines $(x_2, \boldsymbol{R})$ and $(x_3, \boldsymbol{R})$. In particular, we have

Proposition 4.1. *Let T^2 be a torus with metric $ds^2 = dx^2+f^2(x)dy^2$. If the set of all tangent vectors of geodesics which are dense in T^2 is dense in ST^2, then $f(x) = const.$, and, in particular, T^2 is flat.*

The similar situation arises in the case of tori having poles. Let $T^2 = \boldsymbol{R}^2/D$ be a torus having poles. Suppose a point $p \in \boldsymbol{R}^2$ lies over a pole in T^2. Then, d_φ assumes its minimum at p for all $\varphi \in D$ (see [10]). Let φ be an isometry of D and $\gamma : (-\infty, \infty) \to \boldsymbol{R}^2$ be the axis of φ with $\gamma(0) = p$. Then, there passes the unique asymptote to γ through each point q, since if $d_\varphi(p) = c$, $\gamma(nc)$ is a pole for each $n \in \boldsymbol{Z}$ (see [4], [5], [10]). Let p_1 be another pole in $\boldsymbol{R}^2$ and let $\gamma_1 : (-\infty, \infty) \to \boldsymbol{R}^2$ be the axis of φ through p_1. In general, the asymptotes to γ and γ_1 through a point q in the strip bounded by γ and γ_1 are different. Suppose $\alpha, \beta : (-\infty, \infty) \to \boldsymbol{R}^2$ are the asymptotes to γ and γ_1 with $\alpha(0) = \beta(0) = q$, $\alpha \neq \beta$, resp.. Let A be the domain bounded by $\alpha[0, \infty)$ and $\beta[0, \infty)$ which does not contain both γ and γ_1 and S the set of all $v \in S\boldsymbol{R}^2$ inward A. We assert

Lemma 4.2. *All geodesics emanating from q with initial tangent vectors in S lie entirely in A.*

Proof. Suppose there is a $v \in$ S such that the geodesic $\gamma_v : [0, \infty) \to \boldsymbol{R}^2$ with $\dot{\gamma}_v(0) = v$ intersects α or β, say $q_1 = \alpha(s_0) = \gamma_v(t_0)$ for the first intersection point. Since α is an asymptote to γ, the geodesics $\alpha_n : [0, L_n] \to \boldsymbol{R}^2$ connecting q_1 to $\gamma(nc)$ converges to $\alpha[0, \infty)$ as n goes to infinity. This implies that the domain B surrounded by $\beta[0, \infty) \cup \alpha[0, s_0] \cup \alpha_n[0, L_n] \cap \gamma[nc, \infty)$ is locally convex for sufficiently large n, where "locally convex" means that the angle of the domain at each vertex is less than or equal to π. Hence, a shortest curve from q to $\gamma(nc)$ in A is a geodesic $\gamma_2 : [0, L] \to \boldsymbol{R}^2$, which intersects $\alpha[0, \infty)$ twice, at $\alpha(0)$ and another point q_2. Since $\gamma(nc)$ is a pole, γ_2 is minimizing in $\boldsymbol{R}^2$. However, since α is minimizing, the minimizing geodesic connecting $\alpha(0)$ to q_2 must exist uniquely, a contradiction. This completes the proof.

We do not know whether $d(\gamma_v[0, \infty), \gamma(-\infty, \infty)) > 0$ and $d(\gamma_v[0, \infty),$

$\gamma_1(-\infty, \infty)) > 0$ for any geodesic $\gamma_v : [0, \infty) \to \boldsymbol{R}^2$ with $\dot{\gamma}(0) = v \in S$. So we need an additional assumption for analogous rseult to Proposition 4.1.

Proposition 4.3. *Let $T^2 = M/D$ be a torus having two poles which do not lie simultaneously in any closed geodesic. If the set of all tangent vectors of geodesics which are dense in T^2 is dense in ST^2, then T^2 is flat.*

Proof. Let $\varphi \in D$. We shall prove that the displacement function d_φ is constant on M. We may assume that no $\eta \in D$ exists such that $\varphi = \eta^n$, $n \neq \pm 1$, since d_φ is so if d_η is constant on M. Let Ψ be an isometry of D such that φ and Ψ generate D. If $\gamma, \alpha : (-\infty, \infty) \to M$ are axes of φ and Ψ, resp., through a pole $p = \gamma(0) = \alpha(0)$ in M, then the compact domain surrounded by $\gamma(-\infty, \infty)$, $\alpha(-\infty, \infty)$, $\Psi\gamma(-\infty, \infty)$ and $\varphi\alpha(-\infty, \infty)$ is the closure F of a fundamental domain of T^2. Another pole p_1 exists in the interior of F. If $\gamma_1 : (-\infty, \infty) \to M$ is the axis of φ through p_1, then $\gamma_1(-\infty, \infty)$ decomposes the strip B bounded by $\gamma(-\infty, \infty)$ and $\Psi\gamma(-\infty, \infty)$ into two components B_1 and B_2. Suppose there is a point q, in B_1 say, such that the asymptotes to two axes of φ which consist of the boundary of B_1 are different. Since the angular domain in B_1 as in Lemma 4.2 can be constructed, the set of all tangent vectors of geodesics which are dense in T^2 is not dense in ST^2, a contradiction. Thus, B is covered simply by a family of asymptotes to both γ and $\Psi\gamma$. Suppose for an indirect proof that d_φ is not constant. Then, from Lemma 5.1, there exists a geodesic $\tau : (-\infty, \infty) \to M$ contained in the interior of B which is not an axis of φ. Put $x = \tau(0)$. Since τ does not have any self-intersection point, it decomposes B into two components C_1 and C_2 also. By Lemma 1.1, the asymptote to both γ and $\Psi\gamma$ from x is entirely contained in one of them, say C_1 and $C_2 \supset \gamma(-\infty, \infty)$. A shortest curve from x to $\gamma(nc)$ in C_2 must be the unique minimizing geodesic in M, since $\gamma(nc)$ is a pole for any $n \in \boldsymbol{Z}$, where $c = \min d_\varphi$. This implies that the asymptote to γ from x lies in C_2, a contradiction. Thus, d_φ is constant on M. Proposition 4.3 follows from Theorem 3.1.

5. Geodesic rays which avoid crossing closed curves. In the present section we investigate the condition (0.4). We first prepare a lemma. The idea of the proof was used by Bangert ([1], [2]).

Lemma 5.1. *Let φ be an axial isometry of M having two axes and B the strip bounded by the axes. If the displacement function d_φ is not constant in B, then there is a geodesic $\gamma : (-\infty, \infty) \to M$ contained in the interior of*

B which is a lift of a closed geodesic in B/D, but not an axis of φ, where $D=\{\varphi^n; n\in \mathbf{Z}\}$.

Proof. Take a strip $B_1 \subset B$ whose boundary consists axes γ and γ_1 of φ such that $d_\varphi(q) > \min d_\varphi(M)$ for any q in the interior of B_1. We shall find a closed geodesic in the interior of $B_1/D = B_0$. Let h_0 and $h_1 : [0,1] \to B_0$ be over γ and γ_1 and $h_s : [0,1] \to B_0$, $s \in [0,1]$, a homotopy from h_0 to h_1. Let $2c$ be the minimum of the convexity radii of all points in B_0. Take a partition $0 = t_0 < t_1 < \cdots < t_{2n} = 1$ of $[0,1]$ such that for each $s \in [0,1]$ the lengths of $h_s[t_i, t_{i+2}]$ are less than c for $i = 0, 1, \cdots, 2n-1$. Define a new homotopy Dh_s from h_s to h_1 as follows. Dh_s is a closed and broken geodesic which consists of minimizing geodesics connecting $h_s(t_{2i})$ to $h_s(t_{2(i+1)})$, i $= 0, 1, \cdots,$ n-1. Again, define a homotopy D^2h_s from h_0 to h_1 as follows. D^2h_s consists of minimizing geodesics from $Dh_s(t_{2i+1})$ to $Dh_s(t_{2i+3})$, $i = 0, 1, \cdots, n-1$. Repeating the process we have a sequence $\{D^nh_s ; n = 0, 1, \cdots\}$ of homotopies. For each n, since D^nh_s is a homotopy from h_0 and h_1, the set $J_n = \{s \in [0,1]$; the length of D^nh_s is greater then or equal to $\max d_\varphi(B_1)\}$ is closed and not empty. Since $J_n \supset J_{n+1}$ for all n, a real $s_0 \in \bigcap_{n=1}^{\infty} J_n$ exists. Thus, the limit of a converging subsequence of $\{D^nh_{s0}\}$ is a closed geodesic whose length is greater than or equal to $\max d_\varphi(B_1)$, since the number of vertices of each D^nh_{s0} is n. This completes the proof.

We now prove the converse of (0.4).

Theorem 5.2. *If a torus T^2 satisfies the condition(0.4), then T^2 is flat.*

Proof. Let $T^2 = M/D$ and $\varphi \in D$. We may assume as in the proof of Proposition 4.3 that φ and some Ψ can generate D. Let $\gamma : (-\infty, \infty) \to M$ be an axis of φ and B the strip bounded by γ and $\Psi\gamma$. We can consider the γ and $\Psi\gamma$ as two lifts of a non-contractible closed curve in T^2. By the condition (0.4), the asymptotes from any point in B to γ and $\Psi\gamma$ must be identified. Suppose the displacement function d_φ is not constant in B. From Lemma 5.1 there exists a geodesic $\alpha : (-\infty, \infty) \to M$ contained in B such that $\Psi\alpha(t) = \alpha(t+a)$, $a > \min d_\varphi$. The α must be an asymptote to γ because of (0.4). Then it follows that α is an axis of φ(see [5] p. 65, (2)). This contradicts that $a > \min d_\varphi$. This completes the proof.

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FACULTY OF INTEGRATED ARTS AND SCIENCES
HIROSHIMA UNIVERSITY
HIROSHIMA, 730 JAPAN

DEPARTMENT OF MATHEMATICS
NAGOYA INSTITUTE OF TECHNOLOGY
NAGOYA, 466 JAPAN
(Present Address)

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