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ON THE FIXED POINT SET OF S^1-ACTIONS ON THE SPACE WHOSE RATIONAL COHOMOLOGY RING IS GENERATED BY ELEMENTS OF DEGREE 2

Dedicated to Professor Masahiro Sugawara on his 60th birthday

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1. **Introduction.** Let $ES^1 \to BS^1$ be a universal S^1-bundle where S^1 is the circle group. Let us denote X_{S^1} the total space $ES^1 \times_{S^1} X$ of the associated bundle with a fiber X. The cohomology ring of X_{S^1} is called the equivariant cohomology ring of a S^1-space X. We assume the following condition

$$(1.1) \quad H^*(X, Q) = Q[x_1, \dots, x_n]/(\phi_1(x), \dots, \phi_n(x)),$$

where $\phi_i(x)$ is a homogeneous polynomial in $x_1, \dots, x_n$ and $\deg x_i = 2$. Let $\pi: X_{S^1} \to BS^1$ and $i: X \to X_{S^1}$ be the projection and the inclusion of a fiber respectively. Since i^* is a surjection by (1.1), we can take multiplicative generators $x'_1, \dots, x'_n$ and t' such that $i^*(x_i) = x'_i$ and $t' = \pi^*(t)$, where t is a generator of $H^2(BS^1, Q)$. In order to abbreviate the notations let us also use x_i and t in the meaning of x'_i and t' respectively. Then we have

$$(1.2) \quad H^*(X_{S^1}, Q) = Q[x_1, \dots, x_n, t]/(f_1(x, t), \dots, f_n(x, t)),$$

where $f_i(x, t)$ is a homogeneous polynomial and $f_i(x, 0) = \phi_i(x)$.

In [4] we have shown that there is a bijective correspondense between the set of connected components of the fixed point set X^{S^1} and the sol utions of the simultaneous equations $f_1(x, t) = 0, \dots, f_n(x, t) = 0$. Any solutions $\xi = (\xi_1, \dots, \xi_n, 1)$ are rational i.e. ξ_i is a rational number. The ideal $(f_1, \dots, f_n)$ satisfise a local condition concerning the multiplicity of ξ and the cohomology of the connected component F of X_{S^1} corresponding to ξ at each ξ. Conversely, if homogeneous polynomials $f_i(x, t)$ $(1 \leq i \leq n)$ are given which satisfy the conditions, then by V. Puppe's theorem ([8]) there is a S^1-space such that (1.1) and (1.2) hold.

This paper consists of an observation and its consequences. Let F be a connected component of X^{S^1} corresponding to ξ. We observe that there are homogeneous polynomials $\chi_i(x)$ and $A_{ij}(x, t)$ $(1 \leq i, j \leq n)$ such that $f_i(x_1 + \xi_1 t, \dots, x_n + \xi_n t, t) = \sum_j A_{ij}(x, t)\chi_j(x)$. This is equivalent to the local condition stated in the above. $\chi_i(x)$ discribes the cohomology of F and det $(A_{ij}(x, t))$ is considered as the image of $1 \in H^0(F_{S^1}, Q)$ by the equivariant Gysin homomorphism. If the action is smooth, and F is an isolated fixed point, then the tangential representation of S^1 at F is determined up to finite possibilities by $f_1,\dots,f_n$. We say that a connected component F of X^{S^1} is 'generic' if F has the rational cohomology of a point or a sphere S^2. Then, as an consequence we have the following: if the u-resultant of $\phi_1(x),\dots,\phi_{n-1}(x)$ is irreducible over Q and deg $\phi_n(x)$ is very large, then there exist sufficiently many generic connected components however the number of non generic connected components is bounded from the above by a constant depending only on deg $\phi_1,\dots,$beg ϕ_{n-1}.

2. A local condition concerning equivariant cohomology ring. Let us denote $Q[x_1,\dots,x_n]$ (or $Q[x]$) the polynomial ring with rational coefficients in n variables $x_1,\dots,x_n$. Consider a homogeneous ideal $(\phi_1,\dots,\phi_n)$ of $Q[x]$, where deg $\phi_i(x) = r_i$. We assume

(2.1) $\dim_Q Q[x_1,\dots,x_n]/(\phi_1(x),\dots,\phi_n(x)) < \infty.$

This is equivalent to the condition that the simultaneous equations $\phi_1(x)=0$, $\dots, \phi_n(x) = 0$ do not have non-trivial solutions in the field C of complex numbers. Then $\dim_Q Q[x]/(\phi) = \prod_{i=1}^{n} r_i$ and the Jacobian det$(\partial(\phi_1,\dots,\phi_n)/\partial(x_1,\dots,x_n))$ is a generator of the graded ring $Q[x]/(\phi)(\deg x_i = 2)$ in the highest degree $2\sum_{i=1}^{n}(r_i-1)$ (see [6, Th. 18]).

Let us consider an homogeneous ideal $(f_1(x, t),\dots,f_n(x, t))$ of the polynomial ring $Q[x_1,\dots,x_n, t]$ where

(2.2) $f_i(x, 0) = \phi_i(x).$

Then, by (2.1), the number of solutions of the equations $f_1(x, t) = 0,\dots, f_n(x, t) = 0$ is finite. Let us denote $\xi^{(\alpha)} = (\xi_1^{(\alpha)},\dots,\xi_n^{(\alpha)}, 1)$, $1 \leq \alpha \leq \omega$ the set of the solutions. We assume the following

(2.3) $\xi_i^{(\alpha)} \in Q$, $1 \leq i \leq n$, $1 \leq \alpha \leq \omega$.

For each $\xi^{(\alpha)}$, let us denote I_α the homogeneous ideal of $Q[x]$ generated by the coefficients of powers of t in $f_i(x_1+\xi_1^{(\alpha)}t,\dots,x_n+\xi_n^{(\alpha)}t, t)$ $(1 \leq i \leq n)$. I_α contains the ideal (ϕ) and $((f_1),\dots,(f_n))_\xi(\alpha) \leq \dim_Q Q[x]/I_\alpha$ where $((f_1),\dots,(f_n))_\xi(\alpha)$ is the multiplicity of $\xi^{(\alpha)}$ (cf. [9. p. 183] for notation). We assume the following local condition holds at each $\xi^{(\alpha)}$.

(2.4) $((f_1),\ldots,(f_n))_{\xi^{(\alpha)}} = \dim_Q Q[x]/I_\alpha$.

In this situation, we can state the equivariant realization theorem of V. Puppe as follows.

Theorem (V. Puppe [8]). *Let* $\phi_i(x)$ *and* $f_i(x, t)$ $(1 \leq i \leq n)$ *be homogeneous polynomials that satisfy* (2.1),...,(2.4). *Then there is a finite* S^1-*CW complex* X *such that* (1.1) *and* (1.2) *hold.*

Remark. This suggests a problem: Is there non-trivial (f) satisfying (2.2),...,(2.4) for any (ϕ) ?

(2.3) and (2.4) mean the following ([4]). Let j be the inclusion of a connected component F_α of X^{S^1} into X. Then there is a bijective correspondence between $\xi^{(\alpha)}$ and F_α as follows.

$$j^*(x_i) = x_i|F_\alpha + \xi_i^{(\alpha)}t,\ 1 \leq i \leq n,$$

where $x_i|F_\alpha$ is the restriction of $x_i \in H^2(X, Q)$. Let us also use x_i as meaning of $x_i|F_\alpha$. Then, in this notations we have

(2.5) $H^*(F_\alpha, Q) = Q[x]/I_\alpha$.

In this paper we always use the notations ϕ_i, f_i, $\xi^{(\alpha)}$, X etc. in the meaning stated heretofore, even if anything is not stated. Let be a solution of $f_1 = \cdots = f_n = 0$. Without loss of generality we may suppose $\xi = (0,\ldots,0,1)$.

Proposition 2.1. *The condition* (1.4) *at* $\xi = (0,\ldots,0,1)$ *is equivalent to the following: There are homogeneous polynomials* $\chi_i(x)$ *and* $A_{ij}(x, t)$ $(\deg \chi_i = s_i,\ \deg A_{ij} = r_i - s_j,\ 1 \leq i, j \leq n)$ *such that*

(i) $\dim_Q Q[x]/(\chi(x)) < \infty$,

(ii) $\det(a_{ij}) \neq 0$, where $A_{ij}(x, t) = a_{ij}t^{r_i - s_j} +$ the lower terms in t and $a_{ij} \in Q$, and

(iii) $f_i(x, t) = \sum_j A_{ij}(x, t)\chi_j(x) \bmod (x_1,\ldots,x_n)^{s+1}$, $1 \leq i \leq n$, where $s = \sum_i (s_i - 1)$.

Proof. Let us denote J the ideal of $Q[x]$ generated by

$$\tilde{f}_i(x) = f_i(x, 1) = \phi_i^{(r_i)}(x) + \cdots + \phi_i^{(1)}(x),\ 1 \leq i \leq n$$

where deg $\phi_i^{(\alpha)} = d$ and $\phi_i^{(r_i)} = \phi_i$. We may assume that $\phi_1^{(1)}, \dots, \phi_k^{(1)}$ are linearly independent over Q and $\phi_j^{(1)} = \sum_i a_i \phi_i^{(1)}$ $j > k$ and $a_i \in Q$. We replace f_j by $\tilde{f}_j - \sum_{i=1}^{k} a_i \tilde{f}_i$, then we may $J = (\tilde{f}_1, \dots, \tilde{f}_n)$ and $\phi_j^{(1)} = 0$ $(j > k)$. Now assume $\phi_{k+1}^{(2)}, \dots, \phi_{k+l}^{(2)}$ are linearly independent mod $\{L_1\phi_1^{(1)} + \dots + L_k\phi_k^{(1)} \mid L_i(x)$ a linear form$\}$ and

$$\phi_j^{(2)} = \sum_{i=1}^{k} L_i \phi_i^{(1)} + \sum_{h=1}^{l} b_h \phi_{k+h}^{(2)}, \; b_h \in Q, \; j > k+l.$$

Replacing $\tilde{f}_j$ $(j > k+l)$ by $\tilde{f}_j - \sum_i L_i f_i - \sum_h b_h f_{k+h}$, we can suppose farther that $\phi_j^{(2)} = 0$ for $j > k+l$. Iterating at most $n-1$ times, we obtain generators $\tilde{f}_1, \dots, \tilde{f}_n$ of J. Let us denote χ_i the lowest term of f_i From the choice of χ_i we get for any polynomials $B_j(x)$ and $1 \leq i \leq n$

$$(2.6) \quad \chi_i(x) \not\equiv \sum_{j \neq i} B_j(x) f_j(x) \mod (x_1, \dots x_n)^{\deg \chi_i + 1}.$$

Let us consider the primary decomposition $J = \bigcap_j q_j$ of the ideal J. We may assume that the radical of q_1 is the maximal ideal $m = (x_1, \dots, x_n)$, because of $\xi = (0, \dots, 0)$. Let I be the ideal corresponding to ξ. By [4], (2.4) implies $q_1 = I$. In the localized ring $Q[x]_{(m)}$ we have $q_1 Q[x]_{(m)} = (f_1, \dots, f_n) Q[x]_{(m)}$ and hence $\chi_i = \sum_j h_{ij} f_j$ where $h_{ij} \in Q[x]_{(m)}$ since $\chi_i \in I$. Now consider $Q[x]_{(m)}$ as a subring naturally imbedded in the formal power series ring $Q\{x_1, \dots, x_n\}$. If we can write as an element of $Q\{x\}$: $h_{ij} = c_{ij} +$ the higher terms, $c_{ij} \in Q$, then we have $(1 - c_{ii}) \chi_i = \sum_{j \neq i} B_j(x) f_j \mod (x_1, \dots, x_n)^{\deg \chi_i + 1}$, with some polynomials $B_j(x)$. This implies $c_{ii} = 1$ by (2.6). On the other hand by the choice of $\chi_1, \dots, \chi_n$ we have $c_{ij} = 0$ for $i < j$ and hence $\det(h_{ij})$ is a unit in $Q\{x\}$. Moreover, there are polynomials $p_{ij}(x)$ such that $\tilde{f}_i = \sum_j p_{ij} f_j(x, 1)$ and $\det(p_{ij})$ is also an unit in $Q\{x\}$. Thus we can write $f_i(x, 1) = \sum_j A'_{ij} \chi_j$ where the matrix (A'_{ij}) is equal to $((h_{ij})(p_{ij}))^{-1}$. Let us denote A_{ij} the sum of the terms in A'_{ij} of degrees equal or less than $\deg f_i - \deg \chi_j$ and consider as a homogeneous polynomial by introducing the variable t. If we set $A_{ij} t^{r_i - s_j} +$ the lower terms in t, $a_{ij} \in Q$ and $s_j = \deg \chi_j$, then $\det(a_{ij}) \neq 0$. Since $f_i(x, t)$ is a polynomial and $\chi_1, \dots, \chi_n$ are homogeneous, we have $f_i(x, t) = \sum_j A_{ij}(x, t) \chi_j$ from the equality stated in the above. In particular this implies that $\phi_i \in (\chi_1, \dots \chi_n)$ and hence (i) holds for $\chi_1, \dots, \chi_n$.

Next we show the converse. By (i) we can replace A_{ij} so that the congruence in (iii) becomes an epuality. Then $(\tilde{f}_1,\dots,\tilde{f}_n)Q\{x\} = (\chi_1,\dots,\chi_n)Q\{x\}$ where $\tilde{f}_i(x) = f_i(x, 1)$. By the definition of the multiplicity,

$$\begin{aligned}((f_1),\dots,(f_n))_\xi &= \dim_Q Q\{x\}/(\tilde{f}_1,\dots,\tilde{f}_n)Q\{x\} \\ &= \dim_Q Q\{x\}/(\chi_1,\dots,\chi_n)Q\{x\} \\ &= \dim_Q Q[x]/(\chi_1,\dots,\chi_n).\end{aligned}$$

Since $(\chi_1,\dots,\chi_n)$ is the ideal corresponding to ξ in (2.4), this completes the proof.

Assume $r_1 \leq \cdots \leq r_n$ and $s_1 \leq \cdots \leq s_n$ where $\deg \phi_i = r_i$ and $\deg \chi_i = s_i$. Then we have the following

Proposition 2.2. $s_i \leq r_i$, $1 \leq i \leq n$.

Proof. Suppose $s_i > r_i$ for some i. Then $\phi_1,\dots,\phi_i$ are represented by $\chi_1,\dots,\chi_{i-1}$ so that $(\phi_1,\dots,\phi_n)$ are contained in the ideal$(\chi_1,\dots,\chi_{i-1}, \phi_{i+1},\dots, \phi_n)$. Since the simultaneous equations $\chi_1 = \cdots = \chi_{i-1} = \phi_{i+1} = \cdots = \phi_n = 0$ have a non-trivial solution, by [11, p. 11], the equation $\phi_1 = \cdots = \phi_n = 0$ have a non-trivial solution. This contradicts to (2.1) and completes the proof.

Remark. The ideal I in (2.4) is generated by n polynomials by Proposition 2.1, however this follows also from [1].

3. Bredon's orientation for connected components of X^{S^1}. Corresponding to the maps $(X^{S^1})_{S^1} \xrightarrow{j} X_{S^1} \xleftarrow{i} X$ we have the ring homomorphisms

$$Q[x]/(\phi) \xleftarrow{i^*} Q[x, t]/(f) \xrightarrow{j^*} \Big(\bigoplus_{\alpha=1}^{\omega} Q[x]/(\chi^{(\alpha)})\Big) \otimes Q[t],$$

where $i^*(x) = x$, $i^*(t) = 0$, $j^*(t) = t$ and $j^*(x) = \sum_\alpha (x+\xi^{(\alpha)}t)$. Let us denote $l = \sum_i (r_i-1)$ and $l_\alpha = \sum_i (s_i^{(\alpha)}-1)$, where $s_i^{(\alpha)} = \deg \chi_i^{(\alpha)}$. Then $2l$ and $2l_\alpha$ are the formal dimensions of the graded ring $Q[x]/(\phi)$ and $Q[x]/(\chi^{(\alpha)})$ respectively. Let $\xi^{(\alpha)}$ be a solution of the equations $f_1 = \cdots = f_n = 0$. Then we have by Proposition 2.1

(3.1) $f_i(x+\xi^{(\alpha)}t, t) = \sum_j A_{ij}^{(\alpha)}(x, t)\chi_j^{(\alpha)}(x)$, where $A_{ij}^{(\alpha)} = a_{ij}^{(\alpha)} t^{r_i - s_j^{(\alpha)}}$ + the lower terms in t, $a_{ij}^{(\alpha)} \in Q$ and $\det(a_{ij}^{(\alpha)}) \neq 0$. We put $A^{(\alpha)} = \det(A_{ij}^{(\alpha)}(x - \xi^{(\alpha)}t, t))$.

Lemma 3.1. *$j^*(A^{(\alpha)})$ is 0 in $Q[x]/(\chi^{(\beta)}) \otimes Q[t]$ if $\beta \neq \alpha$, and* det *$(a_{ij}^{(\alpha)}) t^{l-l_\alpha}$ + the lower terms in $Q[x]/(\chi^{(\alpha)}) \otimes Q[t]$.*

Proof. The second formula is clear from (3.1) and definitions. We need to show the first. From $0 = j^*(f_i(x, t)) = f_i(x + \xi^{(\beta)}t)$, we have in $Q[x]/(\chi^{(\beta)}) \otimes Q[t]$

$$(3.2) \quad \sum_j A_{ij}^{(\alpha)}(x + (\xi^{(\beta)} - \xi^{(\alpha)})t, t) \chi_j^{(\alpha)}(x + (\xi^{(\beta)} - \xi^{(\alpha)})t) = 0.$$

Since $\xi^{(\alpha)} \neq \xi^{(\beta)}$, we have $\chi_i^{(\alpha)}(x + (\xi^{(\beta)} - \xi^{(\alpha)})t) = d_i t^{s_i^{(\alpha)}}$ + the lower terms in t, $d_i \neq 0$ for some i. Multiplying the matrix of cofactors of the matrix $(A_{ij}^{(\alpha)}(x + (\xi^{(\beta)} - \xi^{(\alpha)})t,)$ to (2.2), we have

$$\det(A_{ij}^{(\alpha)}(x + (\xi^{(\beta)} - \xi^{(\alpha)})t,)(d_i t^{s_i^{(\alpha)}} + \cdots) = 0.$$

From this we hav $j^*(A^{(\alpha)}) = 0$ in $Q[x]/(\chi^{(\beta)}) \otimes Q[t]$.

Lemma 3.2. *If $\phi_i(x) = \sum_j B_{ij}(x) \chi_j$, $1 \leq i \leq n$* where *$B_{ij}(x)$ is homogeneous of degree $r_i - s_j$, then we have*

$$\prod_{i=1}^{n} (s_i/r_i) \det(\partial\phi_i/\partial x_j) \equiv \det(B_{ij}) \det(\partial\chi_i/\partial x_j) \bmod (\phi_1, \ldots, \phi_n).$$

Proof. If we define C_{ij} by the matrix equation

$$(3.3) \quad (B_{ij})((1/s_i)\partial\chi_i/\partial x_j) = ((1/r_i)\partial\phi_i/\partial x_j) + (C_{ij}),$$

then we have $\sum_j C_{ij} x_j = 0$ by using Euler's formula $\sum_j (\partial\phi_i/\partial x_j) x_j = r_i \phi_i$. Then Lemma 3.2 follows from (3.3) and the following : Let (D_{ij}) be a $(n \cdot n)$-matrix where D_{ij} is a polynomial in $x_1, \ldots, x_n$. If $\sum_j D_{1j} x_j = 0$ and $\sum_j D_{ij} x_j \equiv 0 \bmod (\phi_1, \ldots, \phi_n)$ for $i = 2, \ldots, n$ then $\det(D_{ij}) \equiv 0 \bmod (\phi_1, \ldots, \phi_n)$.

In order to show this we may assume that $D_{11} = d_1 x_2 \ldots x_k, \ldots, D_{1k} = d_k x_1 \ldots x_{k-1}$, $D_{1j} = 0$ $(j > k)$ and $d_1 + \ldots + d_k = 0$, since $\det(D_{ij})$ is linear in $D_{11}, \ldots, D_{1n}$. Then we have

$$\begin{vmatrix} 0, & d_2 x_1 x_3 \ldots x_k, \ldots, d_k x_1 \ldots x_{k-1}, & 0 \\ & * & \end{vmatrix} = \begin{vmatrix} 0, & d_2 x_3 \ldots x_k, \ldots, d_k x_2 \ldots x_{k-1}, & 0 \\ D_{i1} x_1, & * & \end{vmatrix}$$

$$= \begin{vmatrix} d_j x_2 \dots x_k, 0, \dots, & 0 \\ 0, & * \end{vmatrix} \bmod (\phi_1, \dots, \phi_n),$$

and hence $\det(D_{ij}) \equiv 0 \bmod (\phi_1, \dots, \phi_n)$.

Now we consider $\det(\partial f_i/\partial x_j)$ as an element $\tilde{\mu}$ of degree $2l$ in the graded ring $Q[x, t]/(f)$. Let us denote $P_\alpha(x) = \det(\partial \chi_i^{(\alpha)}/\partial x_j)$. Then we have the following

Lemma 3.3. $\tilde{\mu} = \sum_{\alpha=1}^{\omega} A^{(\alpha)} P_\alpha(x - \xi^{(\alpha)} t)$.

Proof. By Lemma 3.1 we have

$$j^*\Big(\sum_\alpha A^{(\alpha)} P_\alpha(x-\xi^{(\alpha)}t)\Big) = \sum_\alpha \det(a_{ij}^{(\alpha)} t^{l-l_\alpha} + \cdots) \det(\partial \chi_i^{(\alpha)}/\partial x_j)$$
$$= \sum_\alpha \det(a_{ij}^{(\alpha)}) \det(\partial \chi_i^{(\alpha)}/\partial x_j) t^{l-l_\alpha}.$$

On the other hand, by (3.1)

$$j^*(\tilde{\mu}) = \sum_\alpha \det(A_{ij}^{(\alpha)}(x, t)) \det(\partial \chi_i^{(\alpha)}/\partial x_j)$$
$$= \sum_\alpha \det(a_{ij}^{(\alpha)}) \det(\partial \chi_i^{(\alpha)}/\partial x_j) t^{l-l_\alpha}.$$

Since j^* is injective this implies $\tilde{\mu} = \sum_\alpha A^{(\alpha)} P_\alpha(x - \xi^{(\alpha)} t)$.

By the fact stated in § 1, we can take $\det(\partial \phi_i/\partial x_j)$ as an orientation μ of the graded ring $Q[x]/(\phi)$. Then we define the Bredon's orientation μ_α of the graded ring $Q[x]/(\chi^{(\alpha)})$ as follows : There is an element $\theta \in Q[x, t]/(f)$ of degree 2 such that $j^*(\theta) = \mu_\alpha t^{l-l_\alpha}$ and $i^*(\theta) = \mu$. Since such a θ is unique, μ_α is uniquely determined.

Proposition 3.4. $\mu_\alpha = \prod_{i=1}^{n} (r_i/s_i) \det(a_{ij}^{(\alpha)}) \det(\partial \chi_i^{(\alpha)}/\partial x_j)$.

Proof. If we set $\theta = \prod_i (r_i/s_i) A^{(\alpha)} P_\alpha(x - \xi^{(\alpha)} t)$, then, by Lemma 3.2, $i^*(\theta) = \mu$ and by Lemma 3.1,

$$j^*(\theta) = \prod_i (r_i/s_i) \det(a_{ij}^{(\alpha)}) \det(\partial \chi_i^{(\alpha)}/\partial x_j) t^{l-l_\alpha}.$$

This completes the proof.

Let M be a closed oriented smooth S^1-manifold such that $H^*(M, Q) =$

$Q[x]/(\phi)$ and $H^*(M_{S^1}, Q) = Q[x, t]/(f)$. We assume the following

(3.4) $\langle \det(\partial\phi_i/\partial x_j), [M]\rangle = 1$,

where $\langle , \rangle$ means the Kronecker product. Let F_α be a connected component of M^{S^1} and $N(F_\alpha, M)$ be the normal bundle of F_α in M. We may regard $N(F_\alpha, M)$ as a complex vector bundle such that the representation ϕ_α of S^1 in a fiber is given as follows,

(3.5) $\phi_\alpha = z^{a_1^{(\alpha)}} + \cdots + z^{a_{l-l_\alpha}^{(\alpha)}},\ a_h^{(\alpha)} > 0$,

in the complex representation ring $R(S^1) = Z[z, z^{-1}]$. By this orientation of the nomral bundle, F_α can be naturally oriented. The inclusion j of F_α into M induces the equivariant Gysin homomorphism $j_!: H^*(F_{\alpha S^1}, Q) \to H^*(M_{S^1}, Q)$. Let us denote 1_α the unity in $H^0(F_{\alpha S^1}, Q)$. Then by a property of $j_!$ we have

$$j^*(j_!(1_\alpha)) = \begin{cases} 0 \text{ in } F_\beta & \text{if } \beta \neq \alpha, \\ e^{S^1}(N(F_\alpha, M)) & \text{in } F_\alpha, \end{cases}$$

where $e^{S^1}(N(F_\alpha, M))$ is the equivariant Euler class of $N(F_\alpha, M)$ (see [7]) In this notations, we have the following and (iii) in that is a generalization of [2, Th. 5.5, p.397].

Theorem 3.5. *(i) $j_!(1_\alpha) = cA^{(\alpha)}$, where $c = \prod_{i=1}^{l-l_\alpha} a_i^{(\alpha)}/\det(a_{ij}^{(\alpha)})$, (ii) the equivariant Euler class of M is equal to $\chi(M)\det(\partial f_i/\partial x_j)$, and (iii) $(1/\prod_i a_i^{(\alpha)})\mu_\alpha$ is the cohomology orientation of F_α i. e.*

$$\chi(M)\det(a_{ij}^{(\alpha)})\langle\det(\partial\chi_i^{(\alpha)}/\partial x_j), [F_\alpha]\rangle = \chi(F_\alpha)\prod_{i=1}^{l-l_\alpha} a_i^{(\alpha)}.$$

Proof. Since $e^{S^1}(N(F_\alpha, M)) = \prod_i a_i^{(\alpha)} t^{l-l_\alpha}$ + the lower terms in t, together with Lemma 3.1 and the property of j, we have $j^*(j_!(1_\alpha) - cA^{(\alpha)}) = at^i$ + the lower terms in t where $a \in H^*(F_\alpha, Q)$, $\deg a > 0$. Let μ_α be the cohomology orientation of F_α i. e. $\langle\mu_\alpha, [F_\alpha]\rangle = 1$. If $a \neq 0$, then we can take an element $b = q(x)$ of $H^*(F_\alpha, Q)$ such that $ab = \mu_\alpha$. Then,

$$j^*(q(x - \xi^{(\alpha)}t)(j_!(1_\alpha) - cA^{(\alpha)})) = \mu_\alpha t^i.$$

This implies, by [3, Lemma 3.4], $i > l - l_\alpha$. However this is impossible since $\deg a > 0$, and hence we have shown (i). On the other hand

$$\langle i^*(j_!(1_\alpha)\hat{\mu}_\alpha(x-\xi^{(\alpha)}t),\ [M]\rangle = \langle i^*(j_!(1_\alpha)\hat{\mu}_\alpha(x),\ [M]\rangle$$
$$= \langle \hat{\mu}_\alpha,\ [F_\alpha]\rangle = 1.$$

Therefore by the assumption (3.4), $i^*(j_!(1_\alpha)\hat{\mu}_\alpha(x-\xi^{(\alpha)}t)) = \mu$. Since $j^*(j_!(1_\alpha)\hat{\mu}_\alpha(x-\xi^{(\alpha)}t)) = \prod_i a_i^{(\alpha)}\hat{\mu}_\alpha t^{l-l_\alpha}$, we have $\prod_i a_i^{(\alpha)}\hat{\mu}_\alpha = \mu_\alpha$. Then we obtain (iii) by Proposition 3.4. Moreover we have $j_!(1_\alpha)\hat{\mu}_\alpha(x-\xi^{(\alpha)}t) = (\chi(M)/\chi(F_\alpha))A^{(\alpha)}P_\alpha(x-\xi^{(\alpha)}t)$. Since $j^*(e^{S^1}(M)) = \sum_\alpha j^*(\chi(F_\alpha)j_!(1_\alpha))\hat{\mu}_\alpha$, we get $e^{S^1}(M) = \sum_\alpha \chi(F_\alpha)j_!(1_\alpha)\hat{\mu}_\alpha(x-\xi^{(\alpha)}t)$. From this and Lemma 2.7 we have (ii). This completes the proof.

In particular, if F_α is an isolated fixed point, then $\det(\partial\chi_i^{(\alpha)}/\partial x_j)$ is a rational number and hence (iii) in Theorem 3.5 becomes to

$$(3.6)\quad \prod_{i=1}^{l-l_\alpha} a_i^{(\alpha)} = \pm\chi(M)\det(a_{ij}^{(\alpha)})\det(\partial\chi_i^{(\alpha)}/\partial x_j).$$

Especially, this implies that if the equivariant cohomology ring is given, then there are finite possibilities for the representation of S^1 at an isolated fixed point. The analogous fact holds for the tangential representation of S^1 at any fixed points if we know a number L such that $LH^*(M, Z)$ is contained in the subring of $H^*(M, Q)$ generated by $x_1,\dots,x_n$ over Z.

Corollary 3.6. *Let $\phi_i(x)$ and $f_i(x, t)$ $(1 \le i \le n)$ be homogeneous polynomials satisfying* (2.1),...,(2.4). *Assume that the simultaneous equations $f_1 = \cdots = f_n = 0$ have distinct $\prod_i r_i$ solutions. If ϕ_i and f_i are realized by smooth S^1-actions on closed oriented manifolds such that* (3.4) *holds, then the number of rational total Pontrjagin classes of S^1-manifolds which realize ϕ_i and f_i are finite.*

Proof. Let $P^{S^1}(M)$ be the equivariant Pontrjagin class of S^1-manifold M which realizes ϕ_i and f_i and satisfies (3.4). Then as stated in the above, the tangential representation at each isolated fixed point has only finitely many possibility and hence the image of $P^{S^1}(M)$ is determined by $f_1,\dots,f_n$ up to finite ambiguity at each fixed point. This shows that the possible values of $P^{S^1}(M)$ in $Q[x, t]/(f)$ are finite. This completes the proof.

4. A condition on the ideal $(\phi_1,\dots,\phi_n)$ and the fixed point set. It may be combinient to distinguish the connected components of X^{S^1} into two types, that is, 'generic' if they have the cohomology of a point or a sphere S^2 and non-generic otherwise. In this section we shal estimate the formal dimenstons of non-generic connected components and the number of generic connected components of the fixed point set X^{S^1} under a condition on the ideal $(\phi_1,\dots,\phi_n)$.

We assume $r_1 \leq \cdots \leq r_n$, where $r_i = \deg \phi_i$. Let us consider the u-resultant $R(u)$ of $\phi_1(x),\dots,\phi_{n-1}(x)$. Let $u_1,\dots,u_n$ be variable. Then $R(u)$ is obtained by eliminating $x_1,\dots,x_n$ from the equations $\phi_1(x_1,\dots,x_n) = 0,\dots,$ $\phi_{n-1}(x_1,\dots,x_n) = 0$ and $u_1x_1+\dots+u_nx_n = 0$ and an homogeneous polynomial in $u_1,\dots,u_n$ of degree $\prod_{i=1}^{n-1} r_i$(cf. [11]). Let $(\eta_1^{(l)},\dots,\eta_n^{(l)})$ be a solution of the equations $\phi_1(x) = 0,\dots,\phi_{n-1}(x) = 0$. Then $R(u)$ decomposes into the linear factors :

$$R(u) = c\prod_l (\eta_1^{(l)}u_1+\cdots+\eta_n^{(l)}u_n)^{\rho_l}, \text{ where } \sum \rho_l = \prod_{i=1}^{n-1} r_i.$$

In this section we assume the following

(4.1) $R(u)$ has no multiple factors (i. e. $\rho_l = 1$) and is irreducible over the field Q of rational numbers.

Let X be a compact S^1-space such that $H^*(X, Q) = Q[x]/(\phi)$. Then we say the S^1-action on X is cohomologically trivial if the inclusion of X^{S^1} into X induces an isomorphism between the rational cohomology rings of X^{S^1} and X. This is equivalent to that X^{S^1} is connected i. e. $f_1 = \cdots = f_n = 0$ have a unique solution.

Proposition 4.1. *Let X be a compact S^1-space such that $H^*(X, Q) = Q[x]/(\phi)$ and (4.1) holds. If the S^1-action on X is cohomologically nontrivial, then the formal dimension of any connected components of X^{S^1} is equal or less than* $2\left(\sum_{i=1}^{n-1}(r_i-1)+r_{n-1}-2\right)$.

Proof. Let F be a connected component of X^{S^1}. Then by (2.5), $H^*(F, Q) = Q[x]/(\chi)$ and the formal dimension of F is $2\sum_i (s_i-1)$ where $s_i = \deg \chi_i$. We may assume that $s_1 \leq \cdots \leq s_n$ and show $s_n \leq r_{n-1}$. Let us suppose $s_n > r_{n-1}$ contrary. Then the ideal $(\phi_1,\dots,\phi_{n-1})$ is contained in $(\chi_1,\dots,\chi_{n-1})$. Let $R'(u)$ be the u-resultant of $\chi_1,\dots,\chi_{n-1}$ and $Q(u)$ an irreducible factor of $R'(u)$. Then $Q(u)$ divides a power of $R(u)$. Since $R(u)$ is irreducible by the assumption, we have $R(u) = Q(u)$. On the other hand $\deg R'(u)$

$\leq \deg R(u)$ by Proposition 2.2. and hence $R'(u) = R(u)$. Since this means $(\chi_1, \ldots, \chi_{n-1}) = (\phi_1, \ldots, \phi_{n-1})$ and together with Proposition 2.1, we may assume that $f_1 = \phi_1, \ldots, f_{n-1} = \phi_{n-1}$. Then the equations $f_1 = 0, \ldots, f_n = 0$ have a unique solution $(0, \ldots, 0, 1)$. This contradicts to the cohomological non-triviality of the S^1-action and hence we have $s_n \leq r_{n-1}$.

If $s_n < r_{n-1}$ or $s_n = r_{n-1}$ and $s_i < r_i$ for some $i < n$, then formal $\dim F \leq 2\left(\sum_{i=1}^{n-1}(r_i - 1) + r_{n-1} - 2\right)$. Now suppose $s_i = r_i$, $i < n$ and $s_n = r_{n-1}$. In this case we may assume $\chi_1 = \phi_1, \ldots, \chi_{n-1} = \phi_{n-1}$ and moreover $f_1 = \phi_1, \ldots, f_{n-1} = \phi_{n-1}$. This also implies a contradiction and completes the proof.

Let M be a closed connected manifold such that $\chi(M) \neq 0$ and $m(M)$ be the maximal dimension of the connected components of M^{S^1} for all non-trivial S^1-actions on M.

Lemma 4.2. *If a torus T acts effectively on a closed connected manifold M such that $\chi(M) \neq 0$, then $\dim T \leq m(M)/2 + 1$.*

Proof. This follows immediately from ([5, Th. (IV.7), p.58]).

Now from Proposition 4.1 and Lemma 4.2 we have the following

Theorem 4.3. *Let M be a closed manifold such that $H^*(M, Q) = Q[x]/(\phi)$ and (4.1) holds. If a compact connected Lie group G acts continuously and effectively on M, then*

$$\operatorname{rank} G \leq \sum_{i=1}^{n-1}(r_i - 1) + r_{n-1} - 1.$$

Remark. If the formal dimension of $Q[x]/(\phi)$ is not devided by 4, then there exists a closed smooth manifold M such that $H^*(M, Q) = Q[x]/(\phi)$ (cf. [10]). Hence we see that there exist closed smooth manifolds with comparatively small degree of symmetry with respect to the dimensions.

In the rest of this section we shall estimate the number of generic connected components of the fixed point set of a compact S^1-space X such that $H^*(X, Q) = Q[x]/(\phi)$. We assume (4.1) together with

(4.2) $(r_1 \cdots r_{n-1})r_{n-1} \leq r_n$.

Let the S^1-action on X be cohomologically non-trivial and $H^*(X_{S^1}, Q) =$

$Q[x, t]/(f)$. The homogeneous equations $f_1 = 0, \ldots, f_{n-1} = 0$ define a subvariety C of a complex projective space CP^n. Let $C = \bigcup_j C_j$ be the decomposition of C into irreducible components, E the hyperplane defined by $t = 0$ and $\eta^{(l)}$ a solution of the equations $\phi_1 = 0, \ldots, \phi_{n-1} = 0$. In this notations we have the following

Lemma 4.4. (i) $\dim C_j = 1$. (ii) *For any $\eta^{(l)}$, there is a unique C_j that intersects transversally with E at $\eta^{(l)}$. The degree of C_j is equal to the number of points in $C_j \cap E$.* (iii) *The multiplicity of C_j in C, $(f_1, \ldots, f_{n-1})_{C_j} = 1$.*

Proof. It is clear that $\dim C_j \geq 1$ (cf. [9, Cor.5, p. 57]). Since the hypersurface $f_n = 0$ and C_j intersect in finite points, we see $\dim C_j = 1$. If $C_j \cap E = \phi$, i.e. $C_j \subset A^n = CP^n - E$, then C_j must be a point because C_j is projective)cf. [9, Cor.2, p. 47]). This is impossible and hence C_j contains a point $\eta^{(l)}$. Since each $\eta^{(l)}$ is a point of C, there is a C_j which contains $\eta^{(l)}$. The divisors $(f_1), \ldots, (f_{n-1})$, (t) on CP^n are in general position since the solution of $f_1 = 0, \ldots, f_{n-1} = 0$, $t = 0$ are finite. Then by Bezout's theorem (cf. [9, p. 198]),

$$\prod_{i=1}^{h-1} r_i = ((f_1), \ldots, (f_{n-1}), t)) = \sum_l ((f_1), \ldots, (f_{n-1}), (t))_{\eta^{(l)}}.$$

Now we have $((f_1), \ldots, (f_{n-1}), (t))_{\eta^{(l)}} = 1$ for each $\eta^{(l)}$ since the numder of $\eta^{(l)}$ is equal to $\prod_{i=1}^{n-1} r_i$ by the assumption (4.1). On the other hand by [9, Prop. 1, p. 190],

$$((f_1), \ldots, (f_{n-1}), (t))_{\eta^{(l)}} = \sum_{C_j \ni \eta^{(l)}} (f_1, \ldots, f_{n-1})_{C_j} (\rho_{C_j}(E))_{\eta^{(l)}}.$$

From this formula, noting the integers $(f_1, \ldots, f_{n-1})_{C_j}$ and $(\rho_{C_j}(E))_{\eta^{(l)}}$ if $\eta^{(l)} \in C_j$ are positive, we see that for each $\eta^{(l)}$ there is a unique C_j and its multiplicity in C, $(f_1, \ldots, f_{n-1})_{C_j} = 1$ and moreover $(\rho_{C_j}(E))_{\eta^{(l)}} = 1$. The last equation means that C_j and E intersect transversally at $\eta^{(l)}$ and $\eta^{(l)}$ is a simple point of C_j. Hence the degree of C_j is equal to the number of the points in $C_j \cap E$. This completes the proof.

Lemma 4.5. *Each component C_j of C is not a line.*

Proof. Suppose contrary that C_j is a line. If C_j and the hypersurface

$f_n = 0$ meet in distinct points $(\xi_1, \ldots, \xi_n, 1)$ and $(\xi'_1, \ldots, \xi'_n, 1)$, then ξ_i, $\xi'_i \in Q$ by (2.3). Then $C_j \cap E$ contains a rational point $(\xi_1 - \xi'_1, \ldots, \xi_n - \xi'_n, 0)$. This contradicts to (4.1) and hence C_j and the hypersurface $f_n = 0$ intersect in a unique point ξ. Then we have

$$((f_1), \ldots, (f_n))_\xi = \sum_{C_k \ni \xi} ((f_1), \ldots, (f_{n-1}))_{C_k} (\rho_{C_k}(f_n))_\xi$$
$$\geq (\rho_{C_j}(f_n))_\xi = r_n.$$

Now let $Q[x]/(\chi)$ ($\deg \chi_i = s_i$, $1 \leq i \leq n$) be the cohomology ring of the connected component of X^{S^1} corresponding to ξ. Then $s_i \leq r_i$ and $s_n \leq r_{n-1}$ (see the Proof of Proposition 4.1). Therefore by (2.5) and (4.2), we have $((f_1), \ldots, (f_n))_\xi = \prod_i s_i < r_n$. This contradicts the above inequality and completes the proof.

Let x be a simple point of C and $x \in C_j$. Let us consider all hyperplane H that contain the tangent line L of C_j at x. We define a number $s(x)$ as follows :

$$s(x) = \min_{H \supset L} (C_j, H)_x - 1.$$

Then $1 \leq s(x) \leq \deg C_j - 1$. If C_j is a plane curve, then $s(x)$ is a class of x and x is a flex of C_j provided $s(x) \geq 2$.

Proposition 4.6. *Let ξ be a solution of the equations $f_1 = 0, \ldots, f_n = 0$ and assume that ξ is a simple point of C. If F is the connected component of X^{S^1} corresponding to ξ, then*

$$H^*(F, Q) \cong H^*(CP^k, Q), \text{ where } k \leq s(x).$$

Proof. We can assume $\xi = (0, \ldots, 0, 1)$ by a parallel translation. Since ξ is a simple point of C and C contains no multiple component by Lemma 4.4, ξ is also a simple point of the hypersurface $f_i = 0$ and the tangent spaces of $f_i = 0$, $1 \leq i \leq n-1$, intersect transversally. Then by making use of a linear transformation with rational coefficient if necessary, we can assume that the tangent spaces of the hypersurface $f_i = 0$ at ξ is defined by $x_i = 0$ $(1 \leq i \leq n-1)$. Now applying on $f_i(x, 1)$ a process in the proof of Proposition 2.1, we obtain a generators of the ideal J,

$$\tilde{f}_1 = \ldots + x_1, \ldots, \tilde{f}_{n-1} = \ldots + x_{n-1}, \text{ and } \tilde{f}_n = \ldots + x_n^{k+1}.$$

This shows that $H^*(F, Q) = Q[x]/(x_1, \ldots, x_{n-1}, x_n^{k+1}) = H^*(CP^k, Q)$. On the other hand, by Lemma 4.5, there is a f_i $(i < n)$ which contains a power of x_n^{α}. Assume f_1 contains x_n^{α} where α is the least. Then we have $\alpha \geq k+1$ since $x_n^{\alpha} \in (x_1, \ldots, x_{n-1}, x_n^{k+1})$. It is clear that x_n is a local parameter at ξ of the curve C_j, $C_j \ni \xi$, that is, the maximal ideal m_{ξ} of the local ring of C_j at ξ is generated by x_n. Since any hyperplane H that contains the tangent line L of C_j at ξ is defined by an equation $\sum_{i=1}^{n-1} a_i x_i = 0$, we have $(C_j, H)_{\xi} = \nu_{C_j}(H) \geq \min_i \nu_{C_j}(x_i)$ and hence $s(\xi) = \min_i \nu_{C_j}(x_i) - 1$ (cf. [9, p. 128]). Since $d_i = \nu_{C_j}(x_i)$ means $x_i \in m^{d_i}$ but $x_i \notin m^{d_i+1}$, we have $x_1, \ldots, x_{n-1} \in m^{s+1}$. Then $x_n^{\alpha} \in m^{s+1}$ since $f_1 = 0$ contains x_n. Therefore $\alpha \geq s(\xi)+1$. If $\alpha > S(\xi)+1$, then we have $x_i \in m^{s+2}$ for $i < n$ by $f_i = \cdots + x_i = 0$ on C_j. However this is impossible and hence $\alpha = s(\xi)+1$. Together with $\alpha \geq k+1$, this completes the proof.

Let ξ be a solution of $f_1 = 0, \ldots, f_n = 0$. If ξ is not a singular point or a point such that $s(\xi) \geq 2$ of the curve C, then the connected component of X^{S^1} corresponding to ξ is generic, that is, has the rational cohomology of a point or a sphere S^2. The number of singular points or points such that $s \geq 2$ of C is bounded from the above by a constant $C(r_1, \ldots, r_{n-1})$ depending on $r_1, \ldots, r_{n-1}$, see Lemma 4.8. Then by making use of Proposition 4.6 and [2, Th. 1.6, p. 374] we have

Theorem 4.7. *Let X be a compact S^1-space such that $H^*(X, Q) = Q[x]/(\phi)$ where* (4.1) *and* (4.2) *hold. If the S^1-action on X is cohomologically non-trivial, then the number of generic connected components of X^{S^1} is greater than* $\left\{\prod_{i=1}^{n} r_i - r_{n-1}\left(\prod_{i=1}^{n-1} r_i\right) C(r_1, \ldots, r_{n-1})\right\}/2$.

As an estimation of $C(r_1, \ldots, r_{n-1})$ we have the following

Lemma 4.8. $C(r_1, \ldots, r_{n-1}) = \left(\prod_{i=1}^{n-1} r_i\right)\left(r_{n-1} - 2 + 2\sum_{i=1}^{n-1}(r_i - 1)\right)$.

Proof. Let $x \in C$ be a simple point in $A^n = CP^n - E$. We put $f_k(x) = f_k(x, 1)$ $(1 \leq k \leq n-1)$. Then the equation

$$F_k(X_1, \ldots, X_n) = \sum_i \partial f_k / \partial x_i(x)(X_i - x_i) = 0$$

defines a hyperplane H_k that contains the tangent line L of C at x. Let u be a local parameter of C at x. Then we can express x_i as a power series in u,

(4.3) $x_i = a_i + b_i u + c_i u^2 + \dots,\ 1 \leq i \leq n,$

where $x_i(x) = a_i$ and $(b_1, \dots, b_n) \neq 0$. Since on the curve $C_j \ni x$,

$$0 = \sum_i \partial f_k/\partial x_i(a)(x_i - a_i) + 1/2 \sum_{i,j} \partial^2 f_k/\partial x_i \partial x_j(a)(x_i - a_i)(x_j - a_j) + \dots,$$

we have by (3.12), $F_k(x) \equiv -1/2 \sum_{i,j} \partial^2 f_k/\partial x_i \partial x_j(a) b_i b_j u^2 \bmod m_x^3$. This shows that the condition $(C_j, H_k)_x = \nu_{C_j}(F_k(x)) \geq 3$ is equivalent to $\sum_{i,j} \partial^2 f_k/\partial x_i \partial x_j(a) b_i b_j = 0$. Therefore for a simple point $x \in C \cap A^n$, $s(x) \geq 2$ if and only if there is $(b_1, \dots, b_n) \neq$ such that

$$\sum_i \partial f_k/\partial x_i(x) b_i = 0 \text{ and } \sum_{i,j} \partial^2 f_k/\partial x_i \partial x_j(x) b_i b_j = 0,\ 1 \leq k \leq n-1.$$

Let us denote g_i the determinant of the matrix obtained from the matrix $\partial(f_1, \dots, f_{n-1})/\partial(x_1, \dots, x_n)$ deleting the i-th column. Then because of $b_1 : b_2 : \cdots : b_n = (-)^{n-1} g_1 : (-1)^{n-2} g_2 : \cdots : g_n$, we see that $x \in C$ is a singular point or a point such that $s(x) \geq 2$ if

$$G_k(x) = \sum_{i,j} (-1)^{i+j} \partial^2 f_k/\partial x_i \partial x_j(x) g_i(x) g_j(x) = 0,\ 1 \leq k \leq n-1.$$

$G_k(x)$ is a polynomial of degree less than $r_k - 2 + 2\sum_{i=1}^{n-1}(r_i - 1)$. Since each C_j is not a line by Lemma 4.5, there is a $G_k \not\equiv 0$ on C_j. Therefore we can take a polynomial $G = \sum_k d_k G_k$ $(d_k \in C)$ such that $G \not\equiv 0$ on any C_j. Since $\deg G \leq r_{n-1} - 2 + 2\sum_{i=1}^{n-1}(r_i - 1)$ the number of the solutions of the simultaneous equations $f_1 = 0, \dots, f_{n-1} = 0$ and $G = 0$ is at most $\left(\prod_{i=1}^{n-1} r_i\right)\left(r_{n-1} - 2 + 2\sum_{i=1}^{n-1}(r_i - 1)\right)$ by Bezout's theorem. This completes the proof.

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