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THE CLASSIFICATION OF HOMOGENEOUS STRUCTURES ON 3-DIMENSIONAL SPACE FORMS

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Introduction. In 1958, Ambrose and Singer proved the following theorem ([1])

Theorem. *Let (M, g) be a connected, simply connected, complete Riemannian space. Then (M, g) is a homogeneous Riemannian space if and only if there exists a tensor field T of type $(1, 2)$ such that* (1) $(\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y}$ (2) $g(T_X Y, Z) + g(Y, T_X Z) = 0$ (3) $(\nabla_X R)_{YZ} = [T_X, R_{YZ}] - R_{T_X YZ} - R_{Y T_X Z}$ *for $X, Y, Z \in \mathfrak{X}(M)$, where ∇ denotes the Riemannian connection on (M, g) and R denotes the curvature tensor.*

We note here that this theorem is an extention of the characterization of symmetric spaces given by E. Cartan. Indeed if there exists a tensor field $T = 0$ on (M, g), then (M, g) is a symmetric space. A tensor field of type $(1, 2)$ which satisfies the above (1), (2), (3) is called a homogeneous structure. In [5], Tricerri and Vanhecke studied many properties of homogeneous structures and gave the classification of the homogeneous structures on 2-dimensional space forms. There is only canonical homogeneous structure $T = 0$ on $\boldsymbol{R}^2$, $\boldsymbol{S}^2$, but $\boldsymbol{H}^2$ admits two types of homogeneous structure up to "isomorphisms": one is $T = 0$ and another one is $T_X Y = g(X, Y)\xi - g(\xi, Y)X$ for $X, Y \in \mathfrak{X}(\boldsymbol{H}^2)$, where $\xi = y\dfrac{\partial}{\partial y}$. They also gave the classification of those on Heisenberg group.

In this paper, we give the complete classification of the homogeneous structures on 3-dimensional space forms.

For instance in case of sphere, we have

Theorem. *The 3-dimensional sphere $(S^3 = \{(x_1, x_2, x_3, x_4) \in \boldsymbol{R}^4 \mid \sum_{i=1}^4 x_i^2 = 1\}, g)$ of constant curvature 1 admits two types of homogeneous structures up to the isomorphisms:*

(1) $T^\lambda_{XYZ} = g(T^\lambda_X Y, Z) = \lambda dV(X, Y, Z) \quad (\lambda \geqq 0)$

(2) $T(\lambda)_{XYZ} = (dV + \lambda\theta_1 \otimes (\theta_2 \wedge \theta_3))(X, Y, Z) \quad (\lambda \in \boldsymbol{R} - (0))$

for $X, Y, Z \in \mathfrak{X}(S^3)$, *where we put at* $q = (x_1, x_2, x_3, x_4) \in S^3$

$$\theta_1|_q = -x_2dx_1 + x_1dx_2 + x_4dx_3 - x_3dx_4$$
$$\theta_2|_q = -x_3dx_1 - x_4dx_2 + x_1dx_3 + x_2dx_4$$
$$\theta_3|_q = -x_4dx_1 + x_3dx_2 - x_2dx_3 + x_1dx_4$$

3-dimensional case seems to be most interesting, because the group of all isometries of S^3 with constant curvature is not simple.

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1. Homogeneous structure. In this section we give some facts which are needed in our proof. In the following (M, g) always denotes a connected, simply connected, complete Riemannian space.

(1.1) Let (M, g) be a homogeneous Riemannian space and G be a connected subgroup of the connected component $I_0(M)$ of the isometry group of (M, g) acting transitively on M. If we have a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}_p$ of its Lie algebra, where $\mathfrak{m}$ is a subspace of $\mathfrak{g}$ with $[\mathfrak{m}, \mathfrak{h}_p] \subset \mathfrak{m}$ and $\mathfrak{h}_p$ is the Lie algebra of the isotropy subgroup H_p of G at $p \in M$, then the G-invariant metric connection $\tilde{\nabla}$ called the canonnical connection is defined as follows: Firstly for $\gamma \in \mathfrak{g}$, we denote by γ^* a vector field on M given by $\gamma^*|_q = \frac{d}{dt}(\exp t\gamma)(q)_{t=0}$ at $q \in M$. Then we define for $\beta \in \mathfrak{g}$

$$(\tilde{\nabla}_X\beta^*)_p = [\alpha^*, \beta^*]_p = -[\alpha, \beta]^*_p, \text{ where } \alpha \in \mathfrak{m} \text{ such that } \alpha^*|_p = X.$$

If we define a tensor T by $T = \nabla - \tilde{\nabla}$, then this T is a homogeneous structure (cf. [4], [5]) and we have

$$T_XY|_p = \nabla_Y\alpha^*|_p \text{ for } Y \in T_P(M),$$

i.e.

$$g(T_XY, Z)_p = g([\alpha, \beta]^*, \gamma^*)_p - g([\beta, \gamma]^*, \alpha^*)_p + g([\gamma, \alpha]^*, \beta^*)_p$$

where we take $\gamma \in \mathfrak{g}$ such that $\gamma^*|_p = Z$.
We note here that T is G-invariant.

Conversely, suppose that a homogeneous structure T is given on a Riemannian space (M, g). Then there exists a connected subgroup $\bar{G}$ of $I_0(M)$ which acts simply transitively on the holonomy bundle P(u) of a connection $\tilde{\nabla} = \nabla - T$ through $u \in O(M)$. There exists a reductive decomposition of $\bar{\mathfrak{g}}$, which is defined from the decomposition of $T_u(P(u))$ into the vertical space and horizontal space relative to $\tilde{\nabla}$, so that $\tilde{\nabla}$ coincides with the canonical connection defined from the reductive decomposition. Note here that this group $\bar{G}$ is the minimum connected subgroup of $I_0(M)$ constructing the homogeneous structure T (cf. [4], [5]).

For the homogeneous structure T defined from $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}_p$, the Lie algebra of the isometry group and its reductive decomposition defined from T is given by $\bar{\mathfrak{g}} = \mathfrak{m} \oplus [\mathfrak{m}, \mathfrak{m}]|\mathfrak{h}_p$.

Definition 1.1. Let (M, g) be a homogeneous Riemannian space. Then two homogeneous structures T and T' on (M, g) are said to be *isomorphic* if there exists an isometry φ on (M, g) such that $\varphi_*(T_X Y) = T'_{\varphi_*(X)}\varphi_*(Y)$ for $X, Y \in \mathfrak{X}(M)$.

Theorem 1.2.([5]) *Let (M, g) be a homogeneous Riemannian space and G, G' be the connected Lie subgroups of $I_0(M)$ acting transitively on M. Now assume that the Lie algebra $\mathfrak{g}$ (resp. $\mathfrak{g}'$) of G (resp. G') has a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}_p$ (resp. $\mathfrak{g}' = \mathfrak{m}' \oplus \mathfrak{h}'_q$). Then the homogeneous structure T defined from $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}_p$ and T' from $\mathfrak{g}' = \mathfrak{m}' \oplus \mathfrak{h}'_q$ are isomorphic if and only if there exists a Lie algebra isomorphism Ψ from $\mathfrak{m} \oplus [\mathfrak{m}, \mathfrak{m}]|\mathfrak{h}_p$ to $\mathfrak{m}' \oplus [\mathfrak{m}', \mathfrak{m}']|\mathfrak{h}'_q$ such that $\Psi(\mathfrak{m}) = \mathfrak{m}'$, $\Psi([\mathfrak{m}, \mathfrak{m}]|\mathfrak{h}_p) = ([\mathfrak{m}', \mathfrak{m}']|\mathfrak{h}'_q)$ and $\Psi|\mathfrak{m}$ is an isometry from $\mathfrak{m}$ to $\mathfrak{m}'$ where $\mathfrak{m}$ (resp. $\mathfrak{m}'$) has the inner product induced from $(T_p(M), g_p)$ (resp. $(T_q(M), g_q)$) by the map $\alpha \to \alpha^*|_p$ (resp. $\alpha' \to \alpha'^*|_q$)*

Proof. We refer to Theorem 2.1 and 2.2 in [5].

2. The homogeneous structures on S^3.

Theorem 2.1. *The 3-dimensional sphere $(S^3 = \{(x_1, x_2, x_3, x_4) \in R^4 | \sum_{i=1}^4 x_i^2 = 1\}, g)$ of constant curvature 1 admits two types of homogeneous structures up to the isomorphisms :*

(1) $T^\lambda_{XYZ} = g(T^\lambda_X Y, Z) = \lambda dV(X, Y, Z) \quad (\lambda \geq 0)$

(2) $T(\lambda)_{XYZ} = (dV + \lambda\theta_1 \otimes (\theta_2 \wedge \theta_3))(X, Y, Z) \quad (\lambda \in R-(0))$

for $X, Y, Z \in \mathfrak{X}(S^3)$, where we put at $q = (x_1, x_2, x_3, x_4) \in S^3$

$$\theta_1|_q = -x_2dx_1 + x_1dx_2 + x_4dx_3 - x_3dx_4$$
$$\theta_2|_q = -x_3dx_1 - x_4dx_2 + x_1dx_3 + x_2dx_4$$
$$\theta_3|_q = -x_4dx_1 + x_3dx_2 - x_2dx_3 + x_1dx_4$$

Proof. $I_0(S^3) = SO(4)$ and its Lie algera is $\mathfrak{so}(4)$. Let $H_p = SO(3)$ be the isotropy subgroup of $SO(4)$ at $p = (1, 0, 0, 0) \in S^3$. Then $\mathfrak{h}_p = \langle w_4, w_5, w_6\rangle$, where we put $w_i = E_{1,i+1} - E_{i+1,i} (i = 1, 2, 3)$ $w_4 = E_{43} - E_{34}$, $w_5 = E_{24} - E_{42}$, $w_6 = E_{32} - E_{23}$ for the matrix unit E_{ij}. Further if we set $u_1 = w_4 - w_1$, $u_2 = w_5 - w_2$, $u_3 = w_6 - w_3$, $v_1 = w_4 + w_1$, $v_2 = w_5 + w_2$ and $v_3 = w_6 + w_3$, then we have

$$v_1^*|_p = \frac{\partial}{\partial x_2}\bigg|_p,\ v_2^*|_p = \frac{\partial}{\partial x_3}\bigg|_p,\ v_3^*|_p = \frac{\partial}{\partial x_4}\bigg|_p.$$

We also get

$$(2.1)\quad \begin{array}{l}[u_i, u_j] = 2u_k,\ [v_i, v_j] = 2v_k \text{ for even permutation} \\ (i, j, k) \text{ of } (1, 2, 3),\ [u_m, v_n] = 0\ (m, n = 1, 2, 3).\end{array}$$

Namely we have $\mathfrak{so}(4) = \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2$ (the direct sum as Lie algebra) with $\mathfrak{su}(2)_1 = \langle v_1, v_2, v_3\rangle$ and $\mathfrak{su}(2)_2 = \langle u_1, u_2, u_3\rangle$.

First note that there are no subgroups of dimension 2 in $SO(3)$, because $SO(3)$ is simple. Thus the possible connected subgroups G of $I_0(S^3)$ acting transitively on S^3 are as follows:

(A_1) $\dim G = 6$
(A_2) $\dim G = 4$ and $\dim(G \cap H_p) = 1$
(A_3) $\dim G = 3$ and $\dim(G \cap H_p) = 0$

Case (A_1): $G = SO(4)$ and its Lie algebra is $\mathfrak{g} = \mathfrak{so}(4)$. Let $\mathfrak{m}$ be a complement of $\mathfrak{h}_p$ in $\mathfrak{g}$. Then $\mathfrak{m}$ is expressed as $\langle v_1 + \varphi(v_1), v_2 + \varphi(v_2), v_3 + \varphi(v_3)\rangle$ by means of a linear map φ from $\mathfrak{su}(2)_1$ to $\mathfrak{h}_p$ and by using $[\mathfrak{m}, \mathfrak{h}_p] \subset \mathfrak{m}$, the $\mathfrak{m}$-component of a reductive decomposition of $\mathfrak{g}$ relative to $\mathfrak{h}_p$ is given by

$$\mathfrak{m}^\lambda = \langle v_1 + (\lambda-1)w_4, v_2 + (\lambda-1)w_5, v_3 + (\lambda-1)w_6\rangle \quad (\lambda \in \boldsymbol{R})$$

From (1, 1) in the section 1, the homogeneous structure T^λ defined from this decomposition $\mathfrak{g} = \mathfrak{m}^\lambda \oplus \mathfrak{h}_p$ satisfies

$$\begin{aligned} 2g(T^\lambda_{v_i^*}v_j^*, v_k^*)_p &= g([v_i + (\lambda-1)w_{i+3}, v_j]^*, v_k^*)_p - g([v_j, v_k]^*, v_i^*)_p \\ &\quad + g([v_k, v_i + (\lambda-1)w_{i+3}]^*, v_j^*)_p \\ &= g\left(\left[\frac{\lambda+1}{2}v_i, v_j\right]^*, v_k^*\right)_p - g([v_j, v_k]^*, v_i^*)_p \\ &\quad + g\left(\left[v_k, \frac{\lambda+1}{2}v_i\right]^*, v_j^*\right)_p \end{aligned}$$

Therefore we have $T^{\lambda}_{v_i^* v_j v_k^*}|_p = g(T^{\lambda}_{v_i^*} v_j^*, v_k^*)_p = \operatorname{sgn}(i, j, k)\lambda$. Thus we get $T^{\lambda}|_p = \lambda dV|_p$, where dV denotes the volume form of (S^3, g). Since T and dV are G-invariant, we have $T^{\lambda} = \lambda dV (\lambda \in \boldsymbol{R})$ and it is trivial that T^{λ} and T^{μ} are isomorphic if and only if $\lambda^2 = \mu^2$. This gives the homogeneous structures of the first kind in the theorem.

Case (A_2) : $\dim \mathfrak{g} = 4$ and $\dim(\mathfrak{g} \cap \mathfrak{h}_p) = 1$. We consider more precisely the next three cases :

(A_2^1) $\dim(\mathfrak{g} \cap \mathfrak{su}(2)_1) = 3$
(A_2^2) $\dim(\mathfrak{g} \cap \mathfrak{su}(2)_1) = 2$
(A_2^3) $\dim(\mathfrak{g} \cap \mathfrak{su}(2)_1) = 1$

First note that the case (A_2^2) doesn't occur since there are no subalgebras of dimension 2 in $\mathfrak{su}(2)_1$.

Lemma 2.2. *If* $\mathfrak{g}$ *satisfies the condition* (A_2^1) *(resp.* (A_2^3)*) there exists* w *(resp.* w'*) in* $\mathfrak{h}_p$ *such that* $\mathfrak{g} = \mathfrak{su}(2)_1 \oplus \boldsymbol{R}w$ *(resp.* $\mathfrak{g} = \mathfrak{su}(2)_2 \oplus \boldsymbol{R}w'$*).*

Proof. Case (A_2^1) is trivial because we have $\mathfrak{su}(2)_1 \subset \mathfrak{g}$. In case (A_2^3), it suffices to see that $\mathfrak{g} \cap \mathfrak{su}(2)_2 = \mathfrak{su}(2)_2$. First note that $\dim(\mathfrak{g}|\mathfrak{su}(2)_2) = 3$ because of $\dim(\mathfrak{g} \cap \mathfrak{su}(2)_1) = 1$. Next we have

$$\begin{aligned}\mathfrak{g}|\mathfrak{su}(2)_2 &= [\mathfrak{g}|\mathfrak{su}(2)_2, \mathfrak{g} \cap \mathfrak{h}_p|\mathfrak{su}(2)_2] \\ &= [\mathfrak{g}, \mathfrak{g} \cap \mathfrak{h}_p|\mathfrak{su}(2)_2] \subset \mathfrak{g} \cap \mathfrak{su}(2)_2.\end{aligned}$$

Then clearly $\dim(\mathfrak{g} \cap \mathfrak{su}(2)_2) \geq \dim(\mathfrak{g}|_{\mathfrak{su}(2)_2}) \geq 3$ holds. q. e. d.

Next we shall determine the reductive decompositions of $\mathfrak{g} = \mathfrak{su}(2)_1 \oplus \boldsymbol{R}w$. To do this, choose a basis $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3 \in \mathfrak{su}(2)_1$ and $\tilde{w} \in \boldsymbol{R}w$ such that $[\tilde{w}, \tilde{v}_1] = 0$, $[\tilde{w}, \tilde{v}_2] = \tilde{v}_3$, $[\tilde{w}, \tilde{v}_3] = -\tilde{v}_2$ and $[\tilde{v}_i, \tilde{v}_j] = 2\tilde{v}_k$, where (i, j, k) is an even permutation. Then it is easy to show from these relations that the $\mathfrak{m}$-component of the reductive decompositions of $\mathfrak{g}$ is given by $\langle \tilde{v}_1 + \lambda\tilde{w}, \tilde{v}_2, \tilde{v}_3 \rangle$ $(\lambda \in \boldsymbol{R})$. We note here that the homogeneous structure defined from this decomposition is isomorphic to the homogeneous structures defined from $\mathfrak{su}(2)_1 \oplus \boldsymbol{R}w_4 = \langle v_1 + \lambda w_4, v_2, v_3 \rangle \oplus \boldsymbol{R}w_4$ by Theorem 1.2. In fact, $\{\tilde{v}_1 + \lambda\tilde{w}, \tilde{v}_2, \tilde{v}_3\}$ (resp. $\{v_1 + \lambda w_4, v_2, v_3\}$) is an orthonormal basis relative to the inner product induced from $(T_p(S^3), g_p)$ into $\langle \tilde{v}_1 + \lambda\tilde{w}, \tilde{v}_2, \tilde{v}_3 \rangle$ (resp. $\langle v_1 + \lambda w_4, v_2, v_3 \rangle$). Hence it suffices to determine the homogeneous structures $T(\lambda)$ defined from $\mathfrak{su}(2)_1 \oplus \boldsymbol{R}w_4 = \mathfrak{m}(\lambda) \oplus \boldsymbol{R}w_4$, where $\mathfrak{m}(\lambda) = \langle v_1 + \lambda w_4, v_2, v_3 \rangle$. By using the relations

$$(2.2)\quad \begin{aligned} &v_1^*|_p = \frac{\partial}{\partial x_2}\Big|_p,\ v_2^*|_p = \frac{\partial}{\partial x_3}\Big|_p,\ v_3^*|_p = \frac{\partial}{\partial x_4}\Big|_p \\ &[v_1+\lambda w_4,\ v_1] = 0,\ [v_1+\lambda w_4,\ v_2] = (2+\lambda)v_3, \\ &[v_1+\lambda w_4,\ v_3] = -(2+\lambda)v_2 \end{aligned}$$

we have from (1.1) in section 1 that

$$T(\lambda)_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_3}\Big|_p = (1+\lambda)\frac{\partial}{\partial x_4}\Big|_p,\ T(\lambda)_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_4}\Big|_p = -(1+\lambda)\frac{\partial}{\partial x_3}\Big|_p,$$

$$T(\lambda)_{\frac{\partial}{\partial x_3}}\frac{\partial}{\partial x_2}\Big|_p = -\frac{\partial}{\partial x_4}\Big|_p,\ T(\lambda)_{\frac{\partial}{\partial x_3}}\frac{\partial}{\partial x_4}\Big|_p = \frac{\partial}{\partial x_2}\Big|_p,$$

$$T(\lambda)_{\frac{\partial}{\partial x_4}}\frac{\partial}{\partial x_3}\Big|_p = -\frac{\partial}{\partial x_2}\Big|_p,\ T(\lambda)_{\frac{\partial}{\partial x_4}}\frac{\partial}{\partial x_2}\Big|_p = \frac{\partial}{\partial x_3}\Big|_p$$

and $T(\lambda)_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_i}\Big|_p = 0 \quad (i = 1, 2, 3).$

These relations mean that for $X, Y, Z \in T_p(S^3)$

$$T(\lambda)_{XYZ}|_p = g(T(\lambda)_X Y, Z)_p = (dV + \lambda dx_2 \otimes (dx_3 \wedge dx_4))_p(X, Y, Z).$$

Now since both of $T(\lambda)$ and dV are $SU(2)_1$-invariant, we have

$$T(\lambda)_{XYZ}|_q = (dV + \lambda\varphi^* dx_2 \otimes (\varphi^* dx_3 \wedge \varphi^* dx_4))_q(X, Y, Z)$$
for any point $q \in S^3$,

where $\varphi^{-1}(p) = q$, $\varphi \in SU(2)_1$ and φ^* is the pull back by φ. Let's identify S^3 with $SU(2) = \left\{\begin{pmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{pmatrix} \middle| \begin{matrix} |z_1|^2 + |z_2|^2 = 1 \\ z_1, z_2 \in C \end{matrix}\right\}$ by a map $h: (x_1, x_2, x_3, x_4) \to (z_1 = x_1 + x_2 i,\ z_2 = x_3 + x_4 i)$. Then $SU(2)_1 \subset I_0(S^3)$ acts as left translations on $SU(2)$. In fact, the identifications is given by $v_1 \to \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $v_2 \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $v_3 \to \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$ in terms of Lie algebra. Therefore it follows that

$$\begin{aligned} (\varphi^* dx_2)_q &= -x_2 dx_1 + x_1 dx_2 + x_4 dx_3 - x_3 dx_4 \\ (\varphi^* dx_3)_q &= -x_3 dx_1 - x_4 dx_2 + x_1 dx_3 + x_2 dx_4 \\ (\varphi^* dx_4)_q &= -x_4 dx_1 + x_3 dx_2 - x_2 dx_3 + x_1 dx_4. \end{aligned}$$

We denote $\varphi^* dx_2$, $\varphi^* dx_3$, $\varphi^* dx_4$ by θ_1, θ_2, θ_3, respectively and we have $T(\lambda) = dV + \lambda\theta_1 \otimes (\theta_2 \wedge \theta_3) \quad (\lambda \in R)$

Lemma 2.3. *$T(\lambda)$ and $T(\mu)$ are isomorphic iff $\lambda = \mu$.*

Proof. Suppose that $T(\lambda)$ and $T(\mu)$ are isomorphic. We may assume that $\lambda \neq 0$ and $\mu \neq 0$ by Theorem 1.2. Then there exists a map f from $\mathfrak{m}(\lambda) \oplus \boldsymbol{R}w_4$ to $\mathfrak{m}(\mu) \oplus \boldsymbol{R}w_4$ satisfying the conditions of Theorem 1.2. Since $\{v_1+\lambda w_4, v_2, v_3\}$ (resp. $\{v_1+\mu w_4, v_2, v_3\}$) is an orthonormal basis of $\mathfrak{m}(\lambda)$ (resp. $\mathfrak{m}(\mu)$) relative to the inner product induced from $(T_p(\boldsymbol{S}^3), g_p)$, we have

$$f(v_1+\lambda w_4, v_2, v_3, w_4) = (v_1+\mu w_4, v_2, v_3, w_4)\left(\begin{array}{ccc|c} & (a_{ij}) & & \begin{matrix}0\\0\\0\end{matrix} \\ \hline 0 & 0 & 0 & k \end{array}\right)$$

where $(a_{ij}) \in O(3)$ and $k \in \boldsymbol{R}$. By using the fact that f is a Lie algebra isomorphism, we have $a_{21} = a_{31} = 0$, $a_{11}{}^2 = 1$ and $k^2 = 1$. Further by using it for the relation (2.2), it follows that $(1+\lambda)^2 = (1+\mu)^2$ and $\lambda^2 = \mu^2$. Namely we have $\lambda = \mu$. q. e. d.

Case (A_3) : The possible $\mathfrak{g}$ are as follows :

(A_3^1) $\dim(\mathfrak{g} \cap \mathfrak{su}(2)_1) = 0$
(A_3^2) $\dim(\mathfrak{g} \cap \mathfrak{su}(2)_1) = 1$
(A_3^3) $\dim(\mathfrak{g} \cap \mathfrak{su}(2)_1) = 3$

First Case (A_3^2) can not occur. In fact we have $\mathfrak{g}|_{\mathfrak{su}(2)_1} = \mathfrak{g} \cap \mathfrak{su}(2)_1$ from $\dim(\mathfrak{g} \cap \mathfrak{su}(2)_1) = 1$. Hence it follows from $\dim \mathfrak{g} = 3$ that $\dim(\mathfrak{g} \cap \mathfrak{su}(2)_2) = 2$, a contradiction. The case (A_3^3) implies $\mathfrak{g} = \mathfrak{su}(2)_1$. Finally Case (A_3^1) implies $\mathfrak{g} = \mathfrak{su}(2)_2$. In fact, from $\mathfrak{g} \cap \mathfrak{su}(2)_1 = (0)$, $\mathfrak{g}$ is expressed as $\{f(u)+u \mid u \in \mathfrak{su}(2)_2\}$ by means of a linear map f from $\mathfrak{su}(2)_2$ to $\mathfrak{su}(2)_1$. Since $\mathfrak{g}$ is a Lie algebra, we see that f is a Lie algebra homomorphism. Namely f is a zero map or Lie algebra isomorphism. On the other hand, $\mathfrak{h}_p$ gives a Lie algebra isomorphism $f' : \mathfrak{su}(2)_2 \to \mathfrak{su}(2)_1$ such that $\mathfrak{h}_p = \{f'(u)+u \mid u \in \mathfrak{su}(2)_2\}$. Hence if f is the Lie algebra isomorphism, then $f^{-1}f' : \mathfrak{su}(2)_2 \to \mathfrak{su}(2)_2$ gives an element of $SO(\mathfrak{su}(2)_2)$ and has a non-zero fixed point. Namely $\mathfrak{h}_p \cap \mathfrak{g} = (0)$, a contradiction. Clearly $\mathfrak{su}(2)_1$ and $\mathfrak{su}(2)_2$ defined the isomorphic homogeneous structures. The homogeneous structure defined from $\mathfrak{su}(2)_1$ is the case $\lambda = 0$ in the homogeneous structures $T(\lambda)$. q. e. d.

3. The homogeneous structures on H^3.

Theorem 3.1. *The 3-dimensional hyperbolic space* ($H^3 = \{(x, y, z) \in \boldsymbol{R}^3 \mid z > 0\}$, $g = \frac{1}{z^2}(dx^2+dy^2+dz^2)$) *admits two types of homogeneous struc-*

tures up to the isomorphisms :

$$(1) \quad T^{\lambda}_{XYZ} = g(T^{\lambda}_X Y, Z) = \lambda dV(X, Y, Z) \quad (\lambda \geqq 0)$$

$$(2) \quad T(\lambda)_X Y = \lambda g(X, \xi)\{g(Y, \eta)\zeta - g(Y, \zeta)\eta\} + g(X, Y)\xi - g(\xi, Y)X \quad (\lambda \geqq 0)$$

for $X, Y, Z \in \mathfrak{X}(H^3)$, *where* dV *is the volume form of* (H^3, g) *and* $\eta = z\frac{\partial}{\partial x}$, $\zeta = z\frac{\partial}{\partial y}$, $\xi = z\frac{\partial}{\partial z}$.

Proof. Let's consider as $H^3 = \{x+yi+zj \mid x, y \in R, z > 0\} \subset \{x+yi+zj+wk\}$ (the algebra of quaternions). Then $I_0(H^3) = SL(2, C)/\pm 1$ and the isotropy subgroup K of $I_0(H^3)$ at $j \in H^3$ is $SU(2)/\pm 1 = SO(3)$. Therefore the Lie algebra of $I_0(H^3)$ and K are $\mathfrak{sl}(2, C)$ and $\mathfrak{su}(2)$, respectively. Hence the possible connected subgroups G of $I_0(M)$ acting transitively on H^3 are given as follows as in S^3 :

(B_1) $\dim G = 6$
(B_2) $\dim G = 4$ and $\dim(G \cap K) = 1$
(B_3) $\dim G = 3$ and $\dim(G \cap K) = 0$

Case (B_1) : $G = SL(2, C)$ with Lie algebra $\mathfrak{sl}(2, C)$. Note here that there is a basis $\{u_i, c_i\}$ $(i = 1, 2, 3)$ in $\mathfrak{sl}(2, C)$ such that $u_1^*|_j = \frac{\partial}{\partial z}\Big|_j$, $u_2^*|_j = \frac{\partial}{\partial x}\Big|_j$, $u_3^*|_j = \frac{\partial}{\partial y}\Big|_j$ and $[u_1, u_m] = u_m$ $(m = 2, 3)$ $[u_1, c_1] = 0$, $[u_1, c_m] = u_m - c_m$ $(m = 2, 3)$, $[u_2, u_3] = 0$, $[u_2, c_1] = -u_3$, $[u_2, c_2] = -u_1$, $[u_2, c_3] = c_1$, $[u_3, c_1] = u_2$, $[u_3, c_2] = -c_1$ $[u_3, c_3] = -u_1$, $[c_i, c_j] = c_k$ ((i, j, k) : even permutation of $(1, 2, 3)$) and $\mathfrak{su}(2) = \langle c_1, c_2, c_3\rangle$. Then it is easy to show that the $\mathfrak{m}$-component of a reductive decomposition of $\mathfrak{g}$ relative to $\mathfrak{su}(2)$ is given by $\langle u_1 + \lambda c_1, u_2 - c_2 + \lambda c_3, u_3 - c_3 - \lambda c_2\rangle$ ($\lambda \in R$) and the corresponding homogeneous structure T^{λ} is isomorphic to λdV, where dV denotes the volume form of (H^3, g). T^{λ} and T^{μ} are isomorphic iff $\lambda^2 = \mu^2$.

Case (B_3) : To begin with, we shall state the definitions and theorems given in [2], [3] without proof. By definition, a Riemannian space satisfies the condition ($S.\dot{M}$) if M is a simply connected homogeneous Riemannian space of sectional curvature $\leqq 0$ which admits a connected subgroup S of $I_0(M)$ acting simply transitively on M.

Definition 3.2. Let M be a Riemannian space satisfying the condition

$(S.M)$ without Euclidean factor relative to the de Rham decomposition of M. Then a connected subgroup S' of $I_0(M)$ is said to be *a modification* of S if S' acts simply transitively on M and the Lie algebra $\mathfrak{S}'$ of S' is in the normalizer $\mathfrak{n}(\mathfrak{S})$ of the Lie algebra $\mathfrak{S}$ of S. (cf. [3] p. 43, 44)

We also say that $\mathfrak{S}'$ is a modification of $\mathfrak{S}$.

Remark. S' is a modification of S if and only if S is a modification of S'. (cf. [3] p. 45 remark (b))

Definition 3.3. Let (M, g) be a Riemannian space satisfying the condition $(S.M)$ and B the Killing form of the Lie algebra $\mathfrak{g}$ of $G = I_0(M)$. Then a subgroup S of G is said to be *in standard position* if for some point $p \in M$, $B(V, U) = 0$ for all $V \in [\mathfrak{S}, \mathfrak{S}]^{\perp}$ and all $U \in \mathfrak{h}_p$. In the above $[\mathfrak{S}, \mathfrak{S}]^{\perp}$ denotes the orthogonal complement of $[\mathfrak{S}, \mathfrak{S}]$ in $\mathfrak{S}$ relative to the inner product induced from $(T_p(M), g_p)$ into $\mathfrak{S}$ and $\mathfrak{h}_p$ is the Lie algebra of the isotropy subgroup of G at p. (cf. [3] p. 45)

We also say that $\mathfrak{S}$ is in standard position.

Theorem 3.4. *Let M be a Riemannian space satisfying the condition $(S.M)$ without Euclidean factor. Then for a connected subgroup S' of $I_0(M)$ acting simply transitively on M, there exists a unique subgroup S of $I_0(M)$ being in standard position such that S' is a modification of S.* (cf. [3] p. 48)

Theorem 3.5. *Let M be a Riemannian space satisfying the condition $(S.M)$ without Euclidean factor. Then for subgroups S and S' of $I_0(M)$ being in standard position, there exists $a \in I_0(M)$ such that $aSa^{-1} = S'$. Conversely, if S is in standard position then for any element $a \in I_0(M)$, $aSa^{-1} = S'$ is in standard position.* (cf. [3] p. 46 Remark (a) and p. 44)

Remark. We may state Theorem 3.4 and 3.5 in terms of Lie algebra. First from Theorem 3.5, the Lie algebra being in standard position is obtained from the fixed Lie algebra in standard position by adjoint representation $Ad(a)$ of some $a \in I_0(M)$. Now let $\mathfrak{S}$ be the Lie algebra in standard position. Then $\mathfrak{S}$ and $Ad(a)\mathfrak{S} = \mathfrak{S}'$ $(a \in I_0(M))$ are Lie algebra isomorphic and isometric relative to the inner products induced from $(T_p(M), g_p)$ and $(T_{a(p)}(M), g_{a(p)})$, respectively. Hence we find by Theorem 1.2 that the homogeneous structures defined from these algebras are isomorphic. Furthermore a modification of $\mathfrak{S}$ is transfered to a modification of $\mathfrak{S}'$ by $Ad(a)$. Thus the homogeneous structure defined from the Lie algebra of any subgroup of $I_0(M)$ acting simply transitively on M is isomorphic to that defined from the Lie algebra of a subgroup of $I_0(M)$ which is a modification of a fixed subgroup of S_0 being in standard position.

Now, we shall apply the above fact to the case of H^3.

Lemma 3.6. *Put* $\mathfrak{s} = \left\{ \begin{pmatrix} t & w \\ 0 & -t \end{pmatrix} \middle| t \in \boldsymbol{R},\ w \in \boldsymbol{C} \right\}$. *Then* $\mathfrak{s}$ *is in standard position and* $\mathfrak{s}_\lambda = \left\{ \begin{pmatrix} t(1+\lambda i) & w \\ 0 & -t(1+\lambda i) \end{pmatrix} \middle| \begin{matrix} t \in \boldsymbol{R} \\ w \in \boldsymbol{C} \end{matrix} \right\}$ $(\lambda \in \boldsymbol{R})$ *are modifications of* $\mathfrak{s} = \mathfrak{s}_0$.

Proof. The Killing form B on $\mathfrak{sl}(2, \boldsymbol{C})$ as real Lie algebra is given by $B(X, Y) = 8Re(trace(XY))$. Further $[\mathfrak{s}_\lambda, \mathfrak{s}_\lambda]$ is $\left\{ \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} \middle| w \in \boldsymbol{C} \right\}$ and the orthogonal complement $[\mathfrak{s}_\lambda, \mathfrak{s}_\lambda]^\perp$ of $[\mathfrak{s}_\lambda, \mathfrak{s}_\lambda]$ in $\mathfrak{s}_\lambda$ relative to the inner product induced from $(T_{(J)}(H^3), g_{(J)})$ into $\mathfrak{s}_\lambda$ is equal to

$$\left\{ \begin{pmatrix} t(1+\lambda i) & 0 \\ 0 & -t(1+\lambda i) \end{pmatrix} \middle| t \in \boldsymbol{R} \right\}.$$

Hence $B([\mathfrak{s}_\lambda, \mathfrak{s}_\lambda]^\perp, \mathfrak{su}(2)) = 0$ iff $\lambda = 0$. On the other hand, we have obviously $\dim \mathfrak{s}_\lambda = 3$ and $\mathfrak{s}_\lambda \cap \mathfrak{su}(2) = (0)$ for all $\lambda \in \boldsymbol{R}$. Thus $\mathfrak{s}_0$ is in standard position. If we consider a subalgebra $\bar{\mathfrak{s}}$ of the normalizer $\mathfrak{n}(\mathfrak{s}_0)$ of $\mathfrak{s}_0$ with dimension 3, it is easy to show that there exists $\lambda \in \boldsymbol{R}$ such that $\bar{\mathfrak{s}} = \mathfrak{s}_\lambda$ because of $\mathfrak{n}(\mathfrak{s}_0) = \left\{ \begin{pmatrix} z & w \\ 0 & -z \end{pmatrix} \middle| z, w \in \boldsymbol{C} \right\}$. Hence $\mathfrak{s}_\lambda$ $(\lambda \in \boldsymbol{R})$ are modifications of $\mathfrak{s} = \mathfrak{s}_0$. q. e. d.

Lemma 3.7. *Let* $\mathfrak{s}_\lambda$ *and* $\mathfrak{s}_\mu$ *be as in the above lemma. Then there exists a Lie algebra isomorphism* φ *from* $\mathfrak{s}_\lambda$ *to* $\mathfrak{s}_\mu$ *such that* φ *is an isometry relative to the inner products induced from* $(T_{(J)}(H^3), g_{(J)})$ *iff* $\lambda^2 = \mu^2$.

Proof. First note that $\mathfrak{s}_k$ $(k = \lambda, \mu)$ have orthonormal bases

$$\left\{ v_k = (1+ki)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix},\ u_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ u_3 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \right\}.$$

Now if there exists φ in this lemma, then it follows that $\lambda^2 = \mu^2$ by the same argument with the proof of Lemma 2.3. Conversely if $\lambda^2 = \mu^2$, then the existance of such φ is obvious. q. e. d.

Proposition 3.8. *The homogeneous structures* $T(\lambda)$ *defined from* $\mathfrak{S}_\lambda$ $(\lambda \geqq 0)$ *are given by*

$$T(\lambda)_X Y = \lambda g(X, \xi)\{g(Y, \eta)\zeta - g(Y, \zeta)\eta\} + g(X, Y)\xi - g(\xi, Y)X \quad \text{for } X, Y \in \mathfrak{X}(H^3)$$

where we put $\eta = z\dfrac{\partial}{\partial z}$, $\zeta = z\dfrac{\partial}{\partial y}$, $\xi = z\dfrac{\partial}{\partial z}$.

Proof. By using $[v_\lambda, u_2] = u_2 + \lambda u_3$, $[v_\lambda, u_3] = u_3 - \lambda u_2$, $[u_2, u_3] = 0$ and $v_\lambda^*|_{(J)} = \dfrac{\partial}{\partial z}\Big|_{(J)}$, $u_2^*|_{(J)} = \dfrac{\partial}{\partial x}\Big|_{(J)}$, $u_3^*|_{(J)} = \dfrac{\partial}{\partial y}\Big|_{(J)}$, we see that $T(\lambda)$ satisfies

$$T(\lambda)_{\frac{\partial}{\partial z}}\frac{\partial}{\partial z}\Big|_{(J)} = T(\lambda)_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}\Big|_{(J)} = T(\lambda)_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x}\Big|_{(J)} = 0$$

$$T(\lambda)_{\frac{\partial}{\partial x}}\frac{\partial}{\partial z}\Big|_{(J)} = -\frac{\partial}{\partial x}\Big|_{(J)}, \quad T(\lambda)_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}\Big|_{(J)} = \frac{\partial}{\partial z}\Big|_{(J)}$$

$$T(\lambda)_{\frac{\partial}{\partial y}}\frac{\partial}{\partial z}\Big|_{(J)} = -\frac{\partial}{\partial y}\Big|_{(J)}, \quad T(\lambda)_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}\Big|_{(J)} = \frac{\partial}{\partial z}\Big|_{(J)}$$

$$T(\lambda)_{\frac{\partial}{\partial z}}\frac{\partial}{\partial x}\Big|_{(J)} = \lambda\frac{\partial}{\partial y}\Big|_{(J)}, \quad T(\lambda)_{\frac{\partial}{\partial z}}\frac{\partial}{\partial y}\Big|_{(J)} = -\lambda\frac{\partial}{\partial x}\Big|_{(J)}$$

from (1.1) in section 1.

On the other hand, the connected subgroup of $I_0(H^3)$ corresponding to $\mathfrak{S}_\lambda$ is

$$S_\lambda = \left\{ \begin{pmatrix} e^{t(1+\lambda i)} & 0 \\ 0 & e^{-t(1+\lambda i)} \end{pmatrix} \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \,\middle|\, \begin{matrix} t \in \boldsymbol{R} \\ w \in \boldsymbol{C} \end{matrix} \right\} \quad \text{(semi-direct product)}$$

Let's identify S_λ with H^3. Namely, for $(x, y, z) \in H^3$, we put

$$\Phi(x, y, z) = \begin{pmatrix} e^{(\log z)(1+\lambda i)} & 0 \\ 0 & e^{-(\log z)(1+\lambda i)} \end{pmatrix} \begin{pmatrix} 1 & e^{-(\log z)(x+yi)} \\ 0 & 1 \end{pmatrix}$$

This Φ gives a diffeomorphism between S_λ and H^3, and H^3 admits group structures defined from S_λ Hence we have

$$(L_{(x,y,z)})_*\left(\frac{\partial}{\partial x}\right)_{(J)} = \Phi(x, y, z)_*\left(\frac{\partial}{\partial x}\right)_{(J)}$$
$$= z\cos(\lambda\log z)\frac{\partial}{\partial x}+z\sin(\lambda\log z)\frac{\partial}{\partial y}$$
$$(3.2)\quad (L_{(x,y,z)})_*\left(\frac{\partial}{\partial y}\right)_{(J)} = \Phi(x, y, z)_*\left(\frac{\partial}{\partial y}\right)_{(J)}$$
$$= -z\sin(\lambda\log z)\frac{\partial}{\partial x}+z\cos(\lambda\log z)\frac{\partial}{\partial y}$$
$$(L_{(x,y,z)})_*\left(\frac{\partial}{\partial z}\right)_{(J)} = \Phi(x, y, z)_*\left(\frac{\partial}{\partial z}\right)_{(J)} = z\frac{\partial}{\partial z}.$$

Since $T(\lambda)$ is S_λ-invariant, by using (3.1) and (3.2), it follows easily that

$$T(\lambda)_\xi\xi = T(\lambda)_\zeta\eta = T(\lambda)_\eta\zeta = 0,\ T(\lambda)_\eta\eta = \xi,\ T(\lambda)_\eta\xi = -\eta,$$
$$T(\lambda)_\zeta\zeta = \xi,\ T(\lambda)_\zeta\xi = -\zeta \text{ and } T(\lambda)_\xi\eta = \lambda\zeta,\ T(\lambda)_\xi\zeta = -\lambda\eta.$$

Thus we have

$$T(\lambda)_XY = \lambda g(X, \xi)\{g(Y, \eta)\zeta-g(Y, \zeta)\eta\}+g(X, Y)\xi-g(\xi, Y)X$$
$$\text{for } X, Y\in \mathfrak{X}(H^3)$$

Case (B_2) : The following theorem has been proved in [2].

Theorem 3.9. *Let M be a connected, simply connected homogeneous Riemannian space of sectional curvature ≤ 0 and G a connected subgroup of $I_0(M)$ acting transitively on M. Then there exists a solvable subgroup of G acting simply transitively on M.* (cf. [2] p. 327)

Lemma 3.10. *Let G be a connected subgroup of dimension 4 of $I_0(\boldsymbol{H}^3)$ acting transitively on H^3. then there exists a solvable subgroup S of G acting simply transitively on $\boldsymbol{H}^3$ such that $N(S) = G$, where $N(S)$ is the normalizer of S.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G. Then there exists a solvable Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{s}\cap\mathfrak{su}(2) = (0)$ and $\dim\mathfrak{s} = 3$ from Theorem 3.9. By the way, we have already known the form of such $\mathfrak{s}$ from the remark in this section and Lemma 3.6 : $\mathfrak{s} = Ad(a)\,\mathfrak{s}_\lambda$, where $\lambda\in\boldsymbol{R}$, $a\in I_0(M)$ and $\mathfrak{s}_\lambda = \left\{\begin{pmatrix} t(1+\lambda i) & w \\ 0 & -t(1+\lambda i)\end{pmatrix}\middle|\ \begin{matrix} t\in\boldsymbol{R} \\ w\in\boldsymbol{C}\end{matrix}\right\}$. Therefore we have $\mathfrak{g} = Ad(a)\,\mathfrak{s}_\lambda\oplus\boldsymbol{R}v\ (v\in\mathfrak{su}(2))$ from $\dim\mathfrak{g} = 4$ and $\dim(\mathfrak{g}\cap\mathfrak{su}(2)) = 1$.

This implies $Ad(a^{-1})\mathfrak{g} = \mathfrak{s}_\lambda \oplus \boldsymbol{R}Ad(a^{-1})v$. Now let $Ad(a^{-1})v = \begin{pmatrix} k & b \\ c & -k \end{pmatrix}$ $(k, b, c \in \boldsymbol{C})$. Then for any element $\begin{pmatrix} \alpha & w \\ 0 & -\alpha \end{pmatrix}$ of $\mathfrak{s}_\lambda$, there is $s \in \boldsymbol{R}$ and $\begin{pmatrix} \beta & z \\ 0 & -\beta \end{pmatrix} \in \mathfrak{s}_\lambda$ such that

$$\left[\begin{pmatrix} \alpha & w \\ 0 & -\alpha \end{pmatrix}, \begin{pmatrix} k & b \\ c & -k \end{pmatrix}\right] = \begin{pmatrix} cw & * \\ -2c & -cw \end{pmatrix} = \begin{pmatrix} \beta & z \\ 0 & -\beta \end{pmatrix} + s\begin{pmatrix} k & b \\ c & -k \end{pmatrix}.$$

Note that $c \neq 0$ implies $s = -2\alpha$. If $\lambda \neq 0$, we have $c = 0$ since $\alpha \notin \boldsymbol{R}$. Therefore $Ad(a^{-1})v \in \mathfrak{n}(\mathfrak{s}_\lambda) = \mathfrak{n}(\mathfrak{s}_0)$. Hence we get $Ad(a^{-1})\mathfrak{g} = \mathfrak{n}(\mathfrak{s}_0)$ from $\dim \mathfrak{n}(\mathfrak{s}_0) = 4$. Thus $\mathfrak{g}$ is the normalizer of $Ad(a)\,\mathfrak{s}_0$. If $\lambda = 0$ and $c \neq 0$, then $s = -2\alpha$ and therefore $\beta + sk = \beta - 2\alpha k = cw$ i.e. $\beta = 2k\alpha + cw$. Here $\beta \in \boldsymbol{R}$. Hence $\mathfrak{g}$ is the normalizer of $Ad(a)\,\mathfrak{s}_0$ by using the same argument as the case $\lambda \neq 0$. As consequence, we have $G = N(aS_0a^{-1})$.
q. e. d.

The homogeneous structures defined from the reductive decompositions of the Lie subalgebra $\mathfrak{g}$ of $\mathfrak{sl}(2, \boldsymbol{C})$ satisfying the condition (B_2) coincide with those defined from the Lie algebra $\mathfrak{s}$ satisfying the condition (B_3). In fact, by Lemma 3.10, $\mathfrak{g}$ takes the form $Ad(a)\,\mathfrak{n}(\mathfrak{s}_0)$ and its reductive decompositions is given by $Ad(a)\mathfrak{n}(\mathfrak{s}_0) = Ad(a)\,\mathfrak{s}_\lambda \oplus Ad(a)\,(\mathfrak{n}(\mathfrak{s}_0) \cap \mathfrak{su}(2))$, since we have $\mathfrak{n}(\mathfrak{s}_0) = \left\{\begin{pmatrix} z & w \\ 0 & -z \end{pmatrix} \middle| z, w \in \boldsymbol{C}\right\}$. These are the reductive decompositions of the Lie algebra of $aN(S_0)a^{-1}$ at $a(j) \in \boldsymbol{H}^3$. Thus the required result is obtained from Theorem 1.2. q. e. d.

4. The homogeneous structures on $\boldsymbol{R}^3$.

Theorem 4.1. *The 3-dimensional Euclidean space $(\boldsymbol{R}^3, g)$ admits two types of homogeneous structures up to the isomorphisms :*

(1) $T^\lambda_{XYZ} = g(T^\lambda_X Y, Z) = \lambda dV(X, Y, Z) \quad (\lambda \geqq 0)$

(2) $T(\lambda)_X Y = \lambda g\left(X, \frac{\partial}{\partial x}\right)\left\{g\left(Y, \frac{\partial}{\partial y}\right)\frac{\partial}{\partial z} - g\left(Y, \frac{\partial}{\partial z}\right)\frac{\partial}{\partial y}\right\}$
$(\lambda > 0)$

for $X, Y, Z \in \mathfrak{X}(\boldsymbol{R}^3)$, where dV is a volume form of $(\boldsymbol{R}^3, g)$.

Proof. The possible G's which are connected subgroups of $I_0(\boldsymbol{R}^3) =$

$$R^3 \times SO(3) = \left\{ \begin{pmatrix} SO(3) & \begin{matrix} x \\ y \\ z \end{matrix} \\ 0\ 0\ 0 & 1 \end{pmatrix} \right\}$$ acting transitively on R^3 are as follows as in the case S^3 and H^3 :

(C_1) dim $G = 6$
(C_2) dim $G = 4$ and $\dim(G \cap SO(3)) = 1$
(C_3) dim $G = 3$ and $\dim(G \cap SO(3)) = 0$

Case (C_1) : $G = I_0(R^3) = R^3 \times SO(3)$ with Lie algebra

$$\mathfrak{g} = R^3 \oplus \mathfrak{so}(3) = \begin{pmatrix} \mathfrak{so}(3) & \begin{matrix} x \\ y \\ z \end{matrix} \\ 0\ 0\ 0 & 0 \end{pmatrix}.$$

If we choose $u_1 = E_{32} - E_{23}$, $u_2 = E_{13} - E_{31}$, $u_3 = E_{21} - E_{12}$ *and* $e_1 = E_{14}$, $e_2 = E_{24}$, $e_3 = E_{34}$ as a basis of $\mathfrak{g}$, then we have

$$e_1^*|_0 = \left.\frac{\partial}{\partial x}\right|_0, \quad e_2^*|_0 = \left.\frac{\partial}{\partial y}\right|_0, \quad e_3^*|_0 = \left.\frac{\partial}{\partial z}\right|_0$$

(4.1) $[u_i, u_j] = u_k$ for even permutation (i, j, k) of $(1, 2, 3)$

$$[u_i, e_j] = \begin{cases} 0 & \text{for } i = j \\ \operatorname{sgn}(i, j, k) e_k \end{cases}$$

As in the case S^3 and H^3, by using $[\mathfrak{so}(3), \mathfrak{m}] \subset \mathfrak{m}$, it follows easily that the $\mathfrak{m}$-component of the reductive decomposition of $\mathfrak{g}$ relative to $\mathfrak{so}(3)$ is given by $\mathfrak{m}^\lambda = \langle e_i + \lambda u_i \mid i = 1, 2, 3 \rangle$ $(\lambda \in R)$ and the homogeneous structure defined from $\mathfrak{g} = \mathfrak{m}^\lambda \oplus \mathfrak{so}(3)$ is isomorphic to λdV.

Case (C_3) : By Theorem 3.9, it sufficies to determine the solvable subalgebra $\mathfrak{g}$ of $R^3 \oplus \mathfrak{So}(3)$ such that dim $\mathfrak{g} = 3$ and $\dim(\mathfrak{g} \cap \mathfrak{so}(3)) = 0$. We may express $\mathfrak{g}$ as $\langle e_i + f(e_i) \mid i = 1, 2, 3 \rangle$ by means of a linear map f from R^3 to $\mathfrak{so}(3)$ since $\mathfrak{g}$ is a complement of $\mathfrak{so}(3)$. Now by using the fact that $\mathfrak{g}$ is a solvable Lie algebra, we have $f(e_2) = sf(e_1)$, $f(e_3) = tf(e_1)$, where $t, s \in R$ and further $\mathfrak{g}$ is expressed as $\langle \tilde{e}_1 + u, \tilde{e}_2, \tilde{e}_3 \rangle$, $\{\tilde{e}_i \mid i = 1, 2, 3\}$ is an orthonormal basis of R^3 and $u \in \mathfrak{so}(3)$. Since we may choose a basis $\{u_i \mid i = 1, 2, 3\}$ of $\mathfrak{so}(3)$ such that $\tilde{u}_i$ and $\tilde{e}_j$ $(i, j = 1, 2, 3)$ satisfy the relation (4.1), $\mathfrak{g}$ is given by $\langle \lambda \tilde{u}_1 + \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \rangle$. Here it is clear that the Lie algebras $\langle \lambda \tilde{u}_1 + \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \rangle$ and $\mathfrak{g}_\lambda = \langle \lambda u_1 + e_1, e_2, e_3 \rangle$ are Lie algebra isomorphic and isometric relative to the inner products induced from $(T_0(\mathbf{R}^3), g_0)$. Hence by Theorem 1.2 it suffices to determine the homogeneous structures $T(\lambda)$ defined from $\mathfrak{g}_\lambda$. By (4.1) and (1.1) in section 1, we have

$$T(\lambda)_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}\Big|_0 = 0,\ T(\lambda)_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}\Big|_0 = \lambda\frac{\partial}{\partial z}\Big|_0,\ T(\lambda)_{\frac{\partial}{\partial x}}\frac{\partial}{\partial z}\Big|_0 = -\lambda\frac{\partial}{\partial y}\Big|_0$$

and the other components of $T(\lambda)$ are equal to zero. On the other hand, the connected subgroup of $I_0(\boldsymbol{R}^3)$ corresponding to $\mathfrak{g}_\lambda$ is

$$G_\lambda = \left\{\begin{pmatrix} 1 & 0 & 0 & x \\ 0 & \cos(\lambda x) & -\sin(\lambda x) & y \\ 0 & \sin(\lambda x) & \cos(\lambda x) & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle|\ x, y, z \in \boldsymbol{R}\right\}.$$

Since $T(\lambda)$ is G_λ-invariant, we get $T(\lambda)_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}\Big|_q = \lambda\frac{\partial}{\partial z}\Big|_q$, $T(\lambda)_{\frac{\partial}{\partial x}}\frac{\partial}{\partial z}\Big|_q = -\lambda\frac{\partial}{\partial y}\Big|_q$ by easy computation and the other components of $T(\lambda)$ are zero at any point $q \in \boldsymbol{R}^3$. Thus we have

$$T(\lambda)_X Y = \lambda g\left(X, \frac{\partial}{\partial x}\right)\left\{g\left(Y, \frac{\partial}{\partial y}\right)\frac{\partial}{\partial z} - g\left(Y, \frac{\partial}{\partial y}\right)\frac{\partial}{\partial y}\right\}$$
$$(\lambda \in \boldsymbol{R}) \text{ for } X, Y \in \mathfrak{X}(\boldsymbol{R}^3).$$

By using the same argument as in the proof of Lemma 2.3, it easily follows that $T(\lambda)$ and $T(\mu)$ are isomorphic iff $\lambda^2 = \mu^2$.

Case (C_2) : From Theorem 3.9, the Lie algebra $\mathfrak{g}$ satisfying the condition (C_2) contains a Lie algebra satisfying the condition (C_3). Therefore $\mathfrak{g}$ is expressed as $\langle \lambda\tilde{e}_1+\tilde{e}_2, \tilde{e}_2, \tilde{e}_3\rangle \oplus \boldsymbol{R}\tilde{u}_1$ since there are no subalgebras of dimension 2 in $\mathfrak{so}(3)$. Hence we have $\mathfrak{g} = \boldsymbol{R}^3 \oplus \boldsymbol{R}\tilde{u}_1$. Conversely it is easy to show that the reductive decomposition of $\mathfrak{g}$ is given by $\langle \lambda\tilde{u}_1+\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\rangle \oplus \boldsymbol{R}\tilde{u}_1$. Then from Theorem 1.2, the homogeneous structures defined from the decomposition $\langle \lambda\tilde{u}_1+\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\rangle \oplus \boldsymbol{R}\tilde{u}_1$ is isomorphic to the homogeneous structures defined from $\mathfrak{g}_\lambda$. This completes the proof of theorem 4.1. q. e. d.

5. The quotient space and homogeneous structures. As stated in (1.1) of section 1 we have a one-to-one correspondence between the class of homogeneous structures on a homogeneous Riemannian space M, and the representations of M as quotient space of the minimum connected subgroup of $I_0(M)$.

Here we give these correspondence explicitly in case of S^3, H^3, R^3 in the following table:

The class of homogeneous structures	The representation of S^3
$T^{\lambda}_{XYZ} = g(T^{\lambda}_X Y, Z) = \lambda dV(X, Y, Z)$ $(\lambda \neq 1, \lambda \geqq 0)$	1) $SO(4)/SO(3)$
$T(\lambda)_{XYZ} = (dV + \lambda\theta_1 \otimes (\theta_2 \wedge \theta_3))(X, Y, Z)$ $(\lambda \in R-(0))$	2) $SU(2)\times SO(2)/SO(2)$
$T_{XYZ} = dV(X, Y, Z)$	$SU(2)$

1) $SO(4)/SO(3)$ is the reductive homogeneous space with decomposition $\mathfrak{so}(4) = \mathfrak{m}^{\lambda} \oplus \mathfrak{so}(3)$ such that

$$\mathfrak{m}^{\lambda} = \left\{ \left(\begin{array}{c|ccc} 0 & x & y & z \\ \hline -x & 0 & -\lambda z & \lambda y \\ -y & \lambda z & 0 & -\lambda x \\ -z & -\lambda y & \lambda x & 0 \end{array} \right) \middle| \, x, y, z \in R \right\}.$$

2) $SU(2)\times SO(2)$ is the group with the multiplication $(A, e^{i\alpha})(B, e^{i\beta}) = (A(e^{i\alpha}Be^{-i\alpha}), e^{i(\alpha+\beta)})$, where we embed the group $SO(2)$ into $GL(2, C)$ by a map $e^{i\theta} \to \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix}$, and $SU(2)\times SO(2)/SO(2)$ is the reductive homogeneous space with decomposition $\mathfrak{su}(2) \oplus \mathfrak{so}(2) = \mathfrak{m}(\lambda) \oplus \mathfrak{so}(2)$ such that

$$\mathfrak{m}(\lambda) = \left\{ \begin{pmatrix} t(1+\lambda)i, & -\bar{z} \\ z & -ti \end{pmatrix} \middle| \, t \in R, z \in C \right\}.$$

The class of homogeneous structures	The representation of H^3
$T^{\lambda}_{XYZ} = g(T^{\lambda}_X Y, Z) = \lambda dV(X, Y, Z)$ $(\lambda \geqq 0)$	3) $SL(2, C)/\pm 1/SO(3)$
$T(\lambda)_X Y = \lambda g(X, \xi)\{g(Y, \eta)\zeta - g(Y, \zeta)\eta\} + g(X, Y)\xi - g(\xi, Y)X$ $(\lambda \geqq 0)$	$S_{\lambda} = \left\{ \begin{pmatrix} e^{t(1+\lambda i)} & w \\ 0 & e^{-t(1+\lambda i)} \end{pmatrix} \middle| \, \begin{matrix} t \in R \\ w \in C \end{matrix} \right\}$

3) $SL(2, C)/\pm 1/SO(3)$ is the reductive homogeneous space with decomposition $\mathfrak{sl}(2, C) = \mathfrak{m}\lambda \oplus \mathfrak{su}(2)$ such that

$$\mathfrak{m}\lambda = \left\{ (1+\lambda i)\begin{pmatrix} t & \bar{z} \\ z & -t \end{pmatrix} \middle| \, t \in R, z \in C \right\}.$$

The class of homogeneous structures	The representation of $\mathbf{R}^3$
$T^{\lambda}_{XYZ} = g(T^{\lambda}_X Y, Z) = \lambda dV(X, Y, Z) \quad (\lambda \geqq 0)$	4) $R^3 \times SO(3)/SO(3)$
$T(\lambda)_X Y = \lambda g\left(X, \frac{\partial}{\partial x}\right)\left\{g\left(Y, \frac{\partial}{\partial y}\right)\frac{\partial}{\partial z} - g\left(Y, \frac{\partial}{\partial z}\right)\frac{\partial}{\partial y}\right\} \quad (\lambda > 0)$	$G_\lambda = \left\{\begin{pmatrix} 1 & 0 & 0 & x \\ 0 & \cos(\lambda x) & -\sin(\lambda x) & y \\ 0 & \sin(\lambda x) & \cos(\lambda x) & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle\| \begin{matrix} x \\ y \in C \\ z \end{matrix}\right\}$

4) $R^3 \times SO(3)/SO(3)$ is the reductive homogeneous space with decomposition $R^3 \oplus \mathfrak{so}(3) = \mathfrak{m}^\lambda \oplus \mathfrak{so}(3)$ such that

$$\mathfrak{m}^\lambda = \left\{\left(\begin{array}{ccc:c} 0 & -\lambda z & \lambda y & x \\ \lambda z & 0 & -\lambda x & y \\ -\lambda y & \lambda x & 0 & z \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right) \middle| x, y, z \in R\right\}.$$

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