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IDENTIFICATION OF THE RATIO ERGODIC LIMIT FOR AN INVERTIBLE POSITIVE ISOMETRY ON L_1

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1. Introduction. Let T be an invertible positive isometry on L_1 of a σ-finite measure space. It is proved that if f and p are in L_1 and p is nonnegative, then the ratios $\left(\sum_{i=m}^{n} T^i f\right)/\left(\sum_{i=m}^{n} T^i p\right)$ converge almost everywhere on the set $\left\{\sum_{-\infty}^{+\infty} T^i p > 0\right\}$ as $m \to -\infty$ and $n \to +\infty$, independently ; and the identification of the limit is obtained.

Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and T a linear operator from $L_1 = L_1(X, \mathscr{F}, \mu)$ into itself. T is called *positive* if $f \geq 0$ implies $Tf \geq 0$, a *contraction* if $\|Tf\|_1 \leq \|f\|_1$ for all $f \in L_1$, and an *isometry* if $\|Tf\|_1 = \|f\|_1$ for all $f \in L_1$. For f and p in L_1, with $p \geq 0$, we write

$$R_m^n(f, p)(x) = \left(\sum_{i=m}^{n} T^i f(x)\right)\Big/\left(\sum_{i=m}^{n} T^i p(x)\right).$$

It follows from the Chacon-Ornstein theorem [2] that if T is a positive contraction on L_1 then the pointwise limit

$$\lim_{n\to+\infty} R_0^n(f, p)(x)$$

exists and is finite a.e. on the set $\left\{x : \sum_{0}^{+\infty} T^i p(x) > 0\right\}$; furthermore if C denotes the conservative part of T (i.e. $C = \left\{x : \sum_{0}^{+\infty} T^i g(x) = +\infty\right\}$ for some $g \in L_1$ with $g > 0$ a.e. on X), then the identification of the limit can be done, on C, by the Chacon identification theorem [1]. However, in this paper, we assume T to be an invertible positive isometry on L_1 and consider the ratios $R_m^n(f, p)(x)$, with $m < 0 < n$. Noticing that the conservative parts of T and T^{-1} coincide, we may apply these two theorems to infer that the pointwise limit

$$R_{-\infty}^{+\infty}(f, p)(x) = \lim_{m\to-\infty, n\to+\infty} R_m^n(f, p)(x)$$

exists and is finite a.e. on the set $\left\{x : \sum_{-\infty}^{+\infty} T^i p(x) > 0\right\}$. But the Chacon identification theorem can be applied only on the conservative part C, not on the whole set X. This is the starting point for the study in this paper. We shall obtain the identification of the limit $R_{-\infty}^{+\infty}(f, p)(x)$ on the whole set X. In the process of doing this, the almost everywhere existence of the limit $R_{-\infty}^{+\infty}(f, p)(x)$ is proved as a by-product ; we do not use the Chacon-Ornstein theorem. The method is chiefly dependent upon the argument given in Garsia ([4], pp. 39-41) for the identification of the limit function in the Chacon-Ornstein theorem.

2. Identification of the ratio ergodic limit. Let T be an invertible positive isometry on L_1. Since $\|T^{-1}f\|_1 = \|f\|_1$ for any $f \in L_1$, if $f \geq 0$ then we must have $T^{-1}f \geq 0$. Thus T^{-1} is also a positive isometry on L_1. For any nonnegative function h on $(X, \mathscr{F}, \mu)$, we define $Th = \lim_n Tf_n$, where $f_n \in L_1$ and $0 \leq f_n \uparrow h$. Clearly, this definition is independent of the choice of the sequence. Similarly, $T^{-1}h$ is defined.

$A \in \mathscr{F}$ is called *invariant* if

$$A = \operatorname{supp} T1_A = \{x : T1_A(x) \neq 0\},$$

where 1_A denotes the indicator function of A. It is easy to check that $A \in \mathscr{F}$ is invariant if and only if $TL_1(A) = L_1(A)$, where $L_1(A) = \{f \in L_1 : \operatorname{supp} f \subset A\}$. Therefore, if $0 \leq p \in L_1$ then the set $E(p) = \left\{x : \sum_{-\infty}^{+\infty} T^i p(x) > 0\right\}$ is invariant. The class of all invariant sets is denoted by $\mathscr{I}$. Since T is invertible, $\mathscr{I}$ is a sub-σ-field of $\mathscr{F}$.

We are now in a position to state the theorem.

Theorem. *If T is an invertible positive isometry on L_1, then for any f and p in L_1, with $p \geq 0$, the pointwise limit*

$$\lim_{m \to -\infty, n \to +\infty} R_m^n(f, p)(x) = R_{-\infty}^{+\infty}(f, p)(x)$$

exists and is finite a.e. on the set $E(p) = \left\{x : \sum_{-\infty}^{+\infty} T^i p(x) > 0\right\}$; furthermore, the limit function $R_{-\infty}^{+\infty}(f, p)$ is measurable with respect to $\mathscr{I}$ and satisfies

$$\int_A R_{-\infty}^{+\infty}(f, p) \cdot p \, d\mu = \int_A f \, d\mu$$

for all $A \in \mathscr{I}$ with $A \subset E(p)$.

To prove the theorem, we need some lemmas.

Lemma 1. *Let $h \in L_\infty = L_\infty(X, \mathscr{F}, \mu)$. Then $T^*h = h$ if and only if h is measurable with respect to $\mathscr{I}$.*

Proof. Suppose h is measurable with respect to $\mathscr{I}$. An easy approximation argument shows that for the proof of $T^*h = h$, it suffices to prove that $T^*1_A = 1_A$ for all $A \in \mathscr{I}$. But, $A \in \mathscr{I}$ implies $T^*1_A = 0$ on $X \backslash A$, because $\langle f, T^*1_A \rangle = \langle Tf, 1_A \rangle = \int_A Tf \, d\mu = 0$ for all $f \in L_1(X \backslash A)$. Similarly, $T^*1_{X \backslash A} = 0$ on A. Thus $T^*1_A = 1_A$, since $T^*1 = 1$.

Conversely, suppose $T^*h = h$. (Here we may and will assume without loss of generality that $0 \leq h \leq 1$.) Given an $\alpha > 0$, write $A = \{x : h(x) > \alpha\}$, $h_1(x) = \min\{h(x), \alpha\}$ and $h_2(x) = h(x) - h_1(x)$. Then $h = h_1 + h_2 = T^*h = (T^{-1})^*h = (T^{-1})^*h_1 + (T^{-1})^*h_2$. Since $(T^{-1})^*h_1 \leq \alpha$ and $h > \alpha$ on A, it follows that $(T^{-1})^*h_2 > 0$ on A. Since $\operatorname{supp} h_2 = A$, we then have

$$(T^{-1})^*1_A > 0 \text{ on } A.$$

By this and the fact that $T^*(T^{-1})^*1_A = 1_A$, we see that

$$T^*1_A = 0 \text{ on } X \backslash A.$$

Hence $T^*1_A \leq 1_A$, and by a similar argument, $(T^{-1})^*1_A \leq 1_A$. Consequently,

$$1_A = T^*(T^{-1})^*1_A \leq T^*1_A \leq 1_A,$$

which implies $1_A = T^*1_A$ and hence $A \in \mathscr{I}$. The proof is complete.

Lemma 2. *If $h \in L_\infty$ satisfies $T^*h = h$, then for any $f \in L_1$*

$$T(hf) = h(Tf).$$

Proof. If $A \in \mathscr{I}$ then, clearly, $T(1_A f) = 1_A(Tf)$ for all $f \in L_1$. This, together with Lemma 1 and an easy approximation argument, completes the proof.

Lemma 3. *If f and p are in L_1 and $p > 0$ a.e. on X, define*

$$M(f, p)(x) = \sup_{m \leq 0 \leq n} |R_m^n(f, p)(x)|.$$

Then, for any $\lambda > 0$, $\int_{\{M(f,p)>\lambda\}} p\, d\mu \leq \frac{4}{\lambda} \|f\|_1$.

Proof. Put

$$M_+(f, p)(x) = \sup_{0 \leq n} |R_0^n(f, p)(x)|$$

and

$$M_-(f, p)(x) = \sup_{m \leq 0} |R_m^0(f, p)(x)|.$$

Then we have $M(f, p) \leq M_+(f, p) + M_-(f, p)$, and so

$$\{M(f, p) > \lambda\} \subset \left\{M_+(f, p) > \frac{\lambda}{2}\right\} \cup \left\{M_-(f, p) > \frac{\lambda}{2}\right\}.$$

Since $\left\{M_+(f, p) > \frac{\lambda}{2}\right\} \subset \left\{\sup_{0 \leq n} \sum_{i=0}^{n} T^i\left(|f| - \frac{\lambda}{2} p\right) > 0\right\}$, the Hopf maximal ergodic theorem (see e.g. [4], p. 23) gives

$$\int_{\{M_+(f,p)>\frac{\lambda}{2}\}} p\, d\mu \leq \frac{2}{\lambda} \|f\|_1.$$

Similarly, $\int_{\{M_-(f,p)>\frac{\lambda}{2}\}} p\, d\mu \leq \frac{2}{\lambda} \|f\|_1$, and hence the proof is completed.

Proof of the Theorem. We can easily show that we need only check the validity of the Theorem when $p > 0$ a.e. on X. Thus in the following proof we will assume that $p > 0$ a.e. on X.

Let M be the class of all functions f of the form

$$f = hp + g - Tg, \text{ where } h \in L_\infty,\ g \in L_1 \text{ and } T^*h = h.$$

Making use of Lemma 2, if $f = hp + g - Tg \in M$ then

$$R_m^n(f, p)(x) = h(x) + \frac{T^m g(x) - T^{n+1} g(x)}{\sum_{i=m}^{n} T^i p(x)}.$$

Since $p > 0$ a.e. on X, the Chacon-Ornstein lemma (see e.g. Theorem 2.4.2 in [4]) shows that

$$\lim_{m \to -\infty, n \to +\infty} R_m^n(f, p) = h \text{ a.e. on } X.$$

Next, to prove that M is dense in L_1, let $k \in L_\infty$ be such that $\langle f, k\rangle = 0$ for all $f \in M$. Then we have $\langle g - Tg, k\rangle = \langle g, k - T^*k\rangle = 0$ for all $g \in L_1$. Thus $k = T^*k$, and $\langle kp, k\rangle = \int k^2 p \, d\mu = 0$. It follows that $k = 0$ a.e. on X, which proves the denseness of M in L_1.

For $f = hp + g - Tg \in M$, put

$$Hf = hp = R_{-\infty}^{+\infty}(f, p) \cdot p.$$

Then

$$\| Hf \|_1 = \int (\operatorname{sgn} h) hp \, d\mu = \int (\operatorname{sgn} h)[f - g + Tg] \, d\mu$$

where $\operatorname{sgn} h(x) = h(x)/|h(x)|$ if $h(x) \neq 0$, and is 0 if $h(x) = 0$. Since $T^*(\operatorname{sgn} h) = \operatorname{sgn} h$ by Lemma 1, it follows that

$$\int (\operatorname{sgn} h)[f - g + Tg] \, d\mu = \int (\operatorname{sgn} h) f \, d\mu \leq \| f \|_1.$$

Thus $\| Hf \|_1 \leq \| f \|_1$ $(f \in M)$. Since M is dense in L_1, H can be uniquely extended to a contraction operator on L_1. We will denote this extension by the same letter H. Clearly, if $A \in \mathcal{I}$ then

$$\int_A Hf \, d\mu = \int_A f \, d\mu$$

for all $f \in M$ and thus for all $f \in L_1$.

Now, to finish the proof of the Theorem, it suffices to show that

$$\lim_{m \to -\infty, n \to +\infty} R_m^n(f, p) = (1/p) Hf \text{ a.e. on } X$$

for all $f \in L_1$. To do this, we notice that if $f \in L_1$ and $e \in M$ then

$$\begin{aligned} &|R_m^n(f, p) - (1/p)Hf| \\ &\quad \leq |R_m^n(f - e, p) - (1/p)H(f - e)| + |R_m^n(e, p) - (1/p)He| \\ &\quad \leq M(f - e, p) + |(1/p)H(f - e)| + |R_m^n(e, p) - (1/p)He|. \end{aligned}$$

Since $R_m^n(e, p) \to (1/p)He$ a.e. on X as $m \to -\infty$ and $n \to +\infty$, independently, if we let

$$f^*(x) = \lim_{N \to +\infty} \sup_{m \leq -N, n \geq N} \left| R_m^n(f, p)(x) - \frac{Hf(x)}{p(x)} \right|$$

then, for any $\varepsilon > 0$,

$$\{f^* > 2\varepsilon\} \subset \{M(f-e, p) > \varepsilon\} \cup \{|(1/p)H(f-e)| > \varepsilon\}.$$

By Lemma 3,

$$\int_{\{M(f-e,p)>\varepsilon\}} p\, d\mu \leq \frac{4}{\varepsilon}\|f-e\|_1.$$

On the other hand,

$$\int_{\{|(1/p)H(f-e)|>\varepsilon\}} p\, d\mu \leq \frac{1}{\varepsilon}\|H(f-e)\|_1 \leq \frac{1}{\varepsilon}\|f-e\|_1.$$

Here $\|f-e\|_1$ can be arbitrarily small. Thus $\int_{\{f^*>2\varepsilon\}} p\, d\mu = 0$. Since $\varepsilon > 0$ is arbitrary, it follows that $f^* = 0$ a.e. on X, and this completes the proof.

Remark. If $\{T^n : -\infty < n < +\infty\}$ is a group of positive linear operators on L_1 satisfying $\sup_n \|T^n\| = K < +\infty$, then the convergence result in the Theorem holds. In fact, if L denotes a Banach limit (cf. [3]) and if we define $\lambda(A) = \mathrm{L}\left(\int T^n 1_A\, d\mu\right)$ for $A \in \mathscr{F}$ with $\mu(A) < +\infty$ and $\lambda(A) = \sup \{\lambda(B) : B \in \mathscr{F}$ with $B \subset A$ and $\mu(B) < +\infty\}$ for $A \in \mathscr{F}$ with $\mu(A) = +\infty$, then, as is easily seen, $(X, \mathscr{F}, \lambda)$ is a σ-finite measure space such that $K^{-1}\mu \leq \lambda \leq K\mu$ and T is an invertible positive isometry on $L_1(X, \mathscr{F}, \lambda)$. Since $f \in L_1(X, \mathscr{F}, \mu)$ if and only if $f \in L_1(X, \mathscr{F}, \lambda)$, the convergence result follows from the Theorem.

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