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A REMARK ON THE ERGODIC HILBERT TRANSFORM

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Recently the author [6] proved that if T is an invertible power-bounded Lamperti operator on L_p, $1 < p < \infty$, of a σ-finite measure space then the ergodic Hilbert transform Hf (with respect to T) defined by

$$Hf = \lim_{n\to\infty} \sum_{k=-n}^{n}{}' \frac{1}{k} T^k f \qquad (f \in L_p)$$

exists almost everywhere and in the strong operator topology, where the prime denotes that the term with zero denominator is omitted. In this note we remark that the same results hold under a weaker norm condition.

Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and T an invertible Lamperti operator on $L_p = L_p(X, \mathscr{F}, \mu)$ with $1 < p < \infty$. Then, as is well-known (see e.g. [3] or [4]), there exists an invertible positive linear operator Φ acting on measurable functions, induced by a σ-automorphism of the Boolean σ-algebra $\mathscr{F}(\mu)$, and a sequence $\{h_j\}_{j=-\infty}^{\infty}$ of measurable functions, with $0 < |h_j| < \infty$ a.e. on X for each j, such that for any j, T^j has the form $T^j f(x) = h_j(x)\Phi^j f(x)$. Further, by the Radon-Nikodym theorem, there exists a sequence $\{J_j\}_{j=-\infty}^{\infty}$ of positive measurable functions such that for each j and $f \in L_1$

$$(1) \quad \int J_j(x)\Phi^j f(x)\, d\mu = \int f(x)\, d\mu.$$

It is easy to see that

$$(2) \quad h_{j+k}(x) = h_j(x)\Phi^j h_k(x) \text{ and } J_{j+k}(x) = J_j(x)\Phi^j J_k(x) \text{ a.e. on } X.$$

Let $\tau f = |h_1|\Phi f$. Thus τ is an invertible positive operator on L_p and for each integer j, τ^j has the form $\tau^j f = |h_j|\Phi^j f$. Clearly $\|\tau^j\|_p = \|T^j\|_p$. We are now in a position to state the

Theorem. *If* $\sup_{n\geq 0}\left\| \frac{1}{2n+1}\sum_{k=-n}^{n} \tau^k \right\|_p < \infty$ *then the ergodic Hilbert transform* Hf *(with respect to* T*) exists a.e. on* X *and in the strong operator topology.*

For the proof, we need a lemma.

Lemma. *If* $\sup_{n\geq 0}\left\|\frac{1}{2n+1}\sum_{k=-n}^{n}\tau^k\right\|_p < \infty$ *then there exists a constant* C *such that* $\|H^*f\|_p \leq C\|f\|_p$ *for all* $f\in L_p$, *where* H^*f *is the ergodic maximal Hilbert transform (with respect to* T*) defined by*

$$H^*f = \sup_{n\geq 1}\left|\sum_{k=-n}^{n}{}' \frac{1}{k}T^k f\right|.$$

Proof. For an integer $K\geq 1$, define the truncated maximal operator H_K^*f as

$$H_K^*f = \max_{1\leq n\leq K}\left|\sum_{k=-n}^{n}{}' \frac{1}{k}T^k f\right|.$$

From (1) and (2) we see

$$\begin{aligned}\int (H_K^*f)^p\, d\mu &= \frac{1}{2L+1}\int \sum_{j=-L}^{L} J_j(x)\Phi^j(H_K^*f)^p(x)\, d\mu \\ &= \frac{1}{2L+1}\int \sum_{j=-L}^{L} J_j(x)\left[\max_{1\leq n\leq K}\left|\sum_{k=-n}^{n}{}'\frac{1}{k}\Phi^j h_k(x)\Phi^{j+k}f(x)\right|\right]^p d\mu \\ &= \frac{1}{2L+1}\int \sum_{j=-L}^{L} J_j(x)|h_j|^{-p}(x)\left[\max_{1\leq n\leq K}\left|\sum_{k=-n}^{n}{}'\frac{1}{k}h_{j+k}(x)\Phi^{j+k}f(x)\right|\right]^p d\mu.\end{aligned}$$

On the other hand, since $\sup_{n\geq 0}\left\|\frac{1}{2n+1}\sum_{k=-n}^{n}\tau^k\right\|_p < \infty$ by hypothesis, it follows from Martin-Reyes and de la Torre [5] that there exists C such that for a.e. $x\in X$ the function $w_x(j)$ defined on the integers by $w_x(j) = J_j(x)|h_j|^{-p}(x)$ satisfies A_p with the constant C, i.e. for any i and k with $k\geq 0$ we have

$$\left[\sum_{j=0}^{k} w_x(i+j)\right]\left[\sum_{j=0}^{k} w_x^{-1/(p-1)}(i+j)\right]^{p-1} \leq C(k+1)^p.$$

(Here and in the rest of this note C will denote a constant that may be different at each occurrence.) Hence, by a known result about the classical Hilbert transform on the integers (cf. Theorem 10 in [1]) we see that for a.e. $x\in X$

$$\begin{aligned}&\sum_{j=-L}^{L} J_j(x)|h_j|^{-p}(x)\left[\max_{1\leq n\leq K}\left|\sum_{k=-n}^{n}{}'\frac{1}{k}h_{j+k}(x)\Phi^{j+k}f(x)\right|\right]^p \\ &\leq C\sum_{j=-L-K}^{L+K} J_j(x)|h_j|^{-p}(x)|h_j(x)\Phi^j f(x)|^p\end{aligned}$$

$$= C\sum_{j=-L-K}^{L+K} J_j(x)\Phi^j|f|^p(x)$$

where C does not depend on $x \in X$. Therefore

$$\int (H_K^* f)^p\,d\mu \leq \frac{C}{2L+1}\sum_{j=-L-K}^{L+K}\int J_j(x)\Phi^j|f|^p(x)\,d\mu$$

$$= \frac{C}{2L+1}(2L+2K+1)\int |f|^p\,d\mu.$$

Letting $L\uparrow\infty$ and then $K\uparrow\infty$, we have $\int (H^*f)^p\,d\mu \leq C\int |f|^p\,d\mu$, and the proof is completed.

Proof of the theorem. We first notice that the above lemma, together with Lebesgue's convergence theorem, implies that for the proof of the theorem it suffices to show that for any $f\in L_p$ $\lim_{n\to\infty}\sum_{k=-n}^{n}{}' \frac{1}{k}T^k f$ exists a.e. on X. To do this we then notice that, since $\sup_{n\geq 0}\left\|\frac{1}{2n+1}\sum_{k=-n}^{n}\tau^k\right\|_p < \infty$ by hypothesis, τ satisfies a mean ergodic theorem (cf. [5]). So T also satisfies a mean ergodic theorem, and the set $\{g-(f-Tf) : g = Tg$ and f is a simple function in $L_p\}$ is a dense subspace of L_p. We now apply Banach's convergence theorem and see that it suffices to prove the a.e. convergence of $\sum_{k=-n}^{n}{}'\frac{1}{k}T^k(f-Tf)$, with f a simple function in L_p. To do this, we use the equation

$$\sum_{k=-n}^{n}{}'\frac{1}{k}T^k(f-Tf) = f+Tf-\frac{1}{n}(T^{n+1}f+T^{-n}f)$$

$$-\sum_{k=1}^{n-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)(T^{k+1}f+T^{-k}f).$$

Since f is simple, we see (cf. p. 149 in [5]) that there exists r, with $0<r<1$, such that

$$\lim_{n\to\infty} n^{-r}T^n f = \lim_{n\to\infty} n^{-r}T^{-n}f = 0 \text{ a.e. on } X.$$

Then $\sum_{k=1}^{n-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)|T^{k+1}f+T^{-k}f| \leq \sum_{k=1}^{\infty}\frac{1}{k^2}(|T^{k+1}f|+|T^{-k}f|) < \infty$ a.e. on X, and thus the desired result follows.

A final remark. In [6] it is proved that if T is an invertible Lamperti operator on L_1 such that $\sup_{-\infty<n<\infty} \| T^n \|_1 < \infty$ and also such that $\sup_{-\infty<n<\infty} \| T^n \|_\infty < \infty$, then the limit

$$(3) \quad Hf = \lim_{n\to\infty} \sum_{k=-n}^{n}{}' \frac{1}{k} T^k f \qquad (f \in L_1)$$

exists almost everywhere. But this is not the case, when the assumption $\sup_{-\infty<n<\infty} \| T^n \|_\infty < \infty$ is dropped. More precisely, there is an invertible positive isometry T on L_1 (and hence T is a Lamperti operator) such that the a.e. existence of the limit (3) fails to hold for some $f \in L_1$. This can be seen by modifying a method of Ionescu Tulcea [2]. In fact, let $X = [0, 1]$, $\mathscr{F}$ the Lebesgue measurable subsets of X, μ the Lebesgue measure, and $\mathfrak{G}$ the set of all invertible positive isometries on $L_1 = L_1(X, \mathscr{F}, \mu)$. With the strong operator topology $\mathfrak{G}$ becomes a topological (multiplicative) group, and there exists a complete metric on $\mathfrak{G}$ which is compatible with the topology of $\mathfrak{G}$ (cf. [2]). Hence $\mathfrak{G}$ is of the second category in itself. Now we define for each $T \in \mathfrak{G}$ and $1 \le n \le m < \infty$

$$T^{[n,m]}(x) = \sup_{n \le j \le m} \frac{| T^j 1(x) - T^{-j} 1(x) |}{j} \qquad (x \in X),$$

and for measurable real functions f and g on X

$$\rho(f, g) = \int \frac{|f-g|}{1+|f-g|} d\mu$$

Note that $T^{[n,m]}$ is a measurable real function on X. We then have

Proposition 1. *For each $k \ge 1$ the set*

$$\mathfrak{M}_k = \bigcap_{k \le n \le m} \left\{ T \in \mathfrak{G} : \rho(T^{[n,m]}, 0) \le \frac{1}{5} \right\}$$

is closed and nowhere dense in $\mathfrak{G}$.

Sketch of proof. Using Theorem 1 in [2], if $\mathfrak{B}(T)$ is any neighborhood of $T \in \mathfrak{G}$ and $q \ge 1$ is any integer, there is an $S \in \mathfrak{B}(T)$ of the form $Sf(x) = f(\phi^{-1}x)(d\mu \circ \phi^{-1}/d\mu)(x)$ where ϕ is a strictly periodic, invertible measurable and nonsingular transformation on X with strict period $n \ge q$; using Proposition 3,(1) in [2] we may assume without loss of generality that n is an *odd* integer. Then the proof of Theorem 2 in [2] shows that

$S \notin \mathfrak{M}_k$ if an odd n is large enough. This implies that $\mathfrak{M}_k$ is nowhere dense in $\mathfrak{G}$, because $\mathfrak{M}_k$ is a closed set, as is easily seen.

Proposition 2. *The set of all $T \in \mathfrak{G}$ for which the limit* (3) *exists almost everywhere for all $f \in L_1$ is of the first category in $\mathfrak{G}$.*

Proof. Let $T \in \mathfrak{G}$ be such that the limit

$$\lim_{n \to \infty} \sum_{k=-n}^{n}{}' \frac{1}{k} T^k 1(x)$$

exists a.e. on X. Then $\lim_{k \to \infty} \frac{1}{k} | T^k 1(x) - T^{-k} 1(x) | = 0$ a.e. on X, from which there is an integer $k \geq 1$ such that if $k \leq n \leq m$ then $\rho(T^{[n,m]}, 0) \leq 1/5$. Therefore $T \in \bigcup_{k=1}^{\infty} \mathfrak{M}_k$. This establishes the proposition.

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