Math. J. Okayama Univ. 28 (1986), 151-158

AN O THEOREM FOR A CLASS OF DIRICHLET SERIES

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1. Introduction and statement of the result. In this paper we give an O estimate for the error term of the summatory function of the coefficients of a certain class of Dirichlet series. While Chandrasekharan and Narasimhan [1, Theorem 4.1] have given such a theorem for a wider class of Dirichlet series, we feel that our proof is much simpler than theirs, for the case we consider. By modifying our argument we could prove the more general theorem of Chandrasekharan and Narasimhan, though at the expense of more detail.

Let

[1.1] $f(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ and $g(s) = \sum_{n=1}^{\infty} b(n)n^{-s}$

be two Dirichlet series with finite abcissas of absolute convergence $\sigma_a(f)$ and $\sigma_a(g)$, respectively. Let

$$(1.2) \quad \Delta(s) = \prod_{j=1}^{N} \Gamma(\alpha_j s+\beta_j)$$

where $\alpha_j > 0$ and β_j complex, $1 \leq j \leq N$. Assume there exist real numbers C, θ and r, with $C > 0$, and a complex number δ such that $f(s)$ and $g(s)$ satisfy the functional equation

$$(1.3) \quad \Delta(s)f(s) = C^{\theta s+\delta}\Delta(r-s)g(r-s).$$

By Hilfssatz 10 of [6] we have $\sigma_a(f) \geq r/2 \geq r-\sigma_a(g)$. We assume that $f(s)$ can be continued to the entire complex plane as a meromorphic function, whose only singularities are poles lying in the strip $r-\sigma_a(g) \leq \sigma \leq \sigma_a(f)$.

If

$$(1.4) \quad Q(x) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s} x^s \, ds,$$

where C is a curve enclosing all the singularities of the integrand, let

$$(1.5) \quad E(x) = \sum_{n \leq x} a(n) - Q(x).$$

$E(x)$ is called the error term for the summatory function of the coefficients of the Dirichlet series $f(s)$.

We shall prove the following theorem.

Theorem. *Let* $A = \sum_{j=1}^{N} \alpha_j$. *Suppose that for all* $n \geq 1$ *there is a positive absolute constant* c_1 *such that*

$$(1.6) \qquad |a(n)| \leq c_1 n^a \log^b n,$$

where $a \geq 0$ *and* b *is a nonnegative integer. Suppose that, as* $\sigma \rightarrow (a+1)^+$, *we have*

$$(1.7) \qquad \sum_{n=1}^{\infty} |a(n)| n^{-\sigma} = O\left(\frac{1}{(\sigma-a-1)^d}\right).$$

Then, as $x \rightarrow \infty$, *we have*

$$E(x) \ll x^{1+a-(a+1+\sigma_a(g)-r)/(2A\sigma_a(g)-Ar+1/2)} \log^d x + x^a \log^b x.$$

The method of proof of this theorem follows that used by Titchmarsh in [7], section [2.2] for dealing with the divisor problem.

It should be noted that by comparison with the $(b+1)$st derivative of the Riemann zeta function we can take $d = b+1$. Conversely, we have $b \leq d-1$. Also in (1.6) we can take $a = \sigma_a(f)-1$.

2. Lemmas. In the sequel we will denote by

$$\int_{(C,T)} \text{ the integral } \int_{C-iT}^{C+iT}$$

Lemma 1. *Assume* (1.6) *and* (1.7) *for* $a+1 < \sigma < a+2$. *Then for* $a+1 < \sigma < a+2$, $T > 0$ *and* $x > 2$ *we have*

$$(2.1) \qquad \begin{aligned} &\left| \frac{1}{2\pi i} \int_{(C,T)} \frac{f(s)}{s} x^s \, ds - \sum_{n \leq x} a(n) \right| \\ &\leq c_2 \left\{ \frac{x^\sigma}{T(\sigma-a-1)^d} + \frac{x^{a+1} \log^{b+1} x}{T} + x^a \log^b x \right\}, \end{aligned}$$

where c_2 *is a positive constant independent of* x, σ *and* T.

The proof of this lemma is almost identical to that of Hilfssatz 3 of [2] so we do not give it.

Lemma 2. *Let* $F(x)$ *be a real function, twice differentiable and let* $F''(x) \geq r > 0$ *or* $F''(x) \leq -r < 0$ *throughout the interval* (a, b). *Let*

$G(x)/F'(x)$ be monotonic and $|G(x)| \leq M$ throughout the interval (a, b). Then

$$\left| \int_a^b G(x) e^{iF(x)} dx \right| \leq \frac{8M}{\sqrt{r}}.$$

This is Lemma 4. 5 of [7].

Lemma 3. *Let $\Delta(s)$ be as in (1.2). Then, uniformly for $\sigma_1 \leq \sigma \leq \sigma_2$, we have,* as $t \to +\infty$,

$$\frac{\Delta(r-s)}{\Delta(s)} = \Lambda e^{-D\sigma} t^{Ar-2A\sigma} e^{-it(\log t - 2A + D)} \left(1 + O\left(\frac{1}{t}\right)\right),$$

where Λ is a certain constant and $D = 2 \sum_{j=1}^{N} \alpha_j \log \alpha_j$.

This follows easily from Stirling's formula for the gamma function and details can be found in [6, Hilfssatz 8].

Lemma 4. *Let $f(s)$ and $g(s)$ be defined by (1.1) and satisfy the functional equation (1.3). Let $c > \sigma_a(f)$ and $-h < \min(0, r - \sigma_a(g))$. If $-h \leq Re\, s \leq c$, then we have as $t \to +\infty$,*

$$(2.2) \quad f(s) \ll c^{\theta\sigma} t^{\{A(r+2h)/(c+h)\}(c-\sigma)}.$$

Proof. For $\sigma \geq c$ we have that $f(s)$ is bounded since it is an absolutely convergent Dirichlet series. Similarly for $\sigma \leq -h$ $g(r-s)$ is bounded. Thus for $\sigma = c$ we have $f(\sigma + it) \ll t^0$ and for $\sigma = -h$ we have $f(\sigma + it) \ll t^{A(r+2h)}$ by Lemma 3 and the functional equation (1.3). The result, (2.2), follows by the Phragmén-Lindelöf principle and this completes the proof.

3. Proof of Theorem. Let $A(x) = \sum_{n<x} a(n)$. If $x > 2$, then by Lemma 1 and the hypotheses of the theorem we have

$$(3.1) \quad \begin{aligned} A(x) &= \frac{1}{2\pi i} \int_{(c,T)} f(s) \frac{x^s}{s} ds \\ &+ O\left\{ \frac{x^c}{T(c-a-1)^a} + \frac{x^{a+1} \log^{b+1} x}{T} + x^a \log^b x \right\}. \end{aligned}$$

Let R_T be the rectangle with vertices $-h - iT$, $c - iT$, $c + iT$ and $-h + iT$, where T is now large enough so that all the poles of $f(s)$ are in the interior of R_T. By the residue theorem and (1.4) the integral around R_T gives $Q(x)$. We have, by Lemma 4,

$$(3.2)\quad \frac{1}{2\pi i}\int_{-h+iT}^{c+iT} f(s)\frac{x^s}{s}\,ds \ll \int_{-h}^{c} C^{\theta\sigma}T^{A(r+2h)(c-\sigma)/(c+h)-1}x^\sigma d\sigma$$
$$\ll T^{A(r+2h)-1}x^{-h}+T^{-1}x^c,$$

since the maximum of the integrand on the right hand side of (3.2) occurs at one end of the other of the integration interval. Similarly

$$(3.3)\quad \frac{1}{2\pi i}\int_{-h-iT}^{c-iT} f(s)\frac{x^s}{s}ds \ll T^{A(r+2h)-1}x^{-h}+T^{-1}x^c.$$

Thus, by (1.5) and (3.1)–(3.3), we have

$$(3.4)\quad \begin{aligned} E(x) &= \frac{1}{2\pi i}\int_{(-h,T)} f(s)\frac{x^s}{s}\,ds+O\Big\{\frac{x^c}{T(c-a-1)^d} \\ &+\frac{x^{a+1}\log^{b+1}x}{T}+x^a\log^b x+T^{-1}x^c+T^{A(r+2h)-1}x^{-h}\Big\}. \end{aligned}$$

Since $-h < r-\sigma_a(g)$, we have

$$\int_{(-h,T)} f(s)\frac{x^s}{s}ds = \int_{(-h,T)} \frac{\Delta(r-s)}{\Delta(s)}C^{\theta s+\delta}g(r-s)\frac{x^s}{s}\,ds$$
$$(3.5)\quad = \sum_{n=1}^{\infty} b(n)\int_{(-h,T)} \frac{\Delta(r-s)}{\Delta(s)}\cdot\frac{C^{\theta s+\delta}}{n^{r-s}}\cdot\frac{x^s}{s}\,ds$$
$$= ix^{-h}C^{-\theta h+\delta}\sum_{n=1}^{\infty}\frac{b(n)}{n^{r+h}}\int_{-T}^{T}\frac{\Delta(r+h-it)}{\Delta(h+it)}\cdot\frac{(C^\theta nx)^{it}}{-h+it}\,dt.$$

By Lemma 3, we have, for $t > 0$, as $t\to+\infty$,

$$\frac{\Delta(r+h-it)}{\Delta(h+it)} = Ae^{Dh}t^{A(r+2h)}e^{-it(2A\log t-2A+D)}+O(t^{A(r+2h)-1}).$$

Also, as $t\to\infty$,

$$\frac{1}{-h+it} = \frac{1}{it}+O\Big(\frac{1}{t^2}\Big).$$

Write the integral on the right hand side of (3.5) as $\int_{-T}^{-1}+\int_{-1}^{1}+\int_{1}^{T}$.

Then the integral over the interval $(-1, 1)$ is bounded. We have

$$\int_1^T \frac{\Delta(r+h-it)}{\Delta(h+it)} \cdot \frac{(C^\theta nx)^{it}}{-h+it}\, dt$$
$$= -i\Lambda_1 \int_1^T t^{A(2h+r)-1} e^{-it(2A\log t-2A-D)} (C^\theta nx)^{it} dt + O\left(\int_1^T t^{A(r+2h)-2}\, dt\right),$$

where $\Lambda_1 = \Lambda e^{Dh}$.

If $A(r+2h) > 1$, then the O-term in (3.6) is $O(T^{A(r+2h)-1})$. If $A(r+2h) \le 1$, then the O-term is $O(\log T)$.

Let $F(t) = -t(2A\log t - 2A + D) + t\log(C^\theta nx)$. Then

$$F''(t) = -2A/t \le -2A/T$$

for $t \in (1, T)$. Thus, by Lemma 2, we have

$$(3.7) \qquad \int_1^T \frac{\Delta(r+h-it)}{\Delta(h+it)} \cdot \frac{(C^\theta nx)^{it}}{-h+it}\, dt \ll T^{A(r+2h)-1/2}.$$

We get a similar result for the integral over $(-T, -1)$.

Thus, by (3.5)–(3.7), we have

$$(3.8) \qquad \int_{(-h,T)} f(s)\frac{x^s}{s}\, ds \ll x^{-h} \sum_{n=1}^{\infty} \frac{b(n)}{n^{r+h}} T^{A(r+2h)-1/2} \ll \frac{T^{A(r+2h)-1/2}}{x^h}.$$

Thus, by (3.4) and (3.8), we have

$$E(x) \ll \frac{x^c}{T(c-a-1)^d} + \frac{x^{a+1}\log^{b+1} x}{T} + \frac{T^{A(r+2h)-1/2}}{x^h} + x^a \log^b x.$$

If we choose $c = a+1+1/\log x$ and $h = \sigma_a(g) - r + 1/\log x$, we have

$$E(x) \ll \frac{x^{a+1}\log^d x}{T} + x^{r-\sigma_a(g)} T^{1/2A\sigma_a(g) - Ar + 2A/\log x} + x^a \log^b x.$$

Choosing $T = x^{(a+1+\sigma_a(g)-r)/(-Ar+1/2+2A\sigma_a(g))}$ gives

$$E(x) \ll x^{a+1-(a+1+\sigma_a(g)-r)/(1/2+2A\sigma_a(g)-Ar)} \log^d x + x^a \log^b x.$$

This completes the proof of the theorem.

4. Comparison of results and examples. In [3] Landau considered the problem of the O-estimate for a class of Dirichlet series whose functional equations involve more general gamma factors, which the method used here could easily be modified to handle. Also, he considered the case where, in our notation, $\sigma = \sigma_a(g)$. If we specialize Landau's result [3, p. 214] to our functional equation, then we get the same result.

In [6] Richert considered a functional equation similar to ours, but with the more general gamma factors of Landau. Our result is a little better than his Satz 2. Again we could adapt the method used here to cover the form of the functional equation as used by Richert.

In [1] Chandrasekharan and Narasimhan used more general Dirichlet series, though the same functional equation. In some cases their result [1, Theorem 4.1] is better than the one obtained here, though in the case of positive coefficients we get the same result. Even for some cases where the coefficients are not positive (see example 2 below) we get the result obtainable from their theorem.

Example 1. Let K be an algebraic number field of degree n. Let $a(m)$ be the number of integral ideals in K of norm exactly m. For $Re\ s > 1$, let

$$\zeta_K(s) = \sum_{m=1}^{\infty} a(m) m^{-s}$$

be the Dedekind zeta function of K. Then it is known that $\zeta_K(s)$ satisfies the functional equation [4, p. 27]

$$\zeta_K(s)\Gamma^{r_1}(s/2)\Gamma^{r_2}(s) = B^{2s+1}\Gamma^{r_1}((1-s)/2)\Gamma^{r_2}(1-s)\zeta_K(1-s),$$

where B is a certain constant depending only on the field K, r_1 is the number of real conjugates and $2r_2$ the number of imaginary conjugates of K. Here $r = 1$, $\sigma_a(\zeta_K) = 1$, $A = n/2$ and $\zeta_K(s)$ has a simple pole at $s = 1$. Thus we have, if $E(x)$ is the error term associated with $\zeta_K(s)$ by (1.5),

$$E(x) \ll x^{(n-1)/(n+1)} \log x,$$

which is a result of Landau [4, p. 135].

Example 2. Let Λ be a nonprincipal Grössencharacter on ideals mod f, where f is an integral ideal of K, where K is as in Example 1. for $Re\ s > 1$ let

$$\phi_\Lambda(s) = B^s \zeta(s, \Lambda) = B^s \sum \Lambda(\mathfrak{A}) \mathrm{N}(\mathfrak{A})^{-s},$$

where the sum is over all nonzero integral ideals of K. Let

$$e_q = \begin{cases} 1 & 1 \le q \le r_1 \\ 2 & r_1+1 \le q \le r_1+r_2. \end{cases}$$

Then $\phi_A(s)$ can be continued to an entire function and there exist real numbers $\varepsilon_1, \ldots, \varepsilon_{r_1+r_2}$ and nonnegative d_q, d'_q, $1 \leq q \leq r_1+r_2$, such that $\phi_A(s)$ satisfies the functional equation

$$\phi_A(s)\Gamma_A(s) = L\Gamma_A(1-s)\phi_{\bar{A}}(1-s),$$

where L is a certain constant depending on Λ and

$$\Gamma_A(s) = \prod_{q=1}^{r_1+r_2} \Gamma\{[e_q(s+(d_q+d'_q)/2)+i\varepsilon_q]/2\}.$$

Here $A = n/2$, $r = 1$ and $\sigma_a(\phi_A) = 1$. Thus, if $E(x)$ is the error term associated with $\phi_A(s)$ by (1.5), we have

$$E(x) \ll x^{(n-1)/(n+1)},$$

which is the result obtained by Chandrasekharan and Narasimhan [1, p. 128].

Example 3. Let $d_k(n)$ be the number of ways of writing n as a product of k factors. Then, for $Re\ s > 1$, we have

$$\sum_{n=1}^{\infty} d_k(n)n^{-s} = \zeta^k(s).$$

Here $r = 1$, $\sigma_a(\zeta) = 1$, $A = k/2$ and $d = k$. Thus, if $\Delta_k(x)$ is the error term associated to $\zeta^k(s)$ by (1.5), we have

$$\Delta_k(x) \ll x^{(k-1)/(k+1)} \log^k x,$$

which is Theorem 12.2 of [7].

Example 4. Let $\{a(n)\}$ be the coefficients of a cusp form of weight k. For $Re\ s > k$, let

$$f(s) = \sum_{n=1}^{\infty} a^2(n)n^{-s}$$

and

$$g(s) = \zeta(2s-2k+2)f(s).$$

Then in [5, Theorem 3] it is shown that $g(s)$ satisfies the functional equation

$$\Gamma(s)\Gamma(s-k+1)g(s) = (2\pi)^{4s+2-4k}\Gamma(2k-1-s)\Gamma(k-s)g(2k-1-s).$$

Thus $A = 2$, $r = 2k-1$, $\sigma_a(g) = k$ and $d = 1$ since $f(s)$ is analytic except for a simple pole at $s = k$. Thus, if $E(x)$ is the associated error term, we have

$$E(x) \ll x^{k-2/5} \log x,$$

which is the result of Rankin [5].

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(*Received March 20, 1986*)