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EQUIVALENCE OF MODULE CATEGORIES

To the memory of Professor Tadao Tannaka

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Morita contexts and gamma rings are equivalent concepts ([2]). Therefore, the duality theory obtained in Morita contexts is interpreted in the terms of gamma rings and vice versa. Nobusawa [2] proved directly that when R and L contain the unities the categories of all R-modules and L-modules are equivalent, where R and L are the right operator ring and the left operator ring of a gamma ring of homomorphisms respectively. Furthermore, in [3] he obtained a generalization of one of Morita duality theorems, that is, if $R^2 = R$ and $L^2 = L$, then the categories of properly generated R-modules and L-modules are equivalent.

In this note, without the assumption $R^2 = R$ and $L^2 = L$, we shall prove the following theorem :

Theorem. *Let $(R, L, M, \Gamma, \tau, \mu)$ be a Morita context, in which τ and μ are surjective. It is not assumed that the rings R, L have unities nor that the modules are unitary. The categories of properly generated R-modules and L-modules are equivalent.*

We refer to Jacobson [1, p. 166] for the definition of a Morita context $(R, L, M, \Gamma, \tau, \mu)$, where R, L are rings, $M = {}_LM_R$ is an L-R-bimodule, $\Gamma = {}_R\Gamma_L$ is an R-L-bimodule. We shall use his notations, except that the products $\Gamma \times M$ to R and $M \times \Gamma$ to L will be denoted by γx and $x\gamma$ ($x \in M$ and $\gamma \in \Gamma$), since all relevant associative laws hold. It is not assumed that the rings have unities nor that the modules are unitary. But, we assume that τ and μ are surjective.

Let R be a ring and M be a right R-module. If it satisfies (1) $MR = M$, (2) $\{x \in M \mid xR = 0\} = \{0\}$, then according to Nobusawa [3] we say M is properly generated over R.

Let PGM(R) be a category of properly generated right modules over R where the morphisms are R-module homomorphisms. Similarly, PGM(L) denotes a category of properly generated right modules over L where the morphisms are L-module homomorphisms.

Proof of the theorem. Let $G \in$ ob PGM(R). Let A be a free additive abelian group generated by the set of ordered pairs (g,γ), where $g\in G$, $\gamma\in\Gamma$, and let B be the subgroup of elements $\sum_i m_i(g_i,\gamma_i)\in A$, where m_i are integers such that $\sum_i m_i g_i(\gamma_i x)=0$ for all $x\in M$. Denote by $[G,\Gamma]$ the factor group A/B and, without causing any ambiguity, by $[g,\gamma]$ the coset $(g,\gamma)+B$. Every element in $[G,\Gamma]$ therefore can be expressed as a finite sum $\sum_i [g_i,\gamma_i]$. $[G,\Gamma]$ forms a right L-module with definition

$$\sum_i [g_i,\gamma_i]\sum_j x_j\beta_j=\sum_{i,j}[g_i(\gamma_i x_j),\beta_j]$$

for $\sum_i [g_i,\gamma_i]\in[G,\Gamma]$ and $\sum_j x_j\beta_j\in L$. It is well-defined, because $\sum_i [g_i,\gamma_i]=\sum_j [g'_j,\gamma'_j]$ means $\sum_i g_i(\gamma_i x)=\sum_j g'_j(\gamma'_j x)$ for any $x\in M$.

To see $[G,\Gamma]\in$ ob PGM(L), let $\sum_i [g_i,\gamma_i]$ be an element in $[G,\Gamma]$ such that $(\sum_i [g_i,\gamma_i])L=0$, that is, $(\sum_i [g_i,\gamma_i])M\Gamma=0$. By the definition, $\sum_i [g_i(\gamma_i M),\Gamma]=0$, which implies $(\sum_i g_i(\gamma_i M))\Gamma M=0$, that is, $(\sum_i g_i(\gamma_i M))R=0$. Since $G\in$ ob PGM(R), $\sum_i g_i(\gamma_i M)=0$. Hence, $\sum_i [g_i,\gamma_i]=0$.

In addition, $[G,\Gamma]L=[G,\Gamma]M\Gamma=[G\Gamma M,\Gamma]=[GR,\Gamma]=[G,\Gamma]$. Therefore, $[G,\Gamma]\in$ ob PGM(L). Similarly, for $U\in$ ob PGM(L) we can define a right R-module $[U,M]$ and show that $[U,M]\in$ ob PGM(R).

An R-module homomorphism $f: A_R\to B_R$ determines an L-module homomorphism $g:[A,\Gamma]_L\to[B,\Gamma]_L$ by

$$g(\sum_i [a_i,\gamma_i])=\sum_i [f(a_i),\gamma_i].$$

To see that g is well-defined, let $\sum_i [a_i,\gamma_i]=0$. Then for any $\sum_j x_j\omega_j\in L$, $\sum_i [f(a_i),\gamma_i]\sum_j x_j\omega_j=\sum_{i,j}[f(a_i)(\gamma_i x_j),\omega_j]=\sum_{i,j}[f(a_i(\gamma_i x_j)),\omega_j]=\sum_j [f(\sum_i a_i(\gamma_i x_j)),\omega_j]=0$. Hence, $\sum_i [f(a_i),\gamma_i]=0$.

It is easy to see that g is an L-module homomorphism.

Similarly, an L-module homomorphism $h:U_L\to V_L$ determines an R-module homomorphism $k:[U,M]_R\to[V,M]_R$ by $k(\sum_j [u_j,x_j])=\sum_j [h(u_j),x_j]$.

Let f_1 and f_2 be R-module homomorphisms such that $f_1: A\to B$ and $f_2: B\to C$. Let g_1 and g_2 be L-module homomorphisms determined by f_1 and f_2 respectively. Then, $f_2f_1: A\to C$ determines an L-module homomorphism $p:[A,\Gamma]\to[C,\Gamma]$ such that $p=g_2g_1$. Indeed, for any $\sum_i [a_i,\gamma_i]\in[A,\Gamma]$ we have $p(\sum_i [a_i,\gamma_i])=\sum_i [f_2f_1(a_i),\gamma_i]=\sum_i [f_2(f_1(a_i)),\gamma_i]=g_2(\sum_i [f_1(a_i),\gamma_i])=g_2g_1(\sum_i [a_i,\gamma_i])$.

Clearly, $1_A: A\to A$ determines $1_{[A,\Gamma]}:[A,\Gamma]\to[A,\Gamma]$. Thus, we have functors F: PGM(R) $\to$ PGM(L) and H: PGM(L) $\to$ PGM(R), where for $A\in$ ob PGM(R) $F(A)=[A,\Gamma]$ and for $U\in$ ob PGM(L) $H(U)$

$= [U, M]$.

For any $A \in$ ob PGM(R) and $U \in$ ob PGM(L),

$$HF(A) = H([A, \Gamma]) = [[A, \Gamma], M]$$
$$\text{and } FH(U) = F([U, M]) = [[U, M], \Gamma].$$

Define the mapping $\eta_A : A = A(\Gamma M) \to [[A, \Gamma], M]$ by

$$a = \textstyle\sum_i a_i(\gamma_i x_i) \mapsto \sum_i [[a_i, \gamma_i], x_i].$$

We show that η_A is an isomorphism.

$$\begin{aligned} a = \textstyle\sum_i a_i(\gamma_i x_i) = 0 &\Longleftrightarrow (\textstyle\sum_i a_i(\gamma_i x_i))R = 0 \\ &\qquad \text{(By } A \in \text{ob PGM}(R).) \\ &\Longleftrightarrow (\textstyle\sum_i a_i(\gamma_i x_i))\Gamma M = 0 \\ &\qquad \text{(By } R = \Gamma M.) \\ &\Longleftrightarrow [\textstyle\sum_i a_i(\gamma_i x_i), \Gamma] = 0 \\ &\qquad \text{(By the definition that a coset is 0.)} \\ &\Longleftrightarrow \textstyle\sum_i [a_i, \gamma_i](x_i\Gamma) = 0 \\ &\qquad \text{(By the definition of an } L\text{-module} \\ &\qquad \text{imposed on } [A, \Gamma].) \\ &\Longleftrightarrow \textstyle\sum_i [[a_i, \gamma_i], x_i] = 0 \\ &\qquad \text{(By the definition that a coset is 0.)} \end{aligned}$$

Hence, η_A is a bijection of A onto $HF(A)$. It is easy to see that η_A is an R-module homomorphism.

For an R-module homomorphism $f: A_R \to B_R$ and for $a = \sum_i a_i(\gamma_i x_i) \in A$, we have

$$\begin{aligned} HF(f)\eta_A(a) &= HF(f)\eta_A(\textstyle\sum_i a_i(\gamma_i x_i)) = HF(f)\sum_i [[a_i, \gamma_i], x_i] \\ &= \textstyle\sum_i [F(f)([a_i, \gamma_i]), x_i] = \sum_i [[f(a_i), \gamma_i], x_i] \\ &= \eta_B f(a). \end{aligned}$$

Therefore, we have the following commutative diagram :

$$\begin{array}{ccc} A & \xrightarrow{\quad \eta_A \quad} & HF(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle HF(f)} \\ B & \xrightarrow[\quad \eta_B \quad]{} & HF(B) \end{array}$$

Thus, $HF \cong 1_{\text{PGM}(R)}$.

Similarly, we have $FH \cong 1_{PGM(L)}$.
This completes the proof.

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