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SI-MODULES

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Introduction. A ring R is called a left *SI-ring* if every singular left R-module is injective. SI-rings were introduced and studied by K. R. Goodearl. In this paper we say that a left R-module M is an *SI-module* if every singular left R-module is M-injective. It was shown by K. R. Goodearl [6] that a ring R is a left SI-ring if and only if $Z({}_RR) = 0$ and for every essential left ideal I of R, R/I is semisimple. Commutative SI-rings were also investigated by V. C. Cateforis and F. L. Sandomierski [4] and [5]. It was proved in [5] that for a commutative ring R the following are equivalent :

(i) R is an SI-ring.

(ii) R is (von Neumann) regular and $R/Soc(R)$ is semisimple.

In § 2, we show that results of this type can be obtained for SI-modules. The connections between regular modules, V-modules, Generalized V-modules and SI-modules are studied. We also prove, among other things, that if R is a commutative ring and M is a finitely generated projective R-module then M is an SI-module if and only if M is a finite direct sum of regular modules each of which has at most two essential submodules. In § 3, we say that a ring R is a left *P-SI-ring* if every singular left R-module is P-injective. Known results for SI-rings are extended to P-SI-rings.

1. Preliminaries. Throughout this paper, unless otherwise mentioned, R will always have a unit and all modules are unitary left R-modules. All maps will be R-homomorphisms. For any module M we denote by $Z(M)$, $J(M)$, $Soc(M)$ and $E(M)$ the singular submodule, the Jacobson radical, the socle and the injective hull of M, respectively. An R-module M is semisimple if it is a direct sum of simple R-modules.

Let M and U be R-modules. Following G. Azumaya [2], we say that U is *M-injective* if for each submodule K of M every R-homomorphism from K into U can be extended to an R-homomorphism from M into U.

Following Y. Hirano [7], U is said to be *P-M-injective* if every R-homomorphism of any *cyclic* submodule of M into U can be extended to an R-homomorphism of M into U. If every *simple* (resp. *simple singular*) R-module is M-injective, M is called a *V-module* (resp. a *GV-module*). And if every *simple* (resp. *simple singular*) R-module is P-M-injective, M is called a *P-V-module* (resp. a *P-V'-module*). Following Zelmanowitz [12], a module $_RM$ is called *regular* if given any $m \in M$ there exists $f \in Hom_R(M, R)$ with $(m)fm = m$. Following B. Zimmermann-Huisgen [13] we say that a module $_RM$ is *locally projective* if M satisfies the following condition : For all diagrams

$$\begin{array}{ccc} A & \xrightarrow{f} & B \to 0 \\ & & \uparrow g \\ & F \to & M \end{array}$$

with exact upper row and a finitely generated submodule F of M there is a map $g' \in Hom_R(M, A)$ such that $g|F = f \circ g'|F$. It is known that every regular module is locally projective.

Finally we state the next proposition without proof. For the proof see [7] and [10].

Proposition 1.1. *If R is a commutative ring and M is a projective module then the following are equivalent :*

(i) *M is a regular module.*

(ii) *M is a V-module.*

(iii) *M is a GV-module.*

(iv) *M is a P-V-module.*

(v) *M is a P-V'-module.*

2. *SI*-modules.

Definition 2.1. *A left R-module M is called an SI-module* (*resp. P-SI-module*) *if every singular left R-module is M-injective* (resp. *P-M-injective*). Clearly every *SI*-module (resp. *P-SI*-module) is a *GV*-module (resp. *P-V'*-module). *A ring R is called a left SI-ring* (resp. *P-SI-ring*) *if the left R-module $_RR$ is an SI-module* (*resp. P-SI-module*). By [1, Proposition 16.13, p. 188] the following proposition can easily be verified.

Proposition 2.2. (i) *Submodules and homomorphic images of SI-modules are again SI-modules.*

(ii) *$\bigoplus_{i \in I} M_i$ is an SI-module if and only if each M_i is an SI-module.*

Proposition 2.3. *Suppose that $_RM$ is a left SI-module. Then the following statements are true.*

(i) *Every singular homomorphic image of M is semisimple.*

(ii) *M/N is semisimple for every essential submodule N of M.*

(iii) *$J(M) \subseteq Soc(M)$, $Z(M) \subseteq Soc(M)$ and $J(M) \cap Z(M) = 0$.*

Proof. (i) If L is a singular homomorphic image of M then, by Proposition 2.2(i), L is a singular *SI*-module. Whence every submodule of L, which necessarily has to be singular, is L-injective. Hence every submodule of L is a direct summand of L, and so L is semisimple.

(ii) If N is an essential submodule of M then M/N is a singular homomorphic image of M, whence semisimple from above.

(iii) Since $Soc(M)$ is an intersection of essential submodules of M and every proper essential submodule is an intersection of maximal submodules, it follows that $J(M) \subseteq Soc(M)$. Since $Z(M)$ is a singular *SI*-module (since submodules of *SI*-modules are also *SI*-modules), by (i) we infer that $Z(M)$ is semisimple, and hence $Z(M) \subseteq Soc(M)$. Since every *SI*-module is a *GV*-module, it follows from [7, Theorem 3.15] that $J(M) \cap Z(M) = 0$.

Proposition 2.4. *For a locally projective module $_RM$ the following conditions are equivalent :*

(i) *M is an SI-module.*

(ii) *$Z(M) = 0$ and every singular homomorphic image of M is semisimple.*

(iii) *$Z(M) = 0$ and M/N is semisimple for every essential submodule N of M.*

Proof. (i) ⇒ (ii) : Suppose $Z(M) \neq 0$ and let x be a non-zero element of $Z(M)$. Then Rx is a singular submodule of M and hence a direct summand of M. Since M is locally projective it follows that Rx is projective. Now consider the following exact sequence of left R-modules $0 \to Ann_R(x) \to R \xrightarrow{\eta} Rx \to 0$, where η is given by $\eta(r) = rx$, $\forall r \in R$. Since Rx is projective the sequence splits, and hence $Ann_R(x)$ is not essential in $_RR$, contradicting the choice of x. Now the rest of the assertion follows from Proposition 2.3(i).

(ii) ⇒ (iii) : Clear.

(iii) ⇒ (i) : Let L be a singular R-module. We want to show that L is M-injective. So, let N be a proper essential submodule of M and $f : N \to L$ be any non-zero R-homomorphism. Let $K = Ker(f)$. We claim that K is essential in N. For if $K \cap I = 0$ for some non-zero submodule I of N, then $f|I : I \to L$ is a monomorphism. So I is a non-zero singular submodule of M, a clear contradiction since $Z(M) = 0$. Now, since K is essential in M it follows that M/K is semisimple and N/K is a direct summand of M/K. Whence f can be extended to a map $g : M \to L$ in the obvious way.

Note that along the lines of the above proof we have shown that every locally projective SI-module is non-singular. In fact with the same argument one can prove the following.

Proposition 2.5. *Every locally projective P-SI-module is non-singular.*

Proposition 2.6. *Let M be a non-singular module. Then the following conditions are equivalent :*

(i) *M is an SI-module.*

(ii) $Z(L) \subseteq Soc(L)$, *for every homomorphic image L of M.*

(iii) *Every singular homomorphic image of M is semisimple.*

(iv) M/N *is semisimple, for every essential submodule N of M.*

Proof. (i) ⇒ (ii) : If L is a homomorphic image of M then L is an SI-module and hence $Z(L) \subseteq Soc(L)$, by Proposition 2.3(iii). The rest of the implications are trivial.

Observe that if R is a left SI-ring then for any left R-module M, every singular left R-module is M-injective. As a result of this observation we have the following.

Proposition 2.7. *For any ring R the following are equivalent :*

(i) *R is a left SI-ring.*

(ii) *Every left R-module is an SI-module.*

(iii) *Every cyclic left R-module is an SI-module.*

Proposition 2.8. *For a locally projective module M the following are equivalent :*

(i) *M is an SI-module with essential socle.*

(ii) $Soc(M)$ *is projective and* $M/Soc(M)$ *is semisimple.*

Proof. (i) ⇒ (ii) : Since M is a locally projective SI-module, $Z(M) = 0$ by Proposition 2.4, and hence $Soc(M)$ is projective. Since $Soc(M)$ is essential in M, it follows from Proposition 2.3(ii) that $M/Soc(M)$ is semisimple.

(ii) ⇒ (i) : If $Soc(M) \cap I = 0$ for some non-zero submodule I of M, then $I \cong (I + Soc(M))/Soc(M) \subseteq M/Soc(M)$ which implies that I is semisimple and hence $I \subseteq Soc(M)$, a contradiction. Thus $Soc(M)$ is essential in M. Now, if $Z(M)$ is non-zero, then $Z(M) \cap Soc(M) \neq 0$, a contradiction with the projectivity of $Soc(M)$. Thus $Z(M) = 0$. Now if N is any essential submodule of M then $Soc(M) \subseteq N$ and hence M/N is semisimple, and we can apply Proposition 2.6.

In case we restrict our attention to locally projective modules M such that $M/J(M)$ is Artinian, we obtain the following.

Proposition 2.9. (cf. [10, Proposition 3.5]). *If M is a locally projective module such that $M/J(M)$ is Artinian, then the following are equivalent :*

(i) *M is a GV-module.*

(ii) *M is an SI-module.*

Proof. (i) ⇒ (ii) : Since M is a GV-module, by [7, Theorem 3.15] it follows that $Z(M) \cap J(M) = 0$, and hence $Z(M) \cong (Z(M) \oplus J(M))/J(M)$ is a semisimple module being isomorphic to a submodule of the semisimple module $M/J(M)$. This means that $Z(M) \subseteq Soc(M)$. But since M is a locally projective GV-module, by [10, Proposition 3.4] it follows that $Z(M) \cap Soc(M) = 0$, and hence $Z(M)$ must be zero.

Now let L be any singular R-module, N any essential submodule of M and $f: N \to L$ any non-zero R-homomorphism. If $K = Ker(f)$ then we can easily see that K is essential in M and hence $J(M) \subseteq Soc(M) \subseteq K$. Whence N/K is a direct summand of M/K and the map f can be extended to a map $g: M \to L$. Therefore M is an SI-module.

(ii) ⇒ (i) : Obvious.

It was proved in [5, Theorem 1 and Theorem 5] that for a commutative ring R the following conditions are equivalent :

(i) R is an SI-ring.

(ii) R is a regular ring and $R/Soc(R)$ is semisimple.

In [6, Theorem 3.9] K. R. Goodearl has proved that the above conditions are equivalent to saying that :

(iii) R is a finite direct sum of non-singular rings which have at most two essential ideals.

In our next proposition we shall extend these results to modules. Before doing so, we need the following two results, the first of which can be found in [6, Proposition 1.22] while the second is an extension of [6, Proposition 3.6] to modules.

Lemma 2.10. *A module M is finitely generated semisimple if and only if M is finite dimensional and every cyclic submodule of M is a direct summand.*

Lemma 2.11. *If M is a finitely generated SI-module then $M/Soc(M)$ is Noetherian.*

Proof. Although the proof is similar to that of [6, Proposition 3.6] it will be presented here due to its importance in the proof of the next proposition. We will show that every submodule of $M/Soc(M)$ is finitely generated. Let $J = Soc(M)$ and I be a submodule of M with $I \supseteq J$. Let K be a submodule of I maximal with respect to $K \cap J = 0$. Then $J \oplus K$ is essential in I and $I/(J \oplus K)$ is a singular module. Since $M/(J \oplus K)$ is an SI-module we see that $I/(J \oplus K)$ is a direct summand of $M/(J \oplus K)$. Thus $I/(J \oplus K)$ is finitely generated. Our aim is to show that I/J is finitely generated. From the exactness of $0 \to K \to I/J \to I/(J \oplus K) \to 0$ we see that it suffices to prove that K is finitely generated. We first show that K is finite dimensional. If not, then there exists an infinite direct sum $K_1 \oplus K_2 \oplus \cdots$ of non-zero submodules of K. Since $K \cap J = 0$, none of the K_i are semisimple ; whence each K_i has a proper essential submodule H_i. Inasmuch as $(\bigoplus_{i=1}^{\infty} K_i)/(\bigoplus_{i=1}^{\infty} H_i) \cong \bigoplus_{i=1}^{\infty}(K_i/H_i)$ is a singular module and hence is $M/(\bigoplus_{i=1}^{\infty} H_i)$-injective, it follows that $(\bigoplus_{i=1}^{\infty} K_i)/(\bigoplus_{i=1}^{\infty} H_i)$ is a direct summand of $M/(\bigoplus_{i=1}^{\infty} H_i)$ and so is finitely generated, which contradicts the fact that it is an infinite direct sum of non-zero modules. By the finite dimensionality of K, let $\{E_i\}_{i=1}^{n}$ be a maximal family of non-zero cyclic submodules of K such that the sum $\sum_{i=1}^{n} E_i$ is direct. Clearly $E = \bigoplus_{i=1}^{n} E_i$ is essential in K, and hence K/E is singular. Inasmuch as M/E is an SI-module it follows that K/E is a direct summand of M/E and thus is finitely generated. Whence K is finitely generated.

Corollary 2.12. *If M is a finitely generated regular module then the following statements are equivalent :*

(i) *M is an SI-module.*

(ii) *$M/Soc(M)$ is semisimple.*

Proof. (i) ⇒ (ii) : Note first $M/Soc(M)$ is Noetherian, by Lemma

2.11. We claim that $Soc(M)$ is essential in M, for if $I \cap Soc(M) = 0$ for some non-zero submodule I of M it follows that $I \cong I \oplus Soc(M)/Soc(M) \subseteq M/Soc(M)$, which implies that I is a Noetherian module. And since submodules of regular modules are again regular we conclude that I is semisimple by Lemma 2.10. Whence $I \subseteq Soc(M)$, a clear contradiction. Now by Proposition 2.3(ii) it follows that $M/Soc(M)$ is semisimple.

(ii) ⇒ (i): Since M is a regular module, it follows that every simple submodule is a direct summand and hence projective. Hence $Soc(M)$ is projective. Since $M/Soc(M)$ is semisimple, $Soc(M)$ is essential in M. Inasmuch as M is regular, and hence locally projective, it follows from Proposition 2.8 that M is an SI-module.

Following M. S. Shrikhande [8], a module M is called hereditary (resp. semihereditary) if every submodule (resp. finitely generated submodule) of M is projective.

Proposition 2.13. *If R is a commutative ring and $_RM$ is a finitely generated projective R-module. Then the following conditions are equivalent:*

(i) *M is an SI-module.*

(ii) *M is a regular module and $M/Soc(M)$ is semisimple.*

(iii) *M is a semihereditary module and $M/Soc(M)$ is semisimple.*

(iv) *M is a non-singular module and $M/Soc(M)$ is semisimple.*

(v) *M is a finite direct sum of regular modules each of which has at most two essential submodules.*

(vi) *M is a finite direct sum of non-singular modules each of which has at most two essential submodules.*

Proof. (i) ⇒ (ii): Since every SI-module is a GV-module it follows from Proposition 1.1 that M is a regular module, and hence $M/Soc(M)$ is semisimple by Corollary 2.12.

(ii) ⇒ (iii): Clear since every regular module is semihereditary.

(iii) ⇒ (iv): Clear since every semihereditary module is non-singular.

(v) ⇒ (vi): Obvious since every regular module is non-singular.

(vi) ⇒ (i): Let $M = M_1 \oplus \cdots \oplus M_n$, where M_i is non-singular and has at most two essential submodules. By Proposition 2.2(ii), it is enough to show that each M_i is an SI-module. But if I is any essential submodule of M_i then M_i/I is either zero or simple, and by Proposition 2.4 it follows that each M_i is an SI-module.

(iv) ⇒ (i): Let L be any non-zero singular R-module, N any essential submodule of M and $f: N \to L$ any non-zero R-homomorphism. Let $K = Ker(f)$. Since M is non-singular, it is not difficult to see that K is essential in M, and so $Soc(M) \subseteq K$. Now since $M/K \cong (M/Soc(M))/(K/Soc(M))$ is a semisimple module, we see that N/K is a direct summand of M/K and the map f can be extended to a map $g: M \to L$. Whence every singular R-module is M-injective, and so M is an SI-module.

(ii) $\Rightarrow$ (v) : Since $M/Soc(M)$ is a finite direct sum of simple modules, it has a composition series. We shall prove our assertion by induction on the composition length of $M/Soc(M)$. If $l((M)/Soc(M)) = 0$ then $M = Soc(M)$ and M is a finite direct sum of simple projective modules. Assuming that $l(M/Soc(M)) > 0$, then $M/Soc(M)$ has a non-zero simple submodule $I/Soc(M)$. Let $K = Soc(M)$ and choose some $x \in I$ with $x \notin K$. Thus $Rx/(K \cap Rx) \neq 0$. Hence $Rx/(K \cap Rx) \cong I/K$. Because I/K is simple, it follows that $Soc(Rx) = K \cap Rx$ is a maximal submodule of Rx. Inasmuch as M is a regular module we see that Rx is a projective summand of M. Write $M = Rx \oplus N$. Since $Soc(Rx)$ is an intersection of essential submodules of Rx and $Soc(Rx)$ is a maximal submodule of Rx, it follows that Rx has only two essential submodules, namely Rx and $Soc(Rx)$. Since $M/K = Rx \oplus N/Soc(Rx \oplus N) \cong Rx/Soc(Rx) \oplus N/Soc(N)$, we have $l(N/Soc(N)) = l(M/K)-1$, and hence may use an inductive hypothesis on the module N.

Remark 2.14. The above proposition remains valid if we replace "regular module" by "λ-module", where λ stands for one of the symbols V, GV, P-V, P-V' or P-SI. See Proposition 1.1 and the next proposition.

If ${}_RM$ is regular module then every left R-module is P-M-injective ([7, Proposition 3.8]), and hence every regular module is a P-SI-module. By Proposition 1.1, since every P-SI-module is a P-V'-module, we can easily see the following :

Proposition 2.15. *If R is a commutative ring and M is a projective R-module then the following are equivalent :*

(i) *M is a regular module.*

(ii) *M is a P-SI-module.*

In particular if R is a commutative ring then R is a regular ring if and only if R is a P-SI-ring.

3. *P-SI*-rings. A module ${}_RM$ is said to be *P-injective* if for any principal left ideal I of R and $f \in Hom_R(I, M)$ there exists an element $m \in M$ such that $f(x) = xm$, for all $x \in I$. Equivalently M is P-injective if $Ext^1_R(R/Rx, M) = 0$ for each $x \in R$. It was proved in [11] that a ring R is regular if and only if every R-module is P-injective. A ring R is defined to be a left *P-V-ring* if every *simple* left R-module is P-injective. P-V-rings were introduced and studied by Yue Chi Ming [11] and H. Tominaga [9]. We defined a ring R to be a left P-SI-ring if every *singular* left R-module is P-injective (Definition 2.1). In this section we establish the following characterization :

Proposition 3.1. *For a ring R with essential left socle, the following statements are equivalent* :

(i) *R is a left P-SI-ring.*

(ii) $Soc({}_RR)$ *is projective and* $R/Soc({}_RR)$ *is a regular ring.*

We postpone the proof until some of the ideas involved have been sufficiently developed below. Let $\mathfrak{k}$ be a two-sided ideal of R. G. Azumaya has proved in [2, Proposition 10(ii)] that, every injective right $R/\mathfrak{k}$-module is injective as a right R-module if and only if $R/\mathfrak{k}$ is flat as a left R-module. For P-injective modules we have the following :

Proposition 3.2. *Let* $\mathfrak{k}$ *be a two sided ideal of R. Then every P-injective right* $R/\mathfrak{k}$*-module is P-injective as a right R-module if and only if* $R/\mathfrak{k}$ *is flat as a left R-module.*

Proof. "Only if" part : Adopted from [2, Proposition 10]. Let $a \in \mathfrak{k}$ and consider the right R-modules aR, $a\mathfrak{k}$ and $aR/a\mathfrak{k}$. Let ϕ: $aR \to aR/a\mathfrak{k}$ be the canonical mapping. $aR/a\mathfrak{k}$ is annihilated by $\mathfrak{k}$, and so can be regarded as a right $R/\mathfrak{k}$-module. Let $Q = E(aR/a\mathfrak{k})$ be the injective hull of the right $R/\mathfrak{k}$-module $aR/a\mathfrak{k}$. Then Q is P-injective as a right $R/\mathfrak{k}$-module, whence P-injective as a right R-module, by assumption. Now the map ϕ: $aR \to Q$ can be regarded as a map of R-modules. Therefore ϕ can be extended to an R-homomorphism $\bar{\phi}$: $R \to Q$. Let $\bar{\phi}(1) = y$, $y \in Q$. Then $\phi(x) = yx$, $\forall x \in aR$. But $aR \subseteq \mathfrak{k}$, and Q is annihilated by $\mathfrak{k}$, so $yx = 0$ $\forall x \in aR$. Thus $\phi = 0$, and $aR = a\mathfrak{k}$. Since a were arbitrarily chosen from $\mathfrak{k}$, $a \in a\mathfrak{k}$ $\forall a \in \mathfrak{k}$ and it follows from [2, Proposition 5] that ${}_R(R/\mathfrak{k})$ is flat.

"If" part : Suppose ${}_R(R/\mathfrak{k})$ is flat as a left R-module. And let Q be a P-injective right $R/\mathfrak{k}$-module. We want to show that $Ext^1_R(R/xR, Q) = 0$ for every $x \in R$. So let x be any element of R and consider the following exact sequence of right R-modules $0 \to xR \to R \to R/xR \to 0$. Since ${}_R(R/\mathfrak{k})$ is flat it follows that $(R/\mathfrak{k})/(\mathfrak{k}+xR/\mathfrak{k}) \cong (R/xR) \otimes_R (R/\mathfrak{k})$ and that $Ext^1_R(R/xR, Q) \cong Ext^1_{R/\mathfrak{k}}(R/xR \otimes_R R/\mathfrak{k}, Q)$. Whence $Ext^1_R(R/xR, Q) \cong Ext^1_{R/\mathfrak{k}}(R/(\mathfrak{k}+xR), Q)$. Now since Q is P-injective as a right $R/\mathfrak{k}$-module and $((\mathfrak{k}+xR)/\mathfrak{k})$ is a principal right ideal of $R/\mathfrak{k}$ we get $Ext^1_R(R/xR, Q) = 0$ for every $x \in R$ and Q is P-injective as a right R-module.

With the same argument used in the "if" part of the above proof one can also verify the following.

Proposition 3.3. *Let $\mathfrak{k}$ be a two-sided ideal of R, $R/\mathfrak{k}$ flat as a left R-module and Q a right $R/\mathfrak{k}$-module. If Q is P-injective as a right R-module then it is also P-injective as a right $R/\mathfrak{k}$-module.*

We shall also make use of the following result, which was proved in [3, Proposition 1.4 and Corollary 1.11].

Proposition 3.4. *For every ring R one has $Soc_{\mathfrak{P}}({}_RR) = (Soc_RR)^2$, where $Soc_{\mathfrak{P}}({}_RR)$ denotes the projective homogenous component of the left socle of R. Moreover, if $\mathfrak{k}$ is a two-sided ideal contained in $Soc({}_RR)$, then the following conditions are equivalent :*

(i) $\mathfrak{k}^2 = \mathfrak{k}$.

(ii) $(R/\mathfrak{k})_R$ *is flat.*

Proposition 3.5. *Let M be a left R-module. If $Soc(M)$ is projective and $M/Soc(M)$ is a regular module then M is a P-SI-module.*

Proof. Let N be a cyclic submodule of M, L a singular R-module and $f: N \to L$ a non-zero homomorphism. We want to show that f can be extended to a map $g: M \to L$. Let $K = Ker(f)$. If $K \cap I = 0$ for some non-zero submodule I of N, then $f: I \to L$ is a monomorphism and I is a non-zero singular submodule of M. Thus $I \cap Soc(M) = 0$, and hence $I \cong (I+Soc(M)/Soc(M)) \subseteq M/Soc(M)$, which implies that I is a regular submodule of M. But since every regular module is non-singular, it follows that $Z(I) = 0$, a clear contradiction with the singularity of I. Thus K is essential in N, and hence $Soc(N) \subseteq K$.

Now, define $\phi: N/Soc(N) \to (N+Soc(M))/Soc(M)$, by $\phi(n+Soc(N)) = n+Soc(M)$. Then ϕ is an isomorphism. Let $- : M \to M/Soc(M)$ denotes the canonical quotient map, and write $\bar{M} = M/Soc(M)$. Since $\bar{M}$ is a regular module and $\bar{N}$ is a cyclic submodule of $\bar{M}$, we can write $\bar{M} = \bar{N} \oplus \bar{T}$, for some submodule $\bar{T}$ of $\bar{M}$. Since $Soc(N) \subseteq Ker(f)$, there is a map $\tilde{f}: N/Soc(N) \to L$, such that $\tilde{f}(n+Soc(N)) = f(n)$. Thus $\tilde{f} \circ \phi^{-1} : \bar{N} \to L$. Extend $(\tilde{f} \circ \phi^{-1})$ to a map $\tilde{g} : \bar{M} = \bar{N} \oplus \bar{T} \to L$ in the obvious way. Define $g : M \to L$, by $g(m) = \tilde{g}(\bar{m})$, $\forall m \in M$. Now, if $x \in N$ then : $g(x) = \tilde{g}(x) = \tilde{g}(x+Soc(M)) = (\tilde{f} \circ \phi^{-1})(x+Soc(M)) = \tilde{f}(\phi^{-1}(x+Soc(M)) = \tilde{f}(x+Soc(N)) = f(x)$. Thus the map g is the required map.

We can now prove Proposition 3.1 :

(i) ⇨ (ii) : By Proposition 2.5, since R is a left P-SI-ring, R is left non-singular and so $Soc({}_RR)$ is projective. Now, in order to show that $R/(Soc_RR)$ is a regular ring we must prove that every left $R/(Soc_RR)$-module is P-injective. So let M be a left $R/(Soc_RR)$-module. Since $Soc({}_RR)$ is essential in ${}_RR$ it follows that M is a singular left R-module, whence M is P-injective as a left R-module. Inasmuch as $Soc({}_RR)$ is projective, it follows from Proposition 3.4 that $(R/Soc_RR)_R$ is flat as a right R-module. And by Proposition 3.3 we infer that M is P-injective as a left $R/(Soc_RR)$-module.

(ii) ⇨ (i) : By Proposition 3.5.

It was proved in [6, Proposition 3.5] that if $R/J(R)$ is semisimple, then the following statements are equivalent :

(i) $Z_r(R) = 0$ and R is a right SI-ring.

(ii) $Z_r(R) = 0$ and $[J(R)]^2 = 0$.

(iii) $Z_l(R) = 0$ and $[J(R)]^2 = 0$.

(iv) $Z_l(R) = 0$ and R is a left SI-ring.

However in view of our Proposition 2.5, R is a right SI-ring $\Rightarrow Z_r(R) = 0$ (similarly R a left SI-ring $\Rightarrow Z_l(R) = 0$). Thus in (i) we can remove the condition $Z_r(R) = 0$ (similarly in (iv) we can remove the condition $Z_l(R) = 0$).

In the next proposition we shall prove also that, under the same hypothesis, a ring R is a right P-SI-ring if and only if R is a left P-SI-ring. But first we need the following lemma.

Lemma 3.6. *Suppose that ${}_RM$ is a P-V'-module. Then $J(M) \subseteq Soc(M)$ and $J(M) \cap Z(M) = 0$. In particular if R is left P-V'-ring then $J(R)$ is a direct sum of minimal projective left ideals.*

Proof. Suppose to the contrary there exists an element $x \in M$ such that $x \in J(M)$ and $x \notin Soc(M)$. Since $Soc(M)$ is the intersection of all the essential submodules of M, it follows that $x \notin T$ for some proper essential submodule T of M. By Zorn's lemma, the set of all essential submodules I of M such that $x \notin I$ has a maximal member L. Let $\pi: M \to M/L$ denote the quotient map and $\bar{x} = \pi(x) = x+L$. Writing $\bar{M}$ for the factor module M/L, we see that $0 \neq \bar{x} \in \bar{M}$ and any non-zero submodule of $\bar{M}$ must contain $\bar{x}$. It follows that $R\bar{x}$ is a simple singular submodule of $\bar{M}$. Let η denote the restriction of the map π to the submodule Rx. Clearly $\eta: Rx \to R\bar{x}$ is onto. Since M is a left P-V'-module, η can be extended to a map $\bar{\eta}: M \to Rx$. Clearly $\bar{\eta}$ is onto. If $N = Ker(\bar{\eta})$ then $M/N \cong R\bar{x}$ and N is a maximal submodule of M with $x \notin N$, a contradiction with the fact that $x \in J(M)$.

A similar argument shows that $J(M) \cap Z(M) = 0$.

Proposition 3.7. *If $R/J(R)$ is semisimple then the following conditions are equivalent :*

(i) *R is a right SI-ring.*
(ii) *R is a left SI-ring.*
(iii) $Z_r(R) = 0$ *and* $[J(R)]^2 = 0$.
(iv) $Z_l(R) = 0$ *and* $[J(R)]^2 = 0$.
(v) *R is a right P-SI-ring.*
(vi) *R is a left P-SI-ring.*
(vii) *R is a right GV-ring.*
(viii) *R is a left GV-ring.*
(ix) *R is a right P-V'-ring.*
(x) *R is a left P-V'-ring.*
(xi) *R is right semihereditary and* $[J(R)]^2 = 0$.
(xii) *R is left semihereditary and* $[J(R)]^2 = 0$.
(xiii) *R is right hereditary and* $[J(R)]^2 = 0$.
(xiv) *R is left hereditary and* $[J(R)]^2 = 0$.

Proof. (v) ⇒ (ix) : Clear.

(ix) ⇒ (iii) : Inasmuch as R is a right P-V'-ring, $[J(R)]^2 = 0$ and $J(R) \cap Z_r(R) = 0$, by Lemma 3.6. Hence $Z_r(R) \cong (J(R) \oplus Z_r(R))/J(R) \subseteq R/J(R)$. Whence $Z_r(R)$ is semisimple right R-module and so $Z_r(R) \subseteq Soc(R_R)$. But since R is a right P-V'-ring, it follows that every minimal right ideal of R must be projective. Therefore $Z_r(R) = 0$.

(iii) ⇒ (i) : By [6, Proposition 3.5].

(i) ⇒ (v) : Obvious.

(x) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (ii): By symmetry.

(i) $\Leftrightarrow$ (ii): By [6, Proposition 3.5].

(i) $\Leftrightarrow$ (vii): By Proposition 2.9.

(xiii) $\Rightarrow$ (xi): Clear.

(xi) $\Rightarrow$ (iii): If x is any non-zero element of R then the sequence $0 \to Ann_R(x) \to R \to xR \to 0$ splits, where $Ann_R(x)$ denotes the right annihilator of x in R. Whence $Z_r(R) = 0$.

(i) $\Rightarrow$ (xiii): By [6, Proposition 3.3].

(xiv) $\Leftrightarrow$ (xii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (ii): By symmetry.

Finally we conclude this section with the following.

Proposition 3.8. *For a left self-injective ring R, the following conditions are equivalent:*

(i) *R is a left P-SI-ring.*

(ii) *R is a regular ring.*

Proof. (i) $\Rightarrow$ (ii): By Proposition 2.5, since R is a left P-SI-ring it follows that R is left non-singular. And since R is left self-injective, $J(R) = 0$ and R is a regular ring.

(ii) $\Rightarrow$ (i): Since R is a regular ring, every R-module is P-injective, in particular every singular left R-module is P-injective, and hence R is a left P-SI-ring.

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