Math. J. Okayama Univ. 28 (1986), 105−108

ON COMMUTATIVITY CONDITIONS FOR RINGS

WALTER STREB

Let A be a non-empty subset of a ring R. Many authors studied the commutativity behavior of R under the conditions :

(i) For each $x \in R$, there exists a polynomial $f(\lambda) \in \mathbf{Z}[\lambda]$ such that $x - x^2 f(x) \in A$.

(ii) If $x, y \in R$ and $x - y \in A$, then $x^2 = y^2$ or $x, y \in C(A)$, the centralizer of A in R.

Cherubini and Varisco [1] and Tominaga [3] have proved the following proposition :

(*) *If A is commutative and the conditions* (i) *and* (ii) *are satisfied, then R is commutative.*

In this paper, we shall prove a generalization of (*) together with its variations.

In what follows, R will represent a ring with center $Z = Z(R)$. For $X, Y \subseteq R$, $[X, Y] = \{[x, y] = xy - yx \mid x \in X, y \in Y\}$, $X \circ Y = \{x \circ y = xy + yx \mid x \in X, y \in Y\}$.

We define the *Engel center* $EZ = EZ(R)$ as the set of all $x \in R$ with the property that for each $y \in R$ there exists a positive integer n, positive integers m_i not divisible by 3, and operations $*_i \in \{\circ, [\ \ ]\}$, $1 \leq i \leq n$, such that $(\cdots((x *_1 y^{m_1}) *_2 y^{m_2}) \cdots) *_n y^{m_n} = 0$.

R is called *weakly semiprime* if for each ideal I of R with $I^2 = 0$ there holds $I \subseteq EZ$.

For $X \subseteq R$, $l(X) = \{y \in R \mid yX = 0\}$, $r(X) = \{y \in R \mid Xy = 0\}$, $C(X) = \{y \in R \mid [y, X] = 0\}$, and $C^\circ(X) = \{y \in R \mid y \circ X = 0\}$. Further, $Z^\circ = C^\circ(R)$. In case $l(R) = 0$(or $r(R) = 0$), we see that $Z^\circ \subseteq Z$. Actually, if $b \in Z^\circ$ then, for each $x, y \in R$, $bxy = -xyb = xby$, and so $[b, x]R = 0$. Hence $b \in Z$.

Let $\mathbf{Z}[\lambda, \mu]$ (resp. $\mathbf{Z}\{\lambda, \mu\}$) be the polynomial ring with integer coefficient in the commuting (resp. non-commuting) indeterminates λ and μ.

In case A is commutative, (i) implies the following condition :

(j) For each $x, y \in R$, there exists a polynomial $f(\lambda, \mu)$ in the kernel of the canonical homomorphism $\mathbf{Z}\{\lambda, \mu\} \rightarrow \mathbf{Z}[\lambda, \mu]$ each of whose monomial is of length ≥ 3 such that $[x, y] = f(x, y)$.

Obviously, the condition (ii) can be restated as follows :

(ii) For each $x \in R$ and $a \in A$, either $a \circ x = a^2$ or $x, a \in C(A)$.

Now, we generalize (ii) as follows :

(jj) Given $a \in A$, there exists $c_a \in R$ such that $a \circ x = c_a$ for all $x \in R \backslash C(a)$.

An immediate consequence from (ii), we have

(iii) $a^2 \in Z$.

Actually, if $[a, x] = 0$ then $[a^2, x] = 0$; while, if $a \circ x = a^2$ then $[a^2, x] = [a \circ x, x] = [a, x^2] = 0$ by [3, Lemma 1, p. 729].

In view of (iii), we see that the following strengthening of (jj) is still a generalization of (ii).

(jjj) Given $a \in A$, there exists $c_a \in R$ such that $\{a, c_a\} \cap EZ \neq \emptyset$ and $a \circ x = c_a$ for all $x \in R \backslash C(a)$.

In what follows, we shall prove the following generalization of (*) :

(**) *If* (i), (j) *and* (jjj) *are satisfied, then R is commutative.*

In view of [2], (**) is a direct consequence of the next

Theorem 1. *If* (j) *and* (jjj) *are satisfied, then* $A \subseteq Z$.

We shall prove also the following at the same time.

Theorem 2. *Let R be a* 2*-torsion free ring satisfying* (jj). *If* $l(R) = 0$ (*or* $r(R) = 0$), *then* $A \subseteq Z$.

Theorem 3. *If R is a* 2*-torsion free ring satisfying* (j) *and* (jj), *then* $A \subseteq Z$.

Theorem 4. *If R is a weakly semiprime ring satisfying* (j) *and* (jj), *then* $A \subseteq Z$.

Theorem 5. *Let f be a polynomial with integer coefficients in non-commuting indeterminates such that each of the rings* $T_1 = \begin{pmatrix} GF(2) & GF(2) \\ 0 & 0 \end{pmatrix}$, $T_2 = \begin{pmatrix} 0 & GF(2) \\ 0 & GF(2) \end{pmatrix}$, $T_3 = \left\{ \begin{pmatrix} x & y \\ 0 & x^2 \end{pmatrix} \mid x, y \in GF(4) \right\}$ *fails to satisfy* $f = 0$. *Let R be a ring satisfying* $f = 0$. *If* (j) *and* (jj) *are satisfied, then* $A \subseteq Z$.

Proof of theorems. We may assume that A is a singleton $\{a\}$. Suppose, to the contrary, that $a \notin Z$.

First, suppose that $c_a = 0$. Then $R = C(a) \cup C^{\circ}(a)$, and so $R = C(a)$ or $C^{\circ}(a)$. Hence $R = C^{\circ}(a)$. For any $x, y \in R$, we have $[a, x]y = 0$. Therefore $[a, R]R = 0$, and similarly $R[a, R] = 0$, which contradicts (j) and the hypothesis of Theorem 2. Hence, we have seen that $c_a \neq 0$. For any $x, y \in R \backslash C(a)$, $a \circ (x-y) = 0$, and hence $x-y \in C(a)$ by (jj) or (jjj). This shows that the order of the additive group $[a, R]$ is 2. This contradicts the hypothesis of Theorems 2 and 3. The rest of the proof will be immediate by the next

Lemma. *Let $d = [a, r] \neq 0$. Assume that $[a, R]$ coincides with $\{0, d\}$.*

(1) $(d)^2 = 0$.

(2) *If $d \in C(a, r)$ then there exist $x, y \in C(d)$ such that $d = [x, y]$ and $dx = dy^2 = 0$; in particular, R does not satisfy* (j).

(3) *If $d \notin C(a, r)$ then $\{a, d, a \circ r\} \cap EZ = \emptyset$ and there exists a homomorphic image T of a subring of R which is isomorphic to T_1, T_2 or T_3.*

Proof. (1) Since $[a, R]R \subseteq [a, R] + R[a, R]$, it suffices to show that $d^2 = 0$. If $R \backslash C(a)$ is multiplicatively closed, then it is easy to see that $C(a)$ is an ideal of R, and also $d \in C(a)$. Hence $d^2 = d[a, r] = [a, dr] = 0$. If there exist $x, y \in R \backslash C(a)$ such that $xy \in C(a)$ then $xd = x[a, y] = [a, xy] - [a, x]y = -dy$. Since $ad \in \{0, d\}$, we obtain $d^2 = axd - xad = -ady + ady = 0$.

(2) Since $2d = 0$, we have $[a^2, r] = a \circ [a, r] = [a, d] = 0$, and similarly $[a, r^2] = 0$. We put $x = a^2 + a$ and $y = r^2 + r$. Then $[x, y] = d$ and $dx = 0$, since $da \in \{0, d\}$. On the other hand, since $[a, r^2] = [a, 2r] = 0$, we have $y^2 \in C(a)$. If $r^3 \in C(a)$ then $r^5 + r^3 = r^2 r^3 + r^3 \in C(a)$. If $r^3 \notin C(a)$ then $r^5 + r^3 = r^2(r^3 + r) \in C(a)$. Therefore, we can see that $y^3 \in C(a)$, and hence $dy^2 = [a, y^3] = 0$.

(3) We put $I = \{x \in Z(\langle a, r \rangle) \mid xd = 0\}$. If $x \in I$, then $[xa, r] = [a, xr] = xd = 0$, and also $xa, xr \in I$. Hence I is an ideal of $\langle a, r \rangle$. Put $T = \langle a, r \rangle / I$, and denote the residue class of $x \in \langle a, r \rangle$ by $\bar{x}$. If $[\bar{a}, \bar{x}] = 0$ then $[a, x] \in I$, and hence $[a, x] = 0$. Since $2a, 2r \in I$, we get $2T = 0$. Now suppose $x \in Z(\langle a, r \rangle) \backslash I$. Then $0 \neq xd = [xa, r] = [a, xr] = d$, and hence $xa - a, xr - r \in I$. This shows that $\bar{x}$ is an identity element of T.

First, we consider the case that $d \notin C(a)$. Then $[a^2, r] = [a, d] = [a, r]$, and hence $a^2 - a \in I$. Since $ad, da \in \{0, d\}$ and $[a, d] \neq 0$, $\langle \bar{a}, \bar{d} \rangle$ is isomorphic to T_1 or T_2. Furthermore, since $EZ(T_1) = EZ(T_2) = 0$, we get $\{\bar{a}, \bar{d} = \overline{a \circ r}\} \cap EZ(\langle \bar{a}, \bar{d} \rangle) = \emptyset$, and also $\{a, d, a \circ r\} \cap EZ = \emptyset$.

Next, we consider the case that $d \notin C(r)$. Then $[a, r^2] = [d, r] \neq 0$ implies $x = r^2 - r \in Z(\langle a, r \rangle)$. As was shown above, $\bar{x} = 0$ or an identity element of T. In either case, $[t, \bar{r}^n] = [t, \bar{r}]$ for any $t \in T$ and any natural number n not divisible by 3. Since $[d, r] = [[a, r], r] = [a, r^2] = [a, r] = d$, we obtain $\{\bar{a}, \bar{d} = \overline{a \circ r}\} \cap EZ(T) = \emptyset$, and so $\{a, d, a \circ r\} \cap EZ = \emptyset$. Furthermore, by making use of $[d, r] = d$, we can determine the structure of the ring $\langle \bar{r}, \bar{d} \rangle$. Actually, if $\bar{r}^2 - \bar{r} = 1$ then $\bar{r}\bar{d} = \bar{d}\bar{r} + \bar{d} = \bar{d}\bar{r}^2$, and hence it is easy to see that $\langle \bar{r}, \bar{d} \rangle \simeq T_3$. Suppose now that $\bar{r}^2 = \bar{r}$. Then $\boldsymbol{Z}\bar{d}\bar{r}$ and $\boldsymbol{Z}\bar{r}\bar{d}$ are ideals of $\langle \bar{r}, \bar{d} \rangle$ with $\boldsymbol{Z}\bar{d}\bar{r} \cap \boldsymbol{Z}\bar{r}\bar{d} = 0$. Hence, if $\bar{r}\bar{d} \neq 0$ (resp. $\bar{d}\bar{r} \neq 0$) then $\langle \bar{r}, \bar{d} \rangle / \boldsymbol{Z}\bar{d}\bar{r} \simeq T_1$ (resp. $\langle \bar{r}, \bar{d} \rangle / \boldsymbol{Z}\bar{r}\bar{d} \simeq T_2$).

References

[1] A. Cherubini and A. Varisco : A further generalization of a theorem of Outcalt and Yaqub, Math. Japon. 28 (1983), 701–702.

[2] I. N. Herstein : The structure of a certain class of rings, Amer. J. Math. 75 (1953), 864–871.

[3] H. Tominaga : Note on two commutativity properties for rings, Math. Japon. 28 (1983), 729–732.

Universität-GHS-Essen, Fachbereich 6
Universitätsstrasse 3, 4300 Essen 1
Bundesrepublik Deutschland

(Received October 17, 1985)