Math. J. Okayama Univ. 28 (1986), 101−103

ON PERIODIC RINGS AND RELATED RINGS

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Throughout, R will represent a ring. Let E be the set of idempotents in R, N the set of nilpotent elements in R, and N^* the subset of N consisting of all x with $x^2 = 0$. R is called *normal* if E is central. An element x in R is called a *right* (resp. *left*) *p. p. element* if there exists an $e \in E$ such that $xe = x$ and $r(x) = r(e)$ (resp. $ex = x$ and $l(x) = l(e)$), where $r(*)$ (resp. $l(*)$) denotes the right (resp. left) annihilator of $*$ in R. Obviously, every (von Neumann) regular element is a right and left p. p. element. We denote by P_0 the set of right p. p. elements in R. Also, we denote by S the set of strongly regular elements in R, and by P the set of potent elements in R. A ring R is called a *generalized right p. p. ring* if for each $x \in R$ there exists a positive integer n such that $x^n \in P_0$. Needless to say, every periodic ring is a strongly π-regular ring, and every π-regular ring is a generalized right p. p. ring.

Recently, in [2] and [3], the following has been proved: (1) If R is a generalized right p. p. ring and each $x \in R$ has at most one expression of the form $x = u + a$, where $u \in P_0$ and $a \in N$, then $R = P_0 \oplus N$; strictly speaking, both P_0 and N are ideals of R and R is the direct sum of P_0 and N (and conversely). (2) If R is a π-regular ring and each $x \in R$ has at most one expression of the form $x = u + a$, where $u \in S$ and $a \in N$, then $R = S \oplus N$ (and conversely). (3) If R is a periodic ring and each $x \in R$ has at most one expression of the form $x = u + a$, where $u \in P$ and $a \in N$, then $R = P \oplus N$ (and conversely). More recently, M. Ôhori [5] has proved the following: (1)' A normal ring R is a generalized right p. p. ring if and only if each $x \in R$ has an expression $x = u + a$, where $u \in P_0$, $a \in N$ and $ua = au$. (2)' R is a strongly π-regular ring if and only if each $x \in R$ has an expression $x = u + a$, where $u \in S$, $a \in N$ and $ua = au$. (3)' R is a periodic ring if and only if each $x \in R$ has an expression $x = u + a$, where $u \in P$, $a \in N$ and $ua = au$.

In connection with the above results, we shall prove the following

Theorem 1. (1) *If each $x \in R$ is uniquely expressible as $x = u + a$,*

*Supported by the Natural Sciences and Engineering Research Council of Canada, Grant No. A 3961.

where $u \in P_0$ *and* $a \in N$, *then* $R = P_0 \oplus N$ *(and conversely).*

(2) *If each* $x \in R$ *is uniquely expressible as* $x = u + a$, *where* $u \in S$ *and* $a \in N$, *then* $R = S \oplus N$ *(and conversely).*

(3) *If each* $x \in R$ *is uniquely expressible as* $x = u + a$, *where* $u \in P$ *and* $a \in N$, *then* $R = P \oplus N$ *(and conversely).*

Proof. (1) In view of the uniqueness of the expression, we can easily see that R is normal. Let $e \in E$ and $a \in N^*$. Then $e + ea = (e + ea)^2 (e - ea)$ is strongly regular, and the uniqueness of its expression implies that $ea = 0$, namely $EN^* = 0$. Furthermore, we can easily see that $P_0 N^* = 0 = P_0 \cap N^*$. Now, we shall prove by induction that $P_0 N = NP_0 = 0$. To see this, it suffices to show that $EN = 0$. Let $e \in E$ and $a \in N$. Then, by the induction hypothesis, we have $(ea)^2 = ea^2 = 0$. Hence $ea = e \cdot ea \in EN^* = 0$. Thus, we have shown that $P_0 N = NP_0 = 0$. By making use of this fact and $P_0 \cap N^* = 0$, we can prove that N forms an ideal. Finally, let A be the ideal of R generated by E, and x an arbitrary element in $A \cap N$. We write $x = e_1 x_1 + \cdots + e_k x_k$ with some $e_i \in E$ and $x_i \in R$. As is well known, there exists a (central) idempotent f such that $fe_i = e_i$ $(1 \leq i \leq k)$. Then $x = fx \in EN = 0$, and hence $A \cap N = 0$. Since $P_0 \subseteq A$ and each element in R is expressible as $u + a$ with some $u \in P_0$ and $a \in N$, this proves that $P_0 = A$ and $R = P_0 \oplus N$.

(2) The proof is quite similar to that of (1).

(3) Observe first that if $x = u + a$, with $u \in P$, $a \in N$ and $au = ua$, there exists $n > 1$ such that $x^n - x \in N$ (see the proof of [5, Theorem 3]). Let e be an arbitrary (central) idempotent, and $a \in N^*$. Applying the above observation to $2e$, we get a positive integer k such that $ke = 0$. Hence $e + ea = (e + ea)^{k+1}$ is potent, and the uniqueness of its expression implies that $ea = 0$, namely $EN^* = 0$. Furthermore, we can easily see that $PN^* = 0$. Now, the rest of the proof proceeds in the same way as in the latter part of the proof of (1).

Next, as was noted in [4, Remark], if N is commutative and each element of R is expressible as the product of elements in $E \cup N$, then N forms an ideal. This can be generalized as follows :

Theorem 2. *If* N^* *is commutative and* N *is multiplicatively closed then* $PN \subseteq N$. *In particular, if* N *is commutative and* $P \cup N$ *generates* R *then* N *forms an ideal.*

Proof. First, we claim that $EN \subseteq N$. Let $e \in E$, and $a^{2^\alpha} = 0$. Since both $ea - eae$ and $ae - eae$ belong to N^*, $ea^2e - (eae)^2 = (ea - eae)(ae - eae) = e(ae - eae)(ea - eae) = 0$, i.e., $(eae)^2 = ea^2e$. Repeating this procedure, we see that $(eae)^{2^\alpha} = ea^{2^\alpha}e = 0$, and so $(ea)^{2^\alpha+1} = 0$. Hence $EN \subseteq N$.

Now, let $x \in P$. Then, by the above claim, there exists a positive integer n such that $x^nN \subseteq N$. Let a be an arbitrary element in N, and suppose that $n > 1$. It is easy to see that $x^iax^{n-i} \in N$ $(0 \leq i \leq n)$. If $n = 2m$ then $x^max^m \in N$, and hence $x^ma \in N$. Next, if $n = 2m+1$, then $b = xa(x^{2(m+1)}a)^{2m} = xax^{2m} \cdot x^2ax^{2m-1} \cdots\cdots x^{2m+1}a \in N$, and therefore $(x^{2(m+1)}a)^n = x^nb \in N$. Hence $x^{m+1}N \subseteq N$, as for the case $n = 2m$. We have thus seen that in either case there exists a positive integer $n' < n$ such that $x^{n'}N \subseteq N$; eventually $xN \subseteq N$.

Combining Theorem 2 with a theorem of Chacron (see, e.g., [1, Theorem 1]), we readily obtain.

Corollary. *If each element in R is expressible as the sum of a potent element and a nilpotent element and N is commutative, then R is periodic.*

References

[1] H. E. Bell: On commutativity of periodic rings and near-rings, Acta Math. Acad. Sci. Hung. **36** (1980), 293–302.

[2] H. E. Bell: On commutativity and structure of periodic rings, Math. J. Okayama Univ. **27** (1985), 1–3.

[3] Y. Hirano and H. Tominaga: Rings decomposed into direct sums of nil rings and certain reduced rings, Math. J. Okayama Univ. 27 (1985), 35–38.

[4] I. Mogami: On certain periodic rings, Math. J. Okayama Univ. **27** (1985), 5–6.

[5] M. Ôhori: On strongly π-regular rings and periodic rings, Math. J. Okayama Univ. **27** (1985), 49–52.

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(Received November 5, 1985)