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SOME CONDITIONS FOR COMMUTATIVITY OF RINGS

To Adil Yaqub on his 60th birthday

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Let A be a non-empty subset of the ring $R(\neq 0)$ with center C; let N denote the set of nilpotent elements in R, N^* the subset of N consisting of all x with $x^2 = 0$, and E the set of idempotents in R. Let $q > 1$ be a fixed integer. We consider the following conditions:

(I-A) For each $x \in R$, there exists a polynomial $f(t)$ in $\mathbf{Z}[t]$ such that $x - x^2 f(x) \in A$.

(I′-A) For each $x \in R$, either $x \in C$ or there exists a polynomial $f(t)$ in $\mathbf{Z}[t]$ such that $x - x^2 f(x) \in A$.

(II-A)$_q$ If $x, y \in R$ and $x - y \in A$, then either $x^q = y^q$ or x and y both belong to the centralizer $C_R(A)$ of A in R.

(III-A) For each $x \in R$ and $a \in A$, $[[a, x], x] = 0$.

(III′-A) For each $x \in R$ and $a \in A$, there exists a positive integer $m = m(x, a)$ such that $[a, x]_m = [[a, x]_{m-1}, x] = 0$.

(III″-A) For each $x \in R$ and $a \in A$, there exists a positive integer $n = n(x, a)$ such that $[[a, x^n], x^n] = 0$ and $[[a, x^{n+1}], x^{n+1}] = 0$.

Our objective is to prove the following theorem which is related to a number of recent results by H. Abu-Khuzam, A. Yaqub and the author (see, e. g., [1], [2], [5], and [6]).

Theorem 1. *The following statements are equivalent:*

1) *R is commutative.*

2) *There exists a commutative subset A for which R satisfies* (I-A), (II-A)$_q$ *and* (III-A).

3) *There exists a commutative subset A of N for which R satisfies* (I′-A) *and* (III-A).

4) *There exists a commutative subset A for which R satisfies* (I-A), (II-A)$_q$ *and* (III′-A).

5) *There exists a commutative subset A of N for which R satisfies* (I′-A) *and* (III′-A).

6) *R satisfies* (III′-N^*) *and there exists a commutative subset A for*

which R satisfies (I-A) *and* (II-A)$_q$.

7) *R satisfies* (III$'$-N^*) *and there exists a commutative subset A of N for which R satisfies* (I$'$-A).

8) *There exists a commutative subset A for which R satisfies* (I-A), (II-A)$_q$ *and* (III$''$-A).

9) *There exists a commutative subset A of N for which R satisfies* (I$'$-A) *and* (III$''$-A).

In preparation for proving Theorem 1, we state the next lemma.

Lemma 1. (1) *If R satisfies* (I-C), *then R is commutative.*

(2) *If R satisfies* (I$'$-A), *then $N \subseteq A^+ + C$ and $N^* \subseteq A \cup C$, where A^+ is the additive subsemigroup generated by A.*

(3) (III-A) *implies* (III$''$-A).

(4) *If R satisfies* (I$'$-A) *and* (II-A)$_q$, *then R is normal ; that is, E is central.*

(5) *If R satisfies* (I$'$-A) *and* (III$''$-A), *then R is normal.*

(6) *If A is commutative and R satisfies* (I$'$-A), *then N is a commutative nil ideal containing the commutator ideal of R and is contained in $C_R(A)$.*

(7) *If R satisfies* (III$'$-N^*) *and there exists a commutative subset A for which R satisfies* (I$'$-A), *then R satisfies* (III$'$-A).

(8) *Let R be a normal, subdirectly irreducible ring. If A is a commutative subset of N not contained in C for which R satisfies* (I$'$-A), *then R is of characteristic p^α, where p is a prime.*

Proof. (1) This is a well-known fact as a theorem of Herstein (see [3]).

(2) See [5, Lemma 1 (2)].

(3) Obviously, $[[a, x^2], x^2] = 0$ for all $x \in R$ and $a \in A$.

(4) See [5, Lemma 1 (4)].

(5) Let $e \in E$, and $a^* \in N^*$. By (2), $N^* \subseteq A \cup C$. This together with (III$''$-A) shows that $[[a^*, e], e] = 0$. Hence e is central by [4, Remark 2].

(6) See [5, Lemma 1 (5)].

(7) Let $x \in R$, and $a \in A$. Since $[R, R] \subseteq N$ and $[N, A] = 0$ by (6), we see that $[a, x]^2 = [a, x](ax - xa) = a[ax, x] - [ax, x]a = 0$. Thus, by (III$'$-$N^*$), there exists a positive integer m such that $0 = [[a, x], x]_m = [a, x]_{m+1}$.

(8) See [5, Lemma 2].

Proof of Theorem 1. Obviously, 1) implies 2)−9). By [6, Theorem 1], each of 4) and 5) implies 1). Furthermore, by Lemma 1 (3) and (7), 2), 3), 6) and 7) imply 8), 9), 4) and 5), respectively.

8) ⇨ 1). We may assume that R is subdirectly irreducible. According to Lemma 1 (1) and (I-A), it suffices to show that $A \subseteq C$. Suppose, to the contrary, that there exist $a \in A$ and $x \in R$ such that $[a, x] \neq 0$. By (I-A) and $(\text{II-}A)_q$, $x^q = (x^2 f(x))^q$ with some $f(t) \in Z[t]$. Since $x \notin N$ by Lemma 1 (2), Lemma 1 (4) shows that $e = (xf(x))^q$ is a non-zero central idempotent, and hence $e = 1$ and x is invertible. By (I-A), we can find a non-zero integer k such that $k = k \cdot 1 \in A$. Obviously, $[a, x+ik] \neq 0$ for all $i \in \boldsymbol{Z}$. Hence, by $(\text{II-}A)_q$, every $x+ik$ is a zero of the polynomial $(t+k)^q - t^q$. Note here that $\bar{R} = R/N$ is a subdirect sum of commutative integral domains (Lemma 1 (6)). Then, we can easily see that $q!\, k^q \in N$, and so $h \cdot 1 = 0$ for some positive integer h. This implies that R is of characteristic p^α, p a prime. Then we can easily see that $\langle x \rangle$ is a finite local ring ; hence $\bar{x} = x + N$ generates a finite subfield of $\bar{R}$: $\langle \bar{x} \rangle = GF(p^\beta)$. By (III$''$-$A$), there exists a positive integer n such that $[[a, x^n], x^n] = 0 = [[a, x^{n+1}], x^{n+1}]$. Since $(\bar{x}^n)^{p^{\alpha\beta}} = \bar{x}^n$ and $[A, N] = 0$ by Lemma 1 (6), we see that $[a, x^n] = [a, (x^n)^{p^{\alpha\beta}}] = p^{\alpha\beta}(x^n)^{p^{\alpha\beta}-1}[a, x^n] = 0$. Similarly, $0 = [a, x^{n+1}] = x^n[a, x] + [a, x^n]x = x^n[a, x]$, and hence $[a, x] = 0$. This is a contradiction.

9) ⇨ 1). We may assume that R is subdirectly irreducible. As above, suppose that there exist $a \in A$ and $x \in R$ such that $[a, x] \neq 0$. Since $x \notin N$ by Lemma 1 (2), there exists a non-zero idempotent e in $x\langle x \rangle$, by (I$'$-A). Since the subdirectly irreducible ring R is normal by Lemma 1 (5), we get $e = 1$, and therefore x is invertible. Furthermore, by Lemma 1 (8), R is of characteristic p^α (p a prime), and $\langle x \rangle$ is a finite local ring. Thus we can repeat the above argument to see that $[a, x] = 0$. This contradiction proves that R is commutative (Lemma 1 (1)).

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