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ON COMMUTATIVITY OF S-UNITAL RINGS

To the memory of Professor Akira Hattori

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Throughout, $R(\neq 0)$ will represent a ring with Jacobson radical J, and N the set of nilpotent elements in R. A ring is called s-unital if $x \in Rx \cap xR$ for any $x \in R$. As is well known, if R is an s-unital ring, then for any finite subset F of R there exists a pseudo-identity e of F ($ex = xe = x$ for all $x \in F$) which is not quasi-regular. (Choose a non-zero element y in R, and consider a pseudo-identity of $F \cup \{y\}$.)

Now, let n be a positive integer, and A a subset of R. We consider the following conditions :

$P_0(n, A)$ $(xy)^n = x^n y^n$ for all $x, y \in A$.

$P_3(n, A)$ $(xy)^n = (yx)^n$ for all $x, y \in A$.

$P_3'(A)$ For any $x, y \in A$, there exist positive integers $l = l(x, y)$ and $m = m(x, y)$ with $(l, m) = 1$ such that $(xy)^l = (yx)^l$ and $(xy)^m = (yx)^m$.

$Q(n)$ For any $x, y \in R$, $n[x, y] = 0$ implies $[x, y] = 0$.

Specifically, we write $P_0(n) = P_0(n, R)$, $P_3(n) = P_3(n, R)$, and $P_3' = P_3'(R)$.

Our present objective is to generalize the recent results of Bell and Yaqub [2] as follows :

Theorem 1. *Let R be an s-unital ring.*

(a) *If R satisfies $P_3(n, R\backslash J)$ and $Q(n)$, then R is commutative.*

(b) *If R satisfies $P_0(n+1, R\backslash J)$ and $Q(n(n+1))$, then R is commutative.*

(c) *If R satisfies $P_3'(R\backslash J)$, then R is commutative.*

Theorem 2. *Let R be an s-unital ring.*

(a) *If R satisfies $P_3(n, R\backslash N)$ and $Q(n)$, then R is commutative.*

(b) *If R satisfies $P_0(n+1, R\backslash N)$ and $Q(n(n+1))$, then R is commutative.*

(c) *If R satisfies $P_3'(R\backslash N)$, then R is commutative.*

In preparation for the proof of our theorems, we state the next

Lemma 1. *Let R be an s-unital ring.*

(a) *If R satisfies $P_3(n)$ and $Q(n)$, then R is commutative.*

(b) *If R satisfies $P_0(n+1)$ and $Q(n(n+1))$, then R is commutative.*

(c) *If R satisfies P_3', then R is commutative.*

Proof. (a) This is included in [3, Theorem 1].

(b) This is proved in [1, Theorem 2].

(c) This is included in [4, Theorem].

Proof of Theorem 1. (a) Since R/J satisfies the polynomial identity $(x_1x_2)^n - (x_2x_1)^n = 0$, [3, Proposition 2] proves that $[R, R] \subseteq J$. Put $p(t) = (1-t)^n - 1 \in \boldsymbol{Z}[t]$. Let a, b be elements of J with quasi-inverses a', b', respectively, and choose a pseudo-identity e of $\{a, b\}$ in $R \backslash J$ and a pseudo-identity f of e. Then

$$(f-b)(e-a)^n(f-b') = ((f-b)(e-a)(f-b'))^n$$
$$= ((e-a)(f-b')(f-b))^n = (e-a)^n,$$

and hence $[b, p(a)] = -[f-b, (e-a)^n] = 0$. This shows that R satisfies the polynomial identity $[[x_1, x_2], p([x_1, x_2][x_3, x_4])] = 0$. In any 2×2 matrix ring over any finite prime field, we have

$$[[e_{11}, e_{12}], p([e_{11}, e_{12}][e_{22}, e_{21}])] = [e_{12}, (1-e_{11})^n] = e_{12}.$$

Hence, again by [3, Proposition 2], $[R, R] \subseteq N$ and N forms an ideal. Next, let $c \in N$, and suppose that $[b, c^r] = 0$ for all integers $r \geq k$, k minimal. Suppose $k > 1$, and put $a = c^{k-1}$. Then, by the above, we see that $0 = [f-b, (e-a)^n] = n[b, c^{k-1}]$, and hence $[b, c^{k-1}] = 0$ by $Q(n)$. This contradiction shows that $k = 1$ and hence $[J, N] = 0$. Again, let $a, b \in J$. Since $[[f-b, e-a], e-a] = -[[b, a], a] \in [N, J] = 0$, we have $n(e-a)^{n-1}[b, a] = [f-b, (e-a)^n] = 0$, whence $[b, a] = 0$ follows. Thus J is a commutative ideal, and J^2 is central. If $n > 1$, this enables us to see that $(xy)^n = (yx)^n$ provided $x \in J$ or $y \in J$. Combining this with $P_3(n, R\backslash J)$, we see that R satisfies $P_3(n)$, and hence R is commutative by Lemma 1 (a). On the other hand, if $n = 1$, $x \in J$ and $y \in R\backslash J$, then $y + xy = (e'+x)y = y(e'+x) = y + yx$, where e' is a pseudo-identity of $\{x, y\}$ in $R\backslash J$. This proves that $[J, R\backslash J] = 0$. Since both J and $R\backslash J$ are commutative, R is also commutative.

(b) As in (a), we can see that $[R, R] \subseteq J$. Put $p(t) = (1-t)^{n(n+1)} - 1$. Now, let $a, b \in J$, and choose e, f as in (a). Then

$$(f-b)(e-a)^{n+1}(f-b') = ((f-b)(e-a)(f-b'))^{n+1}$$
$$= (f-b)^{n+1}(e-a)^{n+1}(f-b')^{n+1},$$

and hence $[(f-b)^n, (e-a)^{n+1}] = 0$. Similarly, we have $[(f-b)^n, (e-a)^{n(n+1)}] = 0$ and $[(f-b)^{n+1}, (e-a)^{n(n+1)}] = [(e-b)^{n+1}, (f-a)^{n(n+1)}] = 0$, and hence $[b, p(a)] \doteq -[f-b, (e-a)^{n(n+1)}] = 0$. This shows that R satisfies the polynomial identity $[[x_1, x_2], p([x_1, x_2][x_3, x_4])] = 0$. Then, as in (a), we can see that $[R, R] \subseteq N$ and N forms an ideal. Next, let $c \in N$, and suppose that $[b, c^r] = 0$ for all integers $r \geq k$, k minimal. Suppose $k > 1$, and put $a = c^{k-1}$. Then, by the above, we see that $0 = [f-b, (e-a)^{n(n+1)}] = n(n+1)[b, c^{k-1}]$, and hence $[b, c^{k-1}] = 0$. This contradiction shows that $[J, N] = 0$. Now, repeating the argument employed in (a), we can easily see that $(xy)^{n+1} = x^{n+1}y^{n+1}$ provided $x \in J$ or $y \in J$. This together with $P_0(n+1, R\backslash J)$ implies $P_0(n+1)$ and hence R is commutative by Lemma 1 (b).

(c) Since R/J is commutative by Lemma 1 (c), we have $[R, R] \subseteq J$. Let $a, b \in J$, and choose e, f as in (a). Then there exist positive integers l, m with $(l, m) = 1$ such that $[f-b, (e-a)^l] = 0 = [f-b, (e-a)^m]$. We can easily see that $[a, b] = [f-b, e-a] = 0$. So, J is commutative and J^2 is central. Accordingly, if $x \in J$ or $y \in J$ then $(xy)^i = (yx)^i$ for $i \geq 2$. Hence R satisfies P_3', and R is commutative by Lemma 1 (c).

Proof of Theorem 2. In view of [3, Proposition 1], Theorem 2 is an immediate consequence of the theorems in [2]. However, we shall reduce the proof to Theorem 1 ; actually, we shall show that N is an ideal (contained in J).

(a) Careful scrutiny of the proof of Theorem 1 (a) shows that N is commutative. Suppose, to the contrary, that $NR \nsubseteq N$, say. Then we can find $a \in N$ and $r \in R$ such that $ar \notin N$ and $a^2R \subseteq N$. Let a' be the quasi-inverse of a, and e a pseudo-identity of $\{a, r\}$. Then, noting that $(e-a)ar \notin N$, by $P_3(n, R\backslash N)$ we see that $(e-a)(ar)^n(e-a') = ((e-a)ar(e-a'))^n = (ar)^n$. Hence $[a, (ar)^n] = -[e-a, (ar)^n] = 0$, and therefore $(ar)^{n+1} = a(ar)^n r - [a, (ar)^n]r \in N$, a contradiction. Similarly, we can show $RN \subseteq N$.

(b) Careful scrutiny of the proof of Theorem 1 (b) shows that N is commutative. Suppose, as in (a), that we can find $a \in N$ and $r \in R$ such that $ar \notin N$ and $a^2R \subseteq N$. Let e be a pseudo-identity of $\{a, r\}$, and h the minimal positive integer such that $[a^s, (ar)^{n+1}] = 0$ for all $s \geq h$. Suppose now that $h > 1$. Then, noting that $ar \notin N$ and $(e-a^{h-1})ar \notin N$, by $P_0(n+1, R\backslash N)$ we can easily see that $(e-a^{h-1})(ar)^{n+1}(e-a^*) = (e-a^{h-1})^{n+1}(ar)^{n+1}(e-a^*)^{n+1}$, where a^* is the quasi-inverse of a^{h-1}. Hence $n[a^{h-1}, (ar)^{n+1}] = -[(e-a^{h-1})^n, (ar)^{n+1}] = 0$, whence $[a^{h-1}, (ar)^{n+1}] = 0$ by $Q(n(n+1))$. This contradiction shows that $h = 1$, and hence $(ar)^{n+2} = a(ar)^{n+1}r$

$-[a, (ar)^{n+1}]r \in N$. But this contradicts $ar \notin N$. We have thus seen that $NR \subseteq N$; similarly $RN \subseteq N$.

(c) Let $b, c \in N, x \in R$, and e a pseudo-identity of $\{b, x\}$. Then, careful scrutiny of the proof of Theorem 1 (c) shows that $[b, c] = 0$ and there exist positive integers l, m with $(l, m) = 1$ such that $[b, x^l] = 0 = [b, x^m]$. Now, choose a minimal positive integer h such that $[b, x^h] = 0$. If $h > 1$, then there exist positive integers p, q with $(p, q) = 1$ such that $p[b, x^{h-1}] = [b, (e+x^{h-1})^p] = 0 = [b, (e+x^{h-1})^q] = q[b, x^{h-1}]$. This forces a contradiction $[b, x^{h-1}] = 0$. Thus we have shown that N is central, and hence N is an ideal.

References

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