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THE CATEGORY OF s-UNITAL MODULES

Dedicated to Professor Hisao Tominaga on his 60th birthday

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The category of unital modules over a ring with identity element is characterized as a cocomplete abelian category with a progenerator. More generally, every cocomplete abelian category **C** with a generating set **U** of small projectives is equivalent to a functor category by Freyd's theorem, and Gabriel showed in [5] that this functor category is equivalent to the category $\mathbf{Mod}_R$ of s-unital right modules over the ring $R = \mathrm{End}_{\mathbf{C}}(\mathbf{U})$ (in notation of Section 2). From this point of view, in [6], Harada studied $\mathbf{Mod}_R$. Fuller, in [4], examined the ring $\mathrm{End}_{\mathbf{Mod}_S}(\mathbf{U})$ induced from the skeleton **U** of the category of finitely generated unital modules over a ring S with identity element. In his method, a subfunctor of the functor $\mathrm{Hom}_S\left(\bigoplus_{U\in \mathbf{U}} U, -\right)$ played an important role.

In this paper, we characterize a cocomplete abelian category **C** by a subfunctor F of the functor $\mathrm{Hom}_{\mathbf{C}}(U, -)$ for suitable $U \in \mathbf{C}$. If F is faithful and exact and preserves coproducts, then **C** is equivalent to the category of s-unital right modules over a right s-unital ring (Theorem 2.4). When this is the case, we call the pair (U, F) a subprogenerator of **C**. Using this result, we can get Freyd-Gabriel theorem mentioned above.

On the other hand, s-unital modules and rings appear occasionally even in the theory of rings with identity element, and were systematically studied by Tominaga [10]. In Section 1, we state fundamental results for s-unital modules and rings.

In Section 3, we examine a subprogenerator of a closed subcategory of a module category, and give a generalization of a theorem on equivalences in Fuller [3]. In particular, considering a subprogenerator of the category of s-unital right modules over a right s-unital ring, we can generalize some results on Morita equivalence for right s-unital rings (Sections 3 and 4). In Section 5, we define quotient rings of right s-unital rings and prove that quotient rings of right Morita equivalent right s-unital rings are also right Morita equivalent.

Throughout the present paper, R will represent a (associative) ring. Except as indicated, rings will not be assumed to have identity element. We denote by $\mathfrak{M}_R$ (resp. ${}_R\mathfrak{M}$) the category of right (resp. left) R-modules, and by **Ab** the category of abelian groups. We shall use freely the categorical notions employed in [8] and [9].

1. Fundamental results on s-unital modules. Let A be a subset of R. A right (resp. left) R-module M is said to be *A-s-unital* if for any $u \in M$ there exists $a \in A$ such that $u = ua$ (resp. $u = au$). A ring R is called a *right* (resp. *left*) *A-s-unital ring* if R_R (resp. ${}_RR$) is A-s-unital, and R is called an *A-s-unital* ring if R is both left and right A-s-unital. A right (or left) R-module is called *s-unital* if R-s-unital. In case A is the set of idempotents in R, every A-s-unital module is called an *s^*-unital module*. Now, we denote by $\mathbf{Mod}_R$ the full subcategory of $\mathfrak{M}_R$ whose objects are s-unital right R-modules. If R has an identity element then $\mathbf{Mod}_R$ coincides with the category of unital right R-modules (see Proposition 1.8). Let R^1 be the ring obtained from R by adjoining an identity element in the customary manner. Then as is well known, $\mathfrak{M}_R = \mathbf{Mod}_{R^1}$.

According to [5, p. 395], we say that a full subcategory of $\mathfrak{M}_R$ which is closed under taking submodules, homomorphic images, and direct sums is a *closed subcategory* of $\mathfrak{M}_R$. Let $\mathbf{C}$ be a closed subcategory of $\mathfrak{M}_R$. If M is an arbitrary right R-module, and $\tau(M)$ denotes the sum of all submodules of M belonging to $\mathbf{C}$, then clearly also $\tau(M) \in \mathbf{C}$. In this way, we get a left exact preradical τ of $\mathfrak{M}_R$. Let f be a morphism in $\mathbf{C}$. Then the kernel (resp. cokernel) of f in the category $\mathfrak{M}_R$ is the kernel (resp. cokernel) of f in the category $\mathbf{C}$. Therefore, f is a monomorphism (resp. epimorphism) in the category $\mathbf{C}$ if and only if f is a monomorphism (resp. epimorphism) in the category $\mathfrak{M}_R$. For any family $\{C_\lambda\}_\Lambda$ of objects in $\mathbf{C}$, the direct sum $\bigoplus_{\lambda \in \Lambda} C_\lambda$ is the coproduct of $\{C_\lambda\}_\Lambda$ in $\mathbf{C}$ and it is easy to see that $\tau\left(\prod_{\lambda \in \Lambda} C_\lambda\right)$ is the product of $\{C_\lambda\}_\Lambda$ in $\mathbf{C}$. Moreover, all the factor modules of R^1_R belonging to $\mathbf{C}$ form a generating set of $\mathbf{C}$. Hence, $\mathbf{C}$ is a Grothendieck category. If $\mathbf{C}$ is closed under extensions, then τ is a radical.

Proposition 1.1. *$\mathbf{Mod}_R$ is a closed subcategory of $\mathfrak{M}_R$ and is closed under extensions.*

Proof. All submodules and all homomorphic images of an s-unital module are also s-unital. Conversely, assume that N is a submodule of M_R such that both N_R and M/N_R are s-unital. Let u be an arbitrary element of M, and choose $r_1 \in R$ such that $u+N = (u+N)r_1$, i.e. $u-ur_1 \in N$. Then, there exists $r_2 \in R$ with $u-ur_1 = (u-ur_1)r_2$, which implies $u \in uR$. Hence M_R is s-unital. This shows that any finite direct sum of s-unital modules is also s-unital. Now, it is clear that $\mathbf{Mod}_R$ is closed under direct sums.

We readily obtain the following fundamental result [10, Theorem 1].

Corollary 1.2 (Tominaga). *Let $M_1,\dots,M_n$ be s-unital right R-modules. If $u_i \in M_i$ $(i = 1,\dots,n)$ then there exists $r \in R$ such that $u_i = u_i r$ $(i = 1, \dots, n)$.*

Here, note that $\mathfrak{M}_R = \mathbf{Mod}_{R^1}$ and R is an ideal of R^1. More generally, we consider the relationship between $\mathbf{Mod}_R$ and $\mathbf{Mod}_{R'}$, where R is an ideal of a ring R'. To this end, we state the following lemma whose proof is easy.

Lemma 1.3. *The natural homomorphism $\lambda(M) : M_R \to \mathrm{Hom}_R(R, M)R_R$ for $M \in \mathbf{Mod}_R$, defined by $\lambda(M)(u)(r) = ur$ $(u \in M,\ r \in R)$, is an isomorphism, namely $\lambda : 1_{\mathbf{Mod}_R} \to \mathrm{Hom}_R(R, -)R$ is a natural equivalence.*

Proposition 1.4. *Let R be an ideal of a ring R'.*

(1) *Every s-unital right R-module M has a unique right R'-module structure which preserves the R-module structure.*

(2) *For any R-s-unital right R'-modules M and N,* $\mathrm{Hom}_R(M, N) = \mathrm{Hom}_{R'}(M, N)$.

Proof. (1) Since $\mathrm{Hom}_R({}_{R'}R, M)R$ is a right R'-module, via the isomorphism $\lambda(M)$ in Lemma 1.3, M has a right R'-module structure preserving the R-module structure. Obviously, for any $u \in M$ and $r' \in R'$, $ur' = u(rr')$, where r is an element of R such that $u = ur$. This shows that such an R'-module structure is uniquely determined, which proves (1).

(2) By the proof of (1), we see that $\mathrm{Hom}_R(M, N) \subseteq \mathrm{Hom}_{R'}(M, N)$, proving (2).

Proposition 1.4 states that if R is an ideal of a ring R' then $\mathbf{Mod}_R$ is a full subcategory of $\mathbf{Mod}_{R'}$. Moreover, the proof of Proposition 1.1 enables us to see that $\mathbf{Mod}_R$ is a closed subcategory of $\mathbf{Mod}_{R'}$ and is closed under extensions.

We shall denote by σ_R the left exact radical of $\mathfrak{M}_R$ corresponding to $\mathbf{Mod}_R$. Next, we consider the exactness of σ_R.

Proposition 1.5. *Let* $S = \sigma_R(R)$. *Then the following are equivalent :*

1) σ_R *is exact.*
2) $\sigma_R(M) = MS$ *for any right R-module M.*
3) $\mathbf{Mod}_S = \mathbf{Mod}_R$.

Proof. 2) ⇒ 1). This is obvious.

1) ⇒ 2). Put $\Lambda = \sigma_R(M)$, and define an epimorphism $f : R_R^{(\Lambda)} \twoheadrightarrow \Lambda_R$ by $f((r_u)_\Lambda) = \sum_{u \in \Lambda} u r_u$. Since $\sigma_R(f) : S_R^{(\Lambda)} \to \Lambda_R$ is also an epimorphism, we obtain $\Lambda = f(S^{(\Lambda)}) = \sum_{u \in \Lambda} uS \subseteq MS$. But, since $MS = \sum_{u \in M} uS$ is a sum of homomorphic images of S_R, MS_R is s-unital. Hence $\Lambda = MS$.

2) ⇒ 3). If M is an s-unital right R-module, then for any $u \in M$, we have $u \in uR = \sigma_R(uR) = uRS = uS$. Thus M_S is s-unital. Combining this with Proposition 1.4, we obtain 3).

3) ⇒ 2). Since $\sigma_R(M)$ is in $\mathbf{Mod}_S$, $\sigma_R(M) = \sigma_R(M)S \subseteq MS$, and hence $\sigma_R(M) = MS$.

Corollary 1.6. *Let* $S = \sigma_R(R)$. *Assume that* σ_R *is exact.*

(1) *S is a generator of* $\mathbf{Mod}_R$.
(2) *S is a right s-unital ring.*
(3) $M \otimes_R S_R$ *and* MS_R *are canonically isomorphic for any right R-module M.*

Proof. (1) This is immediate from Proposition 1.5.2).

(2) This is immediate from Proposition 1.5.3).

(3) It suffices to show that, for $u_1, \dots, u_n \in M$ and $s_1, \dots, s_n \in S$, $\sum_{i=1}^{n} u_i s_i = 0$ implies $\sum_{i=1}^{n} u_i \otimes s_i = 0$. Actually, from (2) and Corollary 1.2, there exists $s \in S$ such that $s_i = s_i s$ $(i = 1, \dots, n)$. Then, $\sum_{i=1}^{n} u_i \otimes s_i = \sum_{i=1}^{n} u_i s_i \otimes s = 0$.

Corollary 1.7. *If R is a right s-unital ring then* σ_R *is exact.*

If R is a right s-unital ring then R is a generator of $\mathbf{Mod}_R$. In general, a generator of $\mathbf{Mod}_R$ is not always a faithful module. For example, $R = \begin{pmatrix} \mathbf{Z} & 0 \\ \mathbf{Z} & 0 \end{pmatrix}$ has a right identity element $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, but $R\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0$.

Next, we state some fundamental results for s-unital modules over right s-unital rings.

Proposition 1.8. *Let A be a subset of R. If R is right A-s-unital then the following are equivalent for a right R-module M :*

1) *M_R is A-s-unital.*
2) *M_R is s-unital.*
3) *$M = MR$.*

Proof. 1) ⇒ 2) ⇒ 3). Trivial.

3) ⇒ 2). This is immediate from Corollary 1.7 and Proposition 1. 5.2).

2) ⇒ 1). Let u be an arbitrary element of M, and choose $r \in R$ with $u = ur$. Furthermore, choose $a \in A$ such that $r = ra$. Then, $u = ur = ura = ua$. Hence M_R is A-s-unital.

Corollary 1.9. *Let A be a subset of R. Let $M_1,\dots,M_n$ be right R-modules such that $M_i = M_iR$ $(i = 1,\dots,n)$. If R is right A-s-unital, then for $u_i \in M_i$ $(i = 1,\dots,n)$ there exists $a \in A$ such that $u_i = u_ia$ $(i = 1,\dots,n)$.*

Proof. By Proposition 1.8 and Corollary 1.2, there exists $r \in R$ with $u_i = u_ir$ $(i = 1,\dots,n)$. Choosing $a \in A$ with $r = ra$, we get $u_i = u_ia$.

For s^*-unital rings we obtain the following fine result.

Proposition 1.10. *Let R be an s^*-unital ring and let $M_1,\dots,M_n$ be R-R-bimodules such that $M_i = RM_iR$ $(i = 1,\dots,n)$. If $u_i \in M_i$ $(i = 1,\dots,n)$ then there exists an idempotent e of R such that $eu_i = u_i = u_ie$ $(i = 1,\dots,n)$.*

Proof. By Corollary 1.9, there exists an idempotent f of R such that $u_i = u_if$ $(i = 1,\dots,n)$. Similarly, we can choose an idempotent g of R such that $f = gf$ and $u_i = gu_i$ $(i = 1,\dots,n)$. Then, one will easily see that $e = f+g-fg$ is a desired idempotent.

In the rest of this section, for the sake of later use, we introduce some notions. According to [10], an injective (resp. projective) object of $\mathbf{Mod}_R$ is called an *s-injective* (resp. *s-projective*) *module*. A left R-module M is said to be *s-flat* if the functor $-\bigotimes_R M : \mathbf{Mod}_R \to \mathbf{Ab}$ is exact. Noting that if X is an s-unital right R-module then $\mathrm{Hom}_R(X, Y)$ and $\mathrm{Hom}_R(X, \sigma_R(Y))$ are isomorphic for any right R-module Y, we can apply the argument used in case

R has an identity element to see the following : A left R-module M is s-flat if and only if $\sigma_R(\mathrm{Hom}_Z(M, \boldsymbol{Q}/\boldsymbol{Z}))_R$ is s-injective. Furthermore, by making use of Baer Criterion ([10, Proposition 3]), we can prove the following

Proposition 1.11 (cf. [10, Remarks (3)]). *Let R be a right s-unital ring. Then a left R-module M is s-flat if and only if, for any finitely generated right ideal I of R, $\iota \otimes 1_M : I \otimes_R M \to R \otimes_R M$ is a monomorphism, where $\iota : I \to R$ is the inclusion map.*

If e is an idempotent of R then $eR = eR^1$ is projective in $\mathfrak{M}_R$. We conclude this section with the following

Proposition 1.12. *Let R be a right s-unital ring, and let M be an s-unital right R-module.*

(1) *M_R is s-projective if and only if M is projective in $\mathfrak{M}_R$.*

(2) *If M_R is s^*-unital then M_R is s-projective if and only if M_R is isomorphic to a direct summand of a direct sum $\bigoplus_{\lambda \in \Lambda} e_\lambda R$, where e_λ are idempotents of R.*

Proof. (1) Since every homomorphic image of M_R is s-unital, $\mathrm{Hom}_R(M, -) \simeq \mathrm{Hom}_R(M, \sigma_R(-))$ as functors $\mathfrak{M}_R \to \mathbf{Ab}$. By Corollary 1.7, σ_R is exact. Hence, $\mathrm{Hom}_R(M, -)$ is exact in $\mathbf{Mod}_R$ if and only if so is in $\mathfrak{M}_R$.

(2) Suppose that M_R is s-projective. For each $u \in M$ there exists an idempotent e_u of R such that $u = ue_u$. Therefore, we can find an epimorphism $\bigoplus_{u \in M} e_u R_R \twoheadrightarrow M_R$, which is split by assumption. The converse is trivial.

2. Characterization of categories of s-unital modules over right s-unital rings. Throughout this section, $\mathbf{C}$ will represent a preadditive category. Let U be an object of $\mathbf{C}$, and F a subfunctor of $\mathrm{Hom}_{\mathbf{C}}(U, -) : \mathbf{C} \to \mathbf{Ab}$. Then, for each $C \in \mathbf{C}$, $F(C)$ is a subgroup of $\mathrm{Hom}_{\mathbf{C}}(U, C)$ and for each morphism $f : C \to D$ the following diagram is commutative :

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{C}}(U, C) & \xrightarrow{f_*} & \mathrm{Hom}_{\mathbf{C}}(U, D) \\ \cup\!\shortmid & & \cup\!\shortmid \\ F(C) & \xrightarrow[F(f)]{} & F(D) \end{array}$$

Thus, $\mathrm{Hom}_{\mathbf{C}}(C, D)F(C) \subseteq F(D)$. In particular, $F(U)$ is a left ideal of $\mathrm{Hom}_{\mathbf{C}}(U, U)$ and $F(C)$ is a right $F(U)$-module for any $C \in \mathbf{C}$. It is clear that for any morphism f of $\mathbf{C}$, $F(f)$ is an $F(U)$-homomorphism. Therefore, F induces a functor from $\mathbf{C}$ to $\mathfrak{M}_{F(U)}$. We state some lemmas concerning F.

Lemma 2.1. *If* $\mathbf{C}$ *is abelian and* F *is right exact then* $F(C)_{F(U)}$ *is s-unital for any* $C \in \mathbf{C}$, *and* $F = \mathrm{Hom}_{\mathbf{C}}(U, -)F(U) = \sigma_{F(U)}(\mathrm{Hom}_{\mathbf{C}}(U, -))$.

Proof. Let $f \in F(C)$, and $g : C \to D$ the cokernel of f. Then the sequence $F(U) \xrightarrow{F(f)} F(C) \xrightarrow{F(g)} F(D) \to 0$ is exact and $F(g)(f) = gf = 0$. Therefore, there is $h \in F(U)$ with $f = F(f)(h) = fh$. Hence $F(C)_{F(U)}$ is s-unital. Also, $F(C) = F(C)F(U) \subseteq \mathrm{Hom}_{\mathbf{C}}(U, C)F(U) \subseteq F(C)$, namely $F(C) = \mathrm{Hom}_{\mathbf{C}}(U, C)F(U)$.

Lemma 2.2. *If* $\mathbf{C}$ *is a full subcategory of a cocomplete abelian category* $\mathbf{D}$ *in which* $\mathbf{C}$ *is closed under taking factor objects, then the following are equivalent* :

1) *F is faithful.*

2) *For any* $C \in \mathbf{C}$, *there exists an epimorphism* $f : U^{(\Lambda)} \twoheadrightarrow C$ *in* $\mathbf{D}$ *with some index set* Λ *such that* $f\iota_\lambda \in F(C)$ *for every injection* ι_λ *of the coproduct* $U^{(\Lambda)}$.

3) *U is a generator of* $\mathbf{C}$ *and there exists an epimorphism* $f : U^{(\Lambda)} \twoheadrightarrow U$ *in* $\mathbf{D}$ *with some index set* Λ *such that* $f\iota_\lambda \in F(U)$ *for every injection* ι_λ *of the coproduct* $U^{(\Lambda)}$.

Proof. 1) $\Rightarrow$ 2). Put $\Lambda = F(C)$ and let $\iota_g : U \to U^{(\Lambda)}$ be the g-th injection for each $g \in \Lambda$. From the property of coproducts there exists $f : U^{(\Lambda)} \to C$ such that $f\iota_g = g$ for all $g \in \Lambda$. Suppose that $h : C \to D$ is a non-zero morphism of $\mathbf{D}$. Then h is factored through an epimorphism $h' : C \twoheadrightarrow C'$ and a monomorphism $i : C' \to D$. Since F is faithful, we have $F(h') \neq 0$ and hence $h'g \neq 0$ for some $g \in F(C)$. Since $g = f\iota_g$ and i is a monomorphism, we obtain $hf \neq 0$. Thus we have shown that f is an epimorphism.

2) $\Rightarrow$ 3). Trivial.

3) $\Rightarrow$ 1). Given a non-zero morphism $h : C \to D$ of $\mathbf{C}$, we must prove $F(h) \neq 0$. Since U is a generator, there exists $g : U \to C$ such that $hg \neq 0$. Since f is an epimorphism, we have $hgf \neq 0$ and hence $hgf\iota_\lambda \neq 0$ for some $\lambda \in \Lambda$. Obviously, $gf\iota_\lambda \in F(C)$, by assumption.

Lemma 2.3. *Let* C *be a coproduct of a family* $\{C_\lambda\}_\Lambda$ *of objects of* $\mathbf{C}$ *with injections* ι_λ *and projections* π_λ. *Let* $e_\lambda = \iota_\lambda \pi_\lambda$ $(\lambda \in \Lambda)$. *Suppose that, for any finite subset* Γ *of* Λ, $\mathbf{C}$ *has the coproduct* $\bigoplus_{\gamma\in\Gamma} C_\gamma$. *Then the following are equivalent :*

1) $F(C)$ *is the coproduct of the family* $\{F(C_\lambda)\}_\Lambda$ *with injections* $F(\iota_\lambda)$.

2) *The monomorphism* $\nu : \bigoplus_{\lambda\in\Lambda} F(C_\lambda) \to F(C)$, *defined by* $\nu((f_\lambda)_\Lambda) = \sum_{\lambda\in\Lambda} \iota_\lambda f_\lambda$, *is an isomorphism.*

3) $F(C) = \bigoplus_{\lambda\in\Lambda} e_\lambda F(C)$.

4) *For any* $f \in F(C)$, *there exists a finite subset* Γ *of* Λ *and* $g \in F\left(\bigoplus_{\gamma\in\Gamma} C_\gamma\right)$ *such that* $f = \iota g$, *where* $\iota : \bigoplus_{\gamma\in\Gamma} C_\gamma \to C$ *is the canonical injection.*

Proof. 1) ⟺ 2). Let $i_\lambda : F(C_\lambda) \to \bigoplus_{\mu\in\Lambda} F(C_\mu)$ be the usual injection. Then $F(\iota_\lambda) = \nu i_\lambda$. Hence, the assertion is clear.

2) ⇒ 3). Any element of $F(C)$ is a finite sum of $e_\lambda \iota_\lambda f_\lambda = \iota_\lambda f_\lambda$ $(f_\lambda \in F(C_\lambda))$. Hence, the assertion is clear.

3) ⇒ 2). Since $\pi_\lambda F(C) \subseteq F(C_\lambda)$, we have $e_\lambda F(C) \subseteq \iota_\lambda F(C_\lambda)$, and therefore $\operatorname{Im} \nu = \sum_{\lambda\in\Lambda} \iota_\lambda F(C_\lambda) = F(C)$. Thus ν is an epimorphism.

3) ⇒ 4). Put $\Gamma = \{\lambda \in \Lambda \mid e_\lambda f \neq 0\}$. Then Γ is a finite set and $f = \sum_{\gamma\in\Gamma} e_\gamma f$. We consider a coproduct $D = \bigoplus_{\gamma\in\Gamma} C_\gamma$ with injections ι'_γ. Let $\iota : D \to C$ be the canonical injection. Then, there exists $\pi : C \to D$ such that $\pi\iota = 1_D$. Since $\iota\pi\iota_\gamma = \iota\pi\iota\iota'_\gamma = \iota\iota'_\gamma = \iota_\gamma (\gamma \in \Gamma)$, we have $\iota\pi e_\gamma = e_\gamma$ $(\gamma \in \Gamma)$. Hence $f = \iota\pi f$. Clearly $\pi f \in F(D)$.

4) ⇒ 3). Now suppose that $f \in F(C)$ is of the form in 4). Let ι'_γ be the γ-th injection of $\bigoplus_{\lambda\in\Gamma} C_\lambda$ $(\gamma \in \Gamma)$. Then $\sum_{\lambda\in\Gamma} e_\lambda \iota\iota'_\gamma = \iota_\gamma = \iota\iota'_\gamma (\gamma \in \Gamma)$, and therefore $\iota = \sum_{\lambda\in\Gamma} e_\lambda \iota$. Hence $f = \sum_{\lambda\in\Gamma} e_\lambda f$.

A pair (U, F) of an object U of $\mathbf{C}$ and a subfunctor F of $\mathrm{Hom}_{\mathbf{C}}(U, -) : \mathbf{C} \to \mathbf{Ab}$ will be called a *subprogenerator* of $\mathbf{C}$ if F is faithful and exact and preserves coproducts. For example, if R is right s-unital then $(R, \mathrm{Hom}_R(R, -)R)$ is a subprogenerator of $\mathbf{Mod}_R$ (Lemma 1.3). We are now in a position to state the main theorem of this section.

Theorem 2.4. *Let* $\mathbf{C}$ *be a cocomplete abelian category, and* R *a right*

s-unital ring. Then $\mathbf{C}$ *is equivalent to* $\mathbf{Mod}_R$ *if and only if* $\mathbf{C}$ *has a subprogenerator* (U, F) *such that* $F(U) \simeq R$ *as rings.*

Proof. First, suppose that $\mathbf{C}$ and $\mathbf{Mod}_R$ are equivalent. Let $G : \mathbf{Mod}_R \to \mathbf{C}$ be an equivalence, and $H: \mathbf{C} \to \mathbf{Mod}_R$ the inverse of G. Consider the subfunctor $F = \mathrm{Hom}_{\mathbf{C}}(G(R), -)G(R_L)$ of $\mathrm{Hom}_{\mathbf{C}}(G(R), -)$, where R_L is the set of all left multiplications effected by elements in R. Then $F \simeq \mathrm{Hom}_R(R, H(-))R \simeq H$ (Lemma 1.3). Therefore $(G(R), F)$ is a subprogenerator of $\mathbf{C}$, and moreover, $F(G(R)) \simeq R$. Conversely, suppose that $\mathbf{C}$ has a subprogenerator (U, F). By Lemma 2.1, F induces a functor $\tilde{F} : \mathbf{C} \to \mathbf{Mod}_{F(U)}$ $(\tilde{F}(C) = F(C))$. We shall prove that $\tilde{F}$ is an equivalence. First we show that $\tilde{F}$ is full. Let ϕ be an arbitrary element of $\mathrm{Hom}_{F(U)}(F(C), F(D))$. By Lemma 2.2, we can construct a resolution

$$U^{(A_2)} \xrightarrow{f_2} U^{(A_1)} \xrightarrow{f_1} C \to 0$$

of C such that $f_1\iota_\lambda \in F(C)$ for every λ-th injection $\iota_\lambda : U \to U^{(A_1)}$ $(\lambda \in A_1)$ and $f_2\iota_\lambda \in F(U^{(A_1)})$ for every λ-th injection $\iota_\lambda : U \to U^{(A_2)}$ $(\lambda \in A_2)$. Similarly, we obtain an exact sequence $U_2 \xrightarrow{g_2} U_1 \xrightarrow{g_1} D \to 0$. Applying the exact functor F, we have the following diagram with exact rows :

$$\begin{array}{ccccccc} F(U^{(A_2)}) & \xrightarrow{F(f_2)} & F(U^{(A_1)}) & \xrightarrow{F(f_1)} & F(C) & \to & 0 \\ {\scriptstyle F(h_2)}\downarrow & & {\scriptstyle F(h_1)}\downarrow & & \downarrow{\scriptstyle \phi} & & \\ F(U_2) & \xrightarrow[F(g_2)]{} & F(U_1) & \xrightarrow[F(g_1)]{} & F(D) & \to & 0 \end{array}$$

We shall fill out this diagram to make it commutative. Since $F(g_1)$ is surjective, for each $\lambda \in A_1$ there is $h_\lambda \in F(U_1)$ such that $F(g_1)(h_\lambda) = \phi(f_1\iota_\lambda)$. The family $\{h_\lambda : U \to U_1\}_{A_1}$ gives rise to a morphism $h_1 : U^{(A_1)} \to U_1$ such that $h_\lambda = h_1\iota_\lambda$ for each $\lambda \in A_1$. Then, we have $F(g_1)F(h_1) = \phi F(f_1)$. To show this, it suffices to show that $F(g_1)F(h_1)F(\iota_\lambda) = \phi F(f_1)F(\iota_\lambda)$ for every $\lambda \in A_1$, since $F(\iota_\lambda)$'s are the structure maps of the coproduct $F(U^{(A_1)}) \simeq F(U)^{(A_1)}$. Actually, $F(g_1)F(h_1)F(\iota_\lambda)(k) = F(g_1)(h_\lambda k) = \phi(f_1\iota_\lambda k) = \phi F(f_1)F(\iota_\lambda)(k)$ for all $k \in F(U)$. Similarly, we can find $h_2 : U^{(A_2)} \to U_2$ such that $F(g_2)F(h_2) = F(h_1)F(f_2)$. Since F is faithful, the left square of the following diagram is commutative :

$$\begin{array}{ccccccc} U^{(A_2)} & \xrightarrow{f_2} & U^{(A_1)} & \xrightarrow{f_1} & C & \to & 0 \\ h_2 \downarrow & & h_1 \downarrow & & \downarrow h & & \\ U_2 & \xrightarrow[g_2]{} & U_1 & \xrightarrow[g_1]{} & D & \to & 0 \end{array}$$

There exists a unique morphism $h : C \to D$ such that this diagram is commutative, since f_1 is the cokernel of f_2. Applying F again, we obtain $\phi = F(h)$, since $F(f_1)$ is the cokernel of $F(f_2)$. Thus, we have shown that $\tilde{F}$ is full. It remains therefore to prove that any s-unital right $F(U)$-module M is isomorphic to $F(C)$ for some $C \in \mathbf{C}$. Since $F(U)$ is a generator of $\mathbf{Mod}_{F(U)}$ and F preserves coproducts, we obtain an exact sequence $F(U_2) \xrightarrow{\phi_2} F(U_1) \xrightarrow{\phi_1} M \to 0$, where U_i $(i = 1, 2)$ are coproducts of copies of U. As was shown above, $\tilde{F}$ is full, and so we have $\phi_2 = F(f)$ for some $f : U_2 \to U_1$. Let $g : U_1 \to C$ be the cokernel of f. Then applying F, we have an exact sequence $F(U_2) \xrightarrow{\phi_2} F(U_1) \xrightarrow{F(g)} F(C) \to 0$. Thus $F(C) \simeq M$, which completes the proof.

Now, let $\mathbf{U} = \{U_\lambda\}_\Lambda$ be a family of objects of $\mathbf{C}$. Put $\mathrm{End}_{\mathbf{C}}(\mathbf{U}) = \bigoplus_{\lambda,\mu \in \Lambda} \mathrm{Hom}_{\mathbf{C}}(U_\lambda, U_\mu)$ as module and make $\mathrm{End}_{\mathbf{C}}(\mathbf{U})$ a ring by the composition of morphisms (see [5]). Then $\mathrm{End}_{\mathbf{C}}(\mathbf{U})$ is an s^*-unital ring. We assume that $\mathbf{C}$ has the coproduct $U = \bigoplus_{\lambda \in \Lambda} U_\lambda$ with injections ι_λ and projections π_λ. Put $e_\lambda = \iota_\lambda \pi_\lambda$. Then $\{e_\lambda\}_\Lambda$ is a set of orthogonal idempotents of $\mathrm{Hom}_{\mathbf{C}}(U, U)$. Now, we define a subfunctor $F_{\mathbf{U}}$ of $\mathrm{Hom}_{\mathbf{C}}(U, -) : \mathbf{C} \to \mathbf{Ab}$ as the coproduct of subfunctors $\mathrm{Hom}_{\mathbf{C}}(U, -)e_\lambda$ $(\lambda \in \Lambda)$, i.e. $F_{\mathbf{U}}(C) = \bigoplus_{\lambda \in \Lambda} \mathrm{Hom}_{\mathbf{C}}(U, C)e_\lambda$ $(C \in \mathbf{C})$. Then, it is obvious that $F_{\mathbf{U}}(U)$ is a right s^*-unital ring and $F_{\mathbf{U}}(C) = \mathrm{Hom}_{\mathbf{C}}(U, C)F_{\mathbf{U}}(U)$ $(C \in \mathbf{C})$. Since both $\mathrm{Hom}_{\mathbf{C}}(U, -)$ and $\sigma_{F_{\mathbf{U}}(U)}$ are left exact, $F_{\mathbf{U}} = \sigma_{F_{\mathbf{U}}(U)}(\mathrm{Hom}_{\mathbf{C}}(U, -))$ is also left exact. Harada pointed out in [6] that if $\mathbf{C}$ is a cocomplete abelian category and $\mathbf{U}$ is a generating family of $\mathbf{C}$ consisting of small projectives, then $\mathbf{C}$ is equivalent to $\mathbf{Mod}_{\mathrm{End}_{\mathbf{C}}(\mathbf{U})}$. In order to give a new proof of this result, we state the following

Lemma 2.5. *Under the above notations, if* $\mathbf{C}$ *is abelian there holds the following* :

(1) $F_{\mathbf{U}}$ *is faithful if and only if* U *is a generator of* $\mathbf{C}$.

(2) *F_U is exact if and only if U is projective in* **C**.

(3) *F_U preserves coproducts if and only if U_λ is small in* **C** *for every $\lambda \in \Lambda$. When this is the case, $F_U(U) \simeq \mathrm{End}_C(\mathbf{U})$ as rings.*

Proof. (1) If F_U is faithful then U is a generator of **C** by Lemma 2.2. Conversely, suppose that U is a generator of **C**. Let $f: C \to D$ be a non-zero morphism. Then $fg \neq 0$ for some $g: U \to C$, and hence $fg\iota_\lambda \neq 0$ for some $\lambda \in \Lambda$. Therefore $fge_\lambda \neq 0$. Since $ge_\lambda \in F_U(C)$, this proves that $F_U(f) \neq 0$.

(2) Suppose F_U is exact. It suffices to show that each U_λ is projective. Let $g: C \twoheadrightarrow D$ be an epimorphism, and $f: U_\lambda \to D$ a morphism. Since $f\pi_\lambda \in F_U(D)$, there is $h \in F_U(C)$ with $gh = f\pi_\lambda$. Then $gh\iota_\lambda = f$. Conversely, suppose U is projective. As was mentioned above, F_U is left exact. It remains therefore to prove that F_U preserves epimorphisms. Let $g: C \twoheadrightarrow D$ be an epimorphism, and $f \in F_U(D)$. Then there is $h: U \to C$ with $gh = f$. Now, f is in $\bigoplus_{i=1}^{n} \mathrm{Hom}_C(U, D)e_{\lambda_i}$ with some e_{λ_i}. If $h' = h(e_{\lambda_1} + \ldots + e_{\lambda_n})$ then $h' \in F_U(C)$ and $F_U(g)(h') = gh' = f$. Thus $F_U(g)$ is an epimorphism.

(3) By definition, we have $F_U(C)e_\lambda = \mathrm{Hom}_C(U, C)e_\lambda = \mathrm{Hom}_C(U_\lambda, C)\pi_\lambda$ and $F_U(C)\iota_\lambda = \mathrm{Hom}_C(U_\lambda, C)$ ($C \in$ **C**, $\lambda \in \Lambda$). Hence the assertion is clear by Lemma 2.3.

Combining Lemma 2.5 with Theorem 2.4, we readily obtain the following

Proposition 2.6 (cf. [6, p. 344]). *If a cocomplete abelian category* **C** *has a generating family* **U** *of* **C** *consisting of small projectives then* **C** *is equivalent to* $\mathbf{Mod}_{\mathrm{End}_C(\mathbf{U})}$.

In [1], G. D. Abrams introduced the notion of rings with local units and studied equivalences of module categories over such rings. In the rest of this section, we shall prove a generalization of the main theorem of [1]. A set E of commuting idempotents in R is called a *set of local units* for R if for each $r \in R$ there exists $e \in E$ such that $er = r = re$. Every ring with local units is an s^*-unital ring.

Let Λ be a directed set. Suppose that for each $\lambda \in \Lambda$ there exists $U_\lambda \in$ **C**, and for each pair $\lambda \leq \mu$ in Λ there exist morphisms $\iota_\lambda^\mu: U_\lambda \to U_\mu$ and $\pi_\lambda^\mu: U_\mu \to U_\lambda$. We call the collection $\{U_\lambda, \iota_\lambda^\mu, \pi_\lambda^\mu\}_\Lambda$ a *split direct-inverse system* in **C** over Λ if $\{U_\lambda, \iota_\lambda^\mu\}_\Lambda$ is a direct system, $\{U_\lambda, \pi_\lambda^\mu\}_\Lambda$ is an inverse system, and $\pi_\lambda^\mu \iota_\lambda^\mu = 1_{U_\lambda}$ for each $\lambda \leq \mu$. If $\mathbf{S} = \{U_\lambda, \iota_\lambda^\mu, \pi_\lambda^\mu\}_\Lambda$ is a split direct-inverse system in $\mathbf{C}$ over Λ, then $\{\mathrm{Hom}_\mathbf{C}(U_\lambda, U_\lambda), \mathrm{Hom}_\mathbf{C}(\pi_\lambda^\mu, \iota_\lambda^\mu)\}_\Lambda$ is a direct system of rings, and $\mathrm{End}(\mathbf{S}) = \varinjlim \mathrm{Hom}_\mathbf{C}(U_\lambda, U_\lambda)$ is an s^*-unital ring.

Lemma 2.7. *Let* $\mathbf{S} = \{U_\lambda, \iota_\lambda^\mu, \pi_\lambda^\mu\}_\Lambda$ *be a split direct-inverse system in* $\mathbf{C}$ *over* Λ. *If* $\mathbf{C}$ *has the direct limit* $U = \varinjlim U_\lambda$ *with structure morphisms* $\{\iota_\lambda : U_\lambda \to U\}_\Lambda$ *on the direct system* $\{U_\lambda, \iota_\lambda^\mu\}_\Lambda$, *then there exists a family of morphisms* $\{\pi_\lambda : U \to U_\lambda\}_\Lambda$ *satisfying the following conditions* :

(1) $\pi_\mu \iota_\lambda = \pi_\mu^\nu \iota_\lambda^\nu$ *for all* $\lambda, \mu \leq \nu$.

(2) $\pi_\lambda \iota_\lambda = 1_{U_\lambda}$ *for all* λ.

(3) $\pi_\mu \iota_\lambda = \iota_\lambda^\mu$ *and* $\pi_\lambda \iota_\mu = \pi_\lambda^\mu$ *for all* $\lambda \leq \mu$.

(4) $\pi_\lambda^\mu \pi_\mu = \pi_\lambda$ *for all* $\lambda \leq \mu$.

Proof. Fix $\mu \in \Lambda$. If $\lambda, \mu \leq \nu \leq \nu'$ in Λ, then $\pi_\mu^\nu \iota_\lambda^\nu = \pi_\mu^\nu \pi_\nu^{\nu'} \iota_\nu^{\nu'} \iota_\lambda^\nu = \pi_\mu^{\nu'} \iota_\lambda^{\nu'}$. Hence, we may set $f_\lambda = \pi_\mu^\nu \iota_\lambda^\nu$ for any $\nu \in \Lambda$ with $\lambda, \mu \leq \nu$. It is easy to see that if $\lambda \leq \lambda'$ then $f_\lambda = f_{\lambda'} \iota_\lambda^{\lambda'}$. Hence, there exists $\pi_\mu : U \to U_\mu$ such that $f_\lambda = \pi_\mu \iota_\lambda$ for every $\lambda \in \Lambda$. Thus, we have obtained a family $\{\pi_\lambda : U \to U_\lambda\}_\Lambda$ satisfying (1). (2) and (3) are clear by (1). Finally, we shall prove (4). Let $\lambda \leq \mu$, and ν be an arbitrary element of Λ. Choose $\nu' \in \Lambda$ such that $\mu, \nu \leq \nu'$. Then $\pi_\lambda^\mu \pi_\mu \iota_\nu = \pi_\lambda^\mu \pi_\mu^{\nu'} \iota_\nu^{\nu'} = \pi_\lambda^{\nu'} \iota_\nu^{\nu'} = \pi_\lambda \iota_\nu$. Whence, $\pi_\lambda^\mu \pi_\mu = \pi_\lambda$.

Under the notations used in Lemma 2.7, we put $e_\lambda = \iota_\lambda \pi_\lambda$. Then e_λ is an idempotent of $\mathrm{Hom}_\mathbf{C}(U, U)$. By the condition (3) of Lemma 2.7, we have $e_\mu e_\lambda = e_\lambda = e_\lambda e_\mu$ for all $\lambda \leq \mu$. This implies that $\{e_\lambda \mathrm{Hom}_\mathbf{C}(U, U) e_\lambda\}_\Lambda$ is a direct system of subrings of $\mathrm{Hom}_\mathbf{C}(U, U)$, and clearly this is isomorphic to the direct system $\{\mathrm{Hom}_\mathbf{C}(U_\lambda, U_\lambda), \mathrm{Hom}_\mathbf{C}(\pi_\lambda^\mu, \iota_\lambda^\mu)\}_\Lambda$. Hence $\bigcup_{\lambda \in \Lambda} e_\lambda \mathrm{Hom}_\mathbf{C}(U, U) e_\lambda \simeq \mathrm{End}(\mathbf{S})$ as rings. Now, we shall generalize [1, Theorem 4.2] as follows :

Proposition 2.8. *Let* $\mathbf{C}$ *be a cocomplete abelian category, and* R *an* s^**-unital ring. Then* $\mathbf{C}$ *is equivalent to* $\mathbf{Mod}_R$ *if and only if* $\mathbf{C}$ *has a split direct-inverse system* $\mathbf{S} = \{U_\lambda, \iota_\lambda^\mu, \pi_\lambda^\mu\}_\Lambda$ *such that* $\{U_\lambda\}_\Lambda$ *is a generating family of* $\mathbf{C}$ *consisting of small projectives and* $\mathrm{End}(\mathbf{S}) \simeq R$ *as rings.*

Proof. Let E be a set of all idempotents of R. By Proposition 1.10, E is a directed set with respect to the usual order ($e \leq f \Leftrightarrow fe = e = ef$). For any $e \leq f$ in E, we denote by ι_e^f an inclusion map $eR \subseteq fR$ and by π_e^f : $fR \to eR$ the map induced from the left multiplication e_L. Then, it is easy to see that $\{eR, \iota_e^f, \pi_e^f\}_E$ is a split direct-inverse system in $\mathbf{Mod}_R$, and $\{eR\}_E$ is a generating family of $\mathbf{Mod}_R$ consisting of small projectives. Therefore, if $\mathbf{C}$ is equivalent to $\mathbf{Mod}_R$ then $\mathbf{C}$ has a split direct-inverse system with the property requested. Conversely, suppose that $\mathbf{C}$ has a split direct-inverse system $\mathbf{S} = \{U_\lambda, \iota_\lambda^\mu, \pi_\lambda^\mu\}_\Lambda$ such that $\mathbf{U} = \{U_\lambda\}_\Lambda$ is a generating family of $\mathbf{C}$ consisting of small projectives. Put $U = \varinjlim U_\lambda$ and consider the family $\{e_\lambda\}_\Lambda$ of idempotents in $\mathrm{Hom}_{\mathbf{C}}(U, U)$ defined just before Proposition 2.8. As was mentioned above, the s^*-unital subring $S = \bigcup_{\lambda\in\Lambda} e_\lambda \mathrm{Hom}_{\mathbf{C}}(U, U)e_\lambda$ is isomorphic to $\mathrm{End}(\mathbf{S})$. We shall prove that $\mathbf{C}$ is equivalent to $\mathbf{D} = \mathbf{Mod}_S$. Obviously, $\mathbf{V} = \{e_\lambda S\}_\Lambda$ is a generating family of $\mathbf{D}$ consisting of small projectives. Since $\mathrm{Hom}_{\mathbf{C}}(U_\lambda, U_\mu) \simeq e_\mu \mathrm{Hom}_{\mathbf{C}}(U, U)e_\lambda = e_\mu S e_\lambda \simeq \mathrm{Hom}_S(e_\lambda S, e_\mu S)$ $(\lambda, \mu \in \Lambda)$, it is easy to see that $\mathrm{End}_{\mathbf{C}}(\mathbf{U}) \simeq \mathrm{End}_{\mathbf{D}}(\mathbf{V})$. Furthermore, by Proposition 2.6, $\mathbf{C}$ is equivalent to $\mathbf{Mod}_{\mathrm{End}_{\mathbf{C}}(\mathbf{U})}$ and $\mathbf{D}$ is equivalent to $\mathbf{Mod}_{\mathrm{End}_{\mathbf{D}}(\mathbf{V})}$. In conclusion, $\mathbf{C}$ is equivalent to $\mathbf{D}$.

Remark. Let $\mathbf{C}$ be a cocomplete abelian category, and $\mathbf{S} = \{U_\lambda, \iota_\lambda^\mu, \pi_\lambda^\mu\}_\Lambda$ a split direct-inverse system in $\mathbf{C}$ over a directed set Λ. Put $U = \varinjlim U_\lambda$ and consider the family $\{e_\lambda\}_\Lambda$ of idempotents in $\mathrm{Hom}_{\mathbf{C}}(U, U)$ defined just before Proposition 2.8. We define a subfunctor $F_{\mathbf{S}}$ of $\mathrm{Hom}_{\mathbf{C}}(U, -) : \mathbf{C} \to \mathbf{Ab}$ as the direct limit of a direct system of subfunctors $\{\mathrm{Hom}_{\mathbf{C}}(U, -)e_\lambda\}_\Lambda$, i.e. $F_{\mathbf{S}}(C) = \bigcup_{\lambda\in\Lambda} \mathrm{Hom}_{\mathbf{C}}(U, C)e_\lambda$ $(C \in \mathbf{C})$. By the same argument of Lemma 2.5, we can prove the following :

(1) $F_{\mathbf{S}}$ is faithful if and only if U is a generator of $\mathbf{C}$.

(2) $F_{\mathbf{S}}$ is exact if and only if U_λ is projective in $\mathbf{C}$ for every $\lambda \in \Lambda$.

(3) $F_{\mathbf{S}}$ preserves coproducts if and only if U_λ is small in $\mathbf{C}$ for every $\lambda \in \Lambda$.

Moreover, if $F_{\mathbf{S}}$ is exact and preserves coproducts then we can prove $F_{\mathbf{S}}(U) \simeq \mathrm{End}(\mathbf{S})$ as rings.

Using these facts, we readily obtain the if part of Proposition 2.8.

3. Subprogenerators in a full subcategory of a module category. In this section, we shall consider subprogenerators of a closed subcategory of $\mathfrak{M}_R$. Throughout this section, $\mathbf{C}$ will represent a closed subcategory of $\mathfrak{M}_R$, U an object of $\mathbf{C}$, and F a subfunctor of $\mathrm{Hom}_R(U, -) : \mathbf{C} \to \mathbf{Ab}$. We put $S = F(U)$. For any right S-module M, there exists a canonical epimorphism

$U_R^{(M)} \twoheadrightarrow M \underset{S}{\otimes} U_R$, and so $M \underset{S}{\otimes} U \in \mathbf{C}$. We define a natural homomorphism $\eta(C) : F(C) \underset{S}{\otimes} U_R \to C_R$ for $C \in \mathbf{C}$ by $\eta(C)(f \otimes u) = f(u)$. First, we prove the following

Lemma 3.1. *If $\eta(C)$ is an epimorphism for every $C \in \mathbf{C}$, then, for any S-submodule M of $F(C)$ $(C \in \mathbf{C})$, the map $\eta : M \underset{S}{\otimes} U_R \to C_R$ defined by $\eta(f \otimes u) = f(u)$ is a monomorphism.*

Proof. Let $\sum_{i=1}^{n} f_i \otimes u_i \in \operatorname{Ker} \eta$ $(f_i \in M,\ u_i \in U)$. We consider the direct sum U^n with projections π_i. Put $K = \operatorname{Ker} \sum_{i=1}^{n} f_i \pi_i$, and $\iota : K \to U^n$ the inclusion map. Since $(u_1, \dots, u_n) \in K$ and $\eta(K)$ is an epimorphism, there exist $g_1, \dots, g_m \in F(K)$ and $v_1, \dots, v_m \in U$ such that $\sum_{j=1}^{m} g_j(v_j) = (u_1, \dots, u_n)$. Noting $\pi_i \iota g_j \in S$, we have $\sum_{i=1}^{n} f_i \otimes u_i = \sum_{i=1}^{n} f_i \otimes \pi_i \iota\left(\sum_{j=1}^{m} g_j(v_j)\right) = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} f_i \pi_i\right)\iota g_j \otimes v_j = 0$.

Proposition 3.2. *The following are equivalent :*

1) *F is faithful.*
2) *U is a generator of $\mathbf{C}$ and $U = SU$.*
3) *$\eta(C)$ is an isomorphism for every $C \in \mathbf{C}$.*

Proof. 1) $\Leftrightarrow$ 2). This is obvious by the equivalence 1) $\Leftrightarrow$ 3) in Lemma 2.2.

3) $\Rightarrow$ 2). Trivial.

2) $\Rightarrow$ 3). Let C be an arbitrary object of $\mathbf{C}$, and c an arbitrary element of C. Since U is a generator of $\mathbf{C}$, there exist $f_1, \dots, f_n \in \operatorname{Hom}_R(U, C)$ and $u_1, \dots, u_n \in U$ such that $c = \sum_{i=1}^{n} f_i(u_i)$. Since $U = SU$, each u_i is writen as $\sum_j s_{ij} u_{ij}$ for some $s_{ij} \in S$ and $u_{ij} \in U$. Then $c = \eta(C)\left(\sum_{i,j} f_i s_{ij} \otimes u_{ij}\right)$. Therefore, $\eta(C)$ is an epimorphism and hence an isomorphism by Lemma 3.1.

Corollary 3.3. *Suppose that F is faithful.*

(1) *$\tilde{F}$ is full.*

(2) *If S is right s-unital then $_SU$ is s-flat.*

Proof. (1) Let ϕ be an arbitrary element of $\mathrm{Hom}_S(F(C), F(D))$ $(C, D \in \mathbf{C})$. We put $f = \eta(D)(\phi \otimes 1_U)\eta(C)^{-1} \in \mathrm{Hom}_R(C, D)$. Then, for any $g \in F(C)$ and any $u \in U$, we have $fg(u) = f\eta(C)(g \otimes u) = \eta(D)(\phi \otimes 1_U)(g \otimes u) = \phi(g)(u)$, which shows $F(f) = \phi$. Thus $\tilde{F}$ is full.

(2) Noting that $\eta(U) : S \underset{S}{\otimes} U_R \to U_R$ is an isomorphism, the assertion is immediate from Lemma 3.1 and Proposition 1.11.

Proposition 3.4. *Suppose that F is exact and S is a right A-s-unital ring for some $A \subseteq S$. Then F preserves direct sums if and only if $a(U)_R$ is finitely generated for every $a \in A$.*

Proof. First, suppose that $a(U)_R$ is finitely generated for every $a \in A$. Let $\{C_\lambda\}_\Lambda$ be an arbitrary family of objects of $\mathbf{C}$, and f an arbitrary element of $F\left(\bigoplus_{\lambda \in \Lambda} C_\lambda\right)$. By Lemma 2.1 and Proposition 1.8, we can choose $a \in A$ with $f = fa$. Since $a(U)_R$ is finitely generated, so is $f(U)_R = fa(U)_R$, and therefore $f(U)$ is contained in some finite direct sum of C_λ's. Hence, by Lemma 2.3, F preserves direct sums. Conversely, suppose that F preserves direct sums. Let a be an arbitrary element of A, and define $\tilde{a} \in F(a(U))$ by $\tilde{a}(u) = a(u)$. There exists an epimorphism $f : \bigoplus_{v \in a(U)} | v)_R \twoheadrightarrow a(U)_R$, where $| v)$ is a submodule of U_R generated by v. Then $\tilde{a} = fg$ for some $g \in F\left(\bigoplus_{v \in a(U)} | v)\right)$. By Lemma 2.3, $g(U)$ is contained in a suitable finite direct sum $V = \bigoplus_{i=1}^{n} | v_i)$ $(v_i \in a(U))$. Hence, $a(U) = fg(U) \subseteq f(V)$, and therefore $a(U) = f(V)$. Thus $a(U)_R$ is finitely generated.

We are now in a position to characterize a subprogenerator of $\mathbf{C}$.

Theorem 3.5. *Let $\mathbf{C}$ be a closed subcategory of $\mathfrak{M}_R$, U an object of $\mathbf{C}$, and F a faithful subfunctor of $\mathrm{Hom}_R(U, -) : \mathbf{C} \to \mathbf{Ab}$. Put $S = F(U)$. Then the following are equivalent :*

1) *(U, F) is a subprogenerator of $\mathbf{C}$.*

2) *S is a right s-unital ring, and the natural homomorphism $\theta(M) : M_S \to F(M \underset{S}{\otimes} U)_S$ for $M \in \mathbf{Mod}_S$ defined by $\theta(M)(m)(u) = m \otimes u$ is an*

isomorphism.

3) *$F(C) \in \mathbf{Mod}_S$ for all $C \in \mathbf{C}$, and $M \otimes_S U = 0$ implies $M = 0$ for $M \in \mathbf{Mod}_S$.*

Proof. 2) ⇒ 1) and 3). Let C be an arbitrary object of $\mathbf{C}$. Since $F(C) \otimes_S F(U) \otimes_S U_R \simeq F(C) \otimes_S U_R \simeq C_R$ by Proposition 3.2, we obtain $F(C)_S \simeq F(F(C) \otimes_S S \otimes_S U)_S$, which is isomorphic to $F(C) \otimes_S S_S$, by assumption. Hence, $F(C) \in \mathbf{Mod}_S$. Thus, $F: \mathbf{C} \to \mathbf{Mod}_S$ and $- \otimes_S U$: $\mathbf{Mod}_S \to \mathbf{C}$ are equivalences.

1) ⇒ 2). By Lemma 2.1, S is right s-unital. It is easy to see that $F(\eta(U))\theta(S) = 1_S$, and therefore $\theta(S)$ is an isomorphism. Now, the functor $F(- \otimes_S U) : \mathbf{Mod}_S \to \mathbf{Mod}_S$ is right exact and preserves direct sums. Hence, θ is a natural equivalence by [2, Theorem 2.6].

3) ⇒ 2). By Corollary 3.3, ${}_SU$ is s-flat, and hence $\theta(M)$ is a monomorphism for any $M \in \mathbf{Mod}_S$. Let M be an arbitrary object of $\mathbf{Mod}_S$, and f: $F(M \otimes_S U)_S \to N_S$ the cokernel of $\theta(M)$. Then, we have an exact sequence

$$0 \to M \otimes_S U_R \xrightarrow{\theta(M) \otimes 1} F(M \otimes_S U) \otimes_S U_R \xrightarrow{f \otimes 1} N \otimes_S U_R \to 0.$$

On the other hand, it is easily seen that $\eta(M \otimes_S U)(\theta(M) \otimes 1_U) = 1_{M \otimes_S U}$, and so $\theta(M) \otimes 1_U$ is an isomorphism. This implies $N \otimes_S U = 0$. By assumption, $N = 0$, and hence $\theta(M)$ is an isomorphism.

The rest of this section, we assume that R is a right s-unital ring and $\mathbf{C} = \mathbf{Mod}_R$. We put $U^* = F(R)$, and define an R-R-homomorphism $(\ ,\ )$: $U^* \otimes_S U \to R$ by $(f, u) = f(u)$ and an S-S-homomorphism $[\ ,\ ]$: $U \otimes_R U^* \to S$ by $[u, f] = u(f, -)$. Then $\{{}_SU_R, {}_RU^*_S, (\ ,\ ), [\ ,\ ]\}$ is a Morita context. Canonically, R and S are regarded as subrings of the generalized matrix ring $T = \begin{pmatrix} S & U \\ U^* & R \end{pmatrix}$. For every right T-module N, we define homomorphisms $\Phi_T(N) : NR_R \to \mathrm{Hom}_S(U^*, N)_R$ by $\Phi_T(N)(n)(f) = n\begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix}$ $(n \in NR, f \in U^*)$ and $\Psi_T(N) : NS_S \to \mathrm{Hom}_R(U, N)_S$ by $\Psi_T(N)(n)(u) = n\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}$ $(n \in NS,\ u \in$

U). In what follows, we shall give conditions, in terms of the Morita context $\{{}_SU_R, {}_RU^*_S, (\ ,\ ), [\ ,\ ]\}$ defined above, for F to be faithful or to be exact and preserve direct sums.

Lemma 3.6. *Let N be a right T-module. If NR (resp. NS) is (U^*, U)-s-unital (resp. $[U, U^*]$-s-unital) then $\Phi_T(N)$ (resp. $\Psi_T(N)$) is a monomorphism and its image is* $\mathrm{Hom}_S(U^*, N)(U^*, U)$ (*resp.* $\mathrm{Hom}_R(U, N)[U, U^*]$) ; *moreover, for any S-submodule (resp. R-submodule) N' of N containing $N\begin{pmatrix}0 & 0\\ U^* & 0\end{pmatrix}$ $\left(\text{resp. } N\begin{pmatrix}0 & U\\ 0 & 0\end{pmatrix}\right)$, $\Phi_T(N)$ (resp. $\Psi_T(N)$) gives an isomorphism $NR_R \xrightarrow{\simeq} \mathrm{Hom}_S(U^*, N')(U^*, U)$ (resp. $NS_S \xrightarrow{\simeq} \mathrm{Hom}_R(U, N')[U, U^*]$).*

Proof. By the definition of $\Phi_T(N)$, $\mathrm{Ker}\ \Phi_T(N) = (\mathrm{Ker}\ \Phi_T(N))(U^*, U) = (\mathrm{Ker}\ \Phi_T(N))\begin{pmatrix}0 & 0\\ U^* & 0\end{pmatrix}\begin{pmatrix}0 & U\\ 0 & 0\end{pmatrix} = 0$. Clearly, $\mathrm{Im}\ \Phi_T(N)$ is contained in $\mathrm{Hom}_S(U^*, N)(U^*, U)$. Conversely, if $\phi \in \mathrm{Hom}_S(U^*, N)$ then

$$\phi(f, u)(g) = \phi((f, u)g) = \phi(f[u, g]) = \phi(f)[u, g]$$
$$= \phi(f)\begin{pmatrix}0 & u\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ g & 0\end{pmatrix} = \Phi_T(N)\left(\phi(f)\begin{pmatrix}0 & u\\ 0 & 0\end{pmatrix}\right)(g)\ (f, g \in U^*\ ;\ u \in U).$$

This shows that $\mathrm{Hom}_S(U^*, NS)(U^*, U) \subseteq \mathrm{Im}\ \Phi_T(N)$. The rest of the proof is clear.

Proposition 3.7. *The following are equivalent :*

1) *F is faithful.*

2) *$(\ ,\ )$ is an isomorphism.*

3) (i) *The natural homomorphism $\alpha(M) : M \otimes_S U_R \to \mathrm{Hom}_S(U^*, M)R_R$ for $M \in \mathfrak{M}_S$ defined by $\alpha(M)(m \otimes u) = m[u, -]$ is an isomorphism.*

(ii) *${}_RU^*$ is faithful and R is regarded as a left ideal of* $\mathrm{End}(U^*_S)$ *via the canonical ring homomorphism.*

Proof. 1) ⇒ 2). $(\ ,\ ) = \eta(R)$ is an isomorphism by Proposition 3.2.

2) ⇒ 1). Canonically, we obtain an epimorphism $U_R^{(U^*)} \twoheadrightarrow U^* \otimes_S U_R \xrightarrow{\simeq} R_R$, and so U is a generator of $\mathbf{Mod}_R$. Also, $U = U(U^*, U) = [U, U^*]U \subseteq SU$, and hence $U = SU$. Thus, by Proposition 3.2, F is faithful.

2) ⇒ 3). (i) Let M be an arbitrary right S-module, and consider

the right T-module $N = \begin{pmatrix} M & M \underset{S}{\otimes} U \\ 0 & 0 \end{pmatrix}$. By Lemma 3.6, $\Phi_T(N)$ gives an isomorphism $M \underset{S}{\otimes} U_R \xrightarrow{\simeq} \mathrm{Hom}_S(U^*, M)R_R$, which is equal to $\alpha(M)$ by definition.

(ii) Put $N = \begin{pmatrix} 0 & 0 \\ U^* & R \end{pmatrix}$, which is a right ideal of T. Then, by Lemma 3.6, $\Phi_T(N)$ gives a monomorphism $R \to \mathrm{End}(U^*_S)$ which is the canonical ring homomorphism determined by the R-module structure of ${}_RU^*$, and Im $\Phi_T(N)$ is a left ideal of $\mathrm{End}(U^*_S)$.

3) ⇨ 2). From the condition (ii), the canonical ring homomorphism $R \to \mathrm{End}(U^*_S)$ gives a ring isomorphism $\lambda: R \xrightarrow{\simeq} \mathrm{End}(U^*_S)R$. Obviously, $\lambda(\ , \ ) = \alpha(U^*)$. Thus $(\ , \ )$ is an isomorphism.

Corollary 3.8. *Suppose that F is faithful.*

(1) $\mathrm{Hom}_S(U^*, -)R : \mathfrak{M}_S \to \mathbf{Mod}_R$ *is exact and preserves direct sums.*

(2) ${}_SU$ *is flat.*

(3) *The S-R-homomorphism* $\alpha : U \to \mathrm{Hom}_S(U^*, S)R$ *defined by* $\alpha(u) = [u, -]$ *is an isomorphism.*

Proof. (1) and (2) are obvious from Proposition 3.7.3).

(3) Obviously, $N = \begin{pmatrix} S & U \\ 0 & 0 \end{pmatrix}$ is a right ideal of T and $\Phi_T(N)(u) = \alpha(u)$ $(u \in U)$. By Proposition 3.7, $(U^*, U) = R$. This implies that α is an isomorphism, by Lemma 3.6.

Lemma 3.9. *If S is a right A-s-unital ring for some $A \subseteq S$ then the following are equivalent :*

1) *F is exact.*

2) *$F(C) \in \mathbf{Mod}_S$ for all $C \in \mathbf{C}$, and for any $a \in A$ there exists a family $\{v_\lambda\}_\Lambda$ of elements of $a(U)$ and a family $\{f_\lambda\}_\Lambda$ of homomorphisms in U^* such that, for each $u \in U$, $a(u)$ is a finite sum $a(u) = \sum_{\lambda \in \Lambda} v_\lambda f_\lambda(u)$, where $f_\lambda(u) = 0$ for almost all $\lambda \in \Lambda$.*

3) *$F = \mathrm{Hom}_R(U, -)S$, and for any $a \in A$ there exists a family $\{v_\lambda\}_\Lambda$ of elements of $a(U)$ and a family $\{g_\lambda\}_\Lambda$ of homomorphisms in $\mathrm{Hom}_R(a(U), R)$ such that, for each $v \in a(U)$, $a(v)$ is a finite sum $a(v) = \sum_{\lambda \in \Lambda} v_\lambda g_\lambda(v)$, where $g_\lambda(v) = 0$ for almost all $\lambda \in \Lambda$.*

Proof. 1) ⇒ 2). By Lemma 2.1, $F(C) \in \mathbf{Mod}_S$ for all $C \in \mathbf{C}$. Next, suppose that $a \in A$ and $\{v_\lambda\}_\Lambda$ is a family of generators of $a(U)_R$. Denote by π_λ the λ-th projection of $R^{(\Lambda)}$, and define an epimorphism $g : R_R^{(\Lambda)} \twoheadrightarrow a(U)_R$ by $g(x) = \sum_{\lambda \in \Lambda} v_\lambda \pi_\lambda(x)$ and $\tilde{a} \in F(a(U))$ by $\tilde{a}(u) = a(u)$. Then, by assumption, there exists $f \in F(R^{(\Lambda)})$ with $\tilde{a} = gf$. One will easily see that $\{v_\lambda\}_\Lambda$ and $\{\pi_\lambda f\}_\Lambda$ have the property requested.

2) ⇒ 3). By the proof of Lemma 2.1, we fave $F = \mathrm{Hom}_R(U, -)S$. Given $a \in A$, we choose $\{v_\lambda\}_\Lambda$ and $\{f_\lambda\}_\Lambda$ as in 2). Then, the family $\{g_\lambda\}_\Lambda$ of restrictions of f_λ to $a(U)$ satisfies the condition requested.

3) ⇒ 1). Obviously, F is left exact. Let $g : M_R \twoheadrightarrow N_R$ be an epimorphism of $\mathbf{Mod}_R$, and f an arbitrary element of $F(N)$. By Proposition 1.8, we can choose $a \in A$ with $f = fa$. For this a, there exist families $\{v_\lambda\}_\Lambda$ and $\{g_\lambda\}_\Lambda$ satisfying the condition in 3). Since g is an epimorphism, there exists $m_\lambda \in M$ such that $g(m_\lambda) = f(v_\lambda)$ for every $\lambda \in \Lambda$. We define a homomorphism $h : U_R \to M_R$ by $h(u) = \sum_{\lambda \in \Lambda} m_\lambda g_\lambda a(u)$. Then, $ha \in F(M)$ and $gha(u) = \sum_{\lambda \in \Lambda} g(m_\lambda) g_\lambda a^2(u) = f\Big(\sum_{\lambda \in \Lambda} v_\lambda g_\lambda a^2(u)\Big) = fa^3(u) = f(u)$ $(u \in U)$. Therefore, we obtain $F(g)(ha) = f$, which shows that $F(g)$ is an epimorphism.

In virtue of Proposition 1.12, as a special case of Lemma 3.9, we readily obtain

Corollary 3.10. *If S is a right s^*-unital ring then F is exact if and only if $F = \mathrm{Hom}_R(U, -)S$ and $e(U)_R$ is s-projective for every idempotent e of S.*

Proposition 3.11. *The following are equivalent :*

1) *F is exact and preserves direct sums.*
2) *$[\ ,\ ]$ is an isomorphism, and $F(M) \in \mathbf{Mod}_S$ for every $M \in \mathbf{Mod}_R$.*
3) *$F \simeq - \otimes_R U^*$ as functors from $\mathbf{Mod}_R$ to $\mathfrak{M}_S$.*

Proof. 3) ⇒ 1). Trivial.

1) ⇒ 2). Let s be an arbitrary element of S. By Lemma 2.1 and Proposition 3.4, $s(U)_R$ is finitely generated. Let $\{v_1, \ldots, v_n\}$ be a set of generators of $s(U)_R$. As was shown in the proof of 1) ⇒ 2) in Lemma 3.9, there exists a family $\{f_1, \ldots, f_n\}$ of elements of U^* such that $s(u) = \sum_{i=1}^{n} v_i f_i(u)$ $(u \in U)$. Therefore $s = \sum_{i=1}^{n} [v_i, f_i]$, and hence $[\ ,\ ]$ is an epimorphism.

Now, we put $\theta = [\ ,\ ]$ and suppose $x \in \operatorname{Ker} \theta$. By Lemma 2.1, we can choose $s \in S$ with $x = xs$. Since θ is an epimorphism, there exists $y \in U \underset{R}{\otimes} U^*$ such that $\theta(y) = s$. Then, by the property of Morita context, $x = x\theta(y) = \theta(x)y = 0$. Thus θ is an isomorphism.

2) $\Rightarrow$ 3). For any $M \in \mathbf{Mod}_R$, we consider the right T-module $N = \begin{pmatrix} M \underset{R}{\otimes} U^* & M \\ 0 & 0 \end{pmatrix}$. Then, by Lemma 3.6, $\Psi_T(N)$ gives a natural isomorphism $M \underset{R}{\otimes} U_S^* \xrightarrow{\simeq} \operatorname{Hom}_R(U, M)S_S = F(M)$.

Proposition 3.7 together with Proposition 3.11 gives the following

Theorem 3.12. *Let R be a right s-unital ring, U_R an s-unital module, and F a subfunctor of* $\operatorname{Hom}_R(U, -) : \mathbf{Mod}_R \to \mathbf{Ab}$. *Then, (U, F) is a sub-progenerator of* $\mathbf{Mod}_R$ *if and only if $F(U)$ is a right s-unital ring, $F =$* $\operatorname{Hom}_R(U, -)F(U)$, *and both $(\ ,\ )$ and $[\ ,\ ]$ are isomorphisms.*

4. Invertible modules. Let R and S be rings with canonical isomorphisms $\mu_R : R \underset{R}{\otimes} R \xrightarrow{\simeq} R$ and $\mu_S : S \underset{S}{\otimes} S \xrightarrow{\simeq} S$, where $\mu_* : x \otimes y \mapsto xy$. An S-R-bimodule U is defined to be *invertible* if ${}_SS \underset{S}{\otimes} U \underset{R}{\otimes} R_R \simeq {}_SU_R$ and there exists an R-S-bimodule V such that ${}_RV \underset{S}{\otimes} U_R \simeq {}_RR_R$, ${}_SU \underset{R}{\otimes} V_S \simeq {}_SS_S$, and ${}_RR \underset{R}{\otimes} V \underset{S}{\otimes} S_S \simeq {}_RV_S$. When this is the case, ${}_RV_S$ is unique up to isomorphism and denoted by U^{-1}. Now, suppose that an S-R-bimodule U and an R-S-bimodule V satisfy ${}_RV \underset{S}{\otimes} U_R \simeq {}_RR_R$ and ${}_SU \underset{R}{\otimes} V_S \simeq {}_SS_S$. Then ${}_SS \underset{S}{\otimes} U_R \simeq {}_SU \underset{R}{\otimes} V \underset{S}{\otimes} U_R \simeq {}_SU \underset{R}{\otimes} R_R$, and it is easy to see that ${}_SU \underset{R}{\otimes} R_R$ is invertible and ${}_RV \underset{S}{\otimes} S_S = (U \underset{R}{\otimes} R)^{-1}$. Let $\mathfrak{N}_R$ (resp. ${}_R\mathfrak{N}$) be the full subcategory of $\mathfrak{M}_R$ (resp. ${}_R\mathfrak{M}$) whose objects are right (resp. left) R-modules M such that $M \underset{R}{\otimes} R_R \simeq M_R$ (resp. ${}_RR \underset{R}{\otimes} M \simeq {}_RM$). Note that if there is an isomorphism $\phi : M \underset{R}{\otimes} R_R \xrightarrow{\simeq} M_R$ (resp. $\phi : {}_RR \underset{R}{\otimes} M \xrightarrow{\simeq} {}_RM$) then the canonical homomorphism $\mu : M \underset{R}{\otimes} R_R \to M_R$ (resp. $\mu : {}_RR \underset{R}{\otimes} M \to {}_RM$) is an isomorphism. In fact, the following diagram is commutative :

$$\begin{array}{ccc} M \underset{R}{\otimes} R \underset{R}{\otimes} R & \xrightarrow{1_M \otimes \mu_R} & M \underset{R}{\otimes} R \\ \phi \otimes 1_R \downarrow \simeq & & \simeq \downarrow \phi \\ M \underset{R}{\otimes} R & \xrightarrow[\mu]{} & M \end{array}$$

Lemma 4.1. *Let R and S be rings with canonical isomorphisms μ_R: $R \underset{R}{\otimes} R \xrightarrow{\simeq} R$ and $\mu_S: S \underset{S}{\otimes} S \xrightarrow{\simeq} S$. If an S-R-bimodule U is invertible then the functors $-\underset{S}{\otimes} U: \mathfrak{N}_S \to \mathfrak{N}_R$ and $U \underset{R}{\otimes} -: {}_R\mathfrak{N} \to {}_S\mathfrak{N}$ are equivalences.*

Proof. It is easy to see that the functor $-\underset{R}{\otimes} U^{-1}: \mathfrak{N}_R \to \mathfrak{N}_S$ gives the inverse of $-\underset{S}{\otimes} U$.

From now on, R and S will represent right s-unital rings. In this case, $\mathfrak{N}_R$ coincides with $\mathbf{Mod}_R$ and an S-R-bimodule U is invertible if and only if $U = UR$ and there exists an R-S-bimodule V such that ${}_RV \underset{S}{\otimes} U_R \simeq {}_RR_R$ and ${}_SU \underset{R}{\otimes} V_S \simeq {}_SS_S$.

Theorem 4.2. *Let R and S be right s-unital rings. Let U be an S-R-bimodule, and $U^* = \mathrm{Hom}_R(U, R)S$. Then the following are equivalent* :

1) ${}_SU_R$ *is invertible.*

2) (i) ${}_SU$ *is faithful and S is regarded as a left ideal of* $\mathrm{End}(U_R)$ *via the canonical ring homomorphism.*

(ii) $(U, \mathrm{Hom}_R(U, -)S)$ *is a subprogenerator of* $\mathbf{Mod}_R$.

3) (i) ${}_SU$ *is faithful and S is regarded as a left ideal of* $\mathrm{End}(U_R)$ *via the canonical ring homomorphism.*

(ii) *The functors* $\mathrm{Hom}_R(U, -)S: \mathbf{Mod}_R \to \mathbf{Mod}_S$ *and* $\mathrm{Hom}_S(U^*, -)R: \mathbf{Mod}_S \to \mathbf{Mod}_R$ *are both faithful.*

Proof. 3) ⇒ 2). Since $\mathrm{Hom}_R(U, -)S$ is faithful, R is regarded as a left ideal of $\mathrm{End}(U_S^*)$ (Proposition 3.7) and ${}_SU_R \simeq {}_S\mathrm{Hom}_S(U^*, S)R_R$ (Corollary 3.8.(3)). These facts and the hypothesis that $\mathrm{Hom}_S(U^*, -)R$ is faithful enable us to apply Corollary 3.8 to U_S^* instead of U_R. Then $\mathrm{Hom}_R(U, -)S$ is exact and preserves direct sums. Hence $(U, \mathrm{Hom}_R(U, -)S)$ is a subprogenerator.

2) ⇒ 1). This is obvious from Theorem 3.12.

1) ⇒ 3). (i) If $s \in S$ annihilates U then $sS \simeq s(U \underset{R}{\otimes} V) = 0$, and hence $s = 0$. Therefore ${}_SU$ is faithful and we may assume that S is a subring of End(U_R). Let ϕ be an arbitrary element of End(U_R), and μ: $S \underset{S}{\otimes} U \to U$ the canonical isomorphism. Since the map $\alpha \mapsto \alpha \otimes 1_U$ from End(S_S) to End$(S \underset{S}{\otimes} U_R)$ is an isomorphism by Lemma 4.1, there exists $\alpha \in$ End(S_S) with $\mu^{-1}\phi\mu = \alpha \otimes 1_U$, namely $\phi = \mu(\alpha \otimes 1_U)\mu^{-1}$. This implies that, for any $s \in S$, $\phi s = \alpha(s) \in S$. Thus S is a left ideal of End(U_R).

(ii) Since $U_R^{(U^{-1})} \twoheadrightarrow U^{-1} \underset{S}{\otimes} U_R \xrightarrow{\simeq} R_R$, U is a generator of $\mathbf{Mod}_R$, and $U = SU$ from $U \simeq S \underset{S}{\otimes} U$. Therefore, Hom$_R(U, -)S$ is faithful by Proposition 3.2. By Proposition 3.7, we have ${}_RU^* \underset{S}{\otimes} U_R \simeq {}_RR_R$, and so ${}_RU_S^* \simeq {}_RU^* \underset{S}{\otimes} S_S \simeq {}_RU^* \underset{S}{\otimes} U \underset{R}{\otimes} U_S^{-1} \simeq {}_RR \underset{R}{\otimes} U_S^{-1} \simeq {}_RU_S^{-1}$. Hence ${}_RU_S^*$ is invertible. Now, the argument employed in proving that Hom$_R(U, -)S$ is faithful enables us to see that Hom$_S(U^*, -)R$ is faithful.

Theorem 4.2 says that an invertible S-R-bimodule U is nothing but a subprogenerator of $\mathbf{Mod}_R$ and Hom$_R(U, R)S = U^{-1}$.

If $\mathbf{Mod}_R$ and $\mathbf{Mod}_S$ are equivalent then we say that R and S are right Morita equivalent. Combining Theorems 2.4, 3.12, and 4.2, we obtain at once

Corollary 4.3. *The following are equivalent :*

1) *R and S are right Morita equivalent.*

2) *There exists an invertible S-R-bimodule.*

3) *There exists a Morita context* $\{{}_SU_R, {}_RV_S, (,), [,]\}$ *such that both* $(,)$ *and* $[,]$ *are isomorphisms.*

4) $\mathbf{Mod}_R$ *has a subprogenerator* (U, F) *with* $F(U) \simeq S$ *as rings.*

Next, we shall show that if ${}_SU_R$ is invertible then ${}_SS_S$ and ${}_RR_R$ have a closed relationship to ${}_SU_R$.

Proposition 4.4. *Suppose that an S-R-bimodule U is invertible.*

(1) *There exists a ring isomorphism* $\tilde{}$: End$({}_SU) \xrightarrow{\simeq}$ End$({}_RR)$ *with* $\phi(ur) = u\tilde{\phi}(r)$ ($\phi \in$ End$({}_SU)$, $u \in U$, $r \in R$), *and* $\widetilde{\text{End}({}_SU_R)} =$ End$({}_RR_R)$.

(2) *There exists uniquely a ring isomorphism* $\hat{}$: End$(U_R) \xrightarrow{\simeq}$ End(S_S)

with $\phi(su) = \hat{\phi}(s)u$ ($\phi \in \mathrm{End}(U_R)$, $u \in U$, $s \in S$), *and* $\widehat{\mathrm{End}({}_SU_R)} = \mathrm{End}({}_SS_S)$.

(3) *If* e *is an idempotent of* $\mathrm{End}({}_SU_R)$ *then both* $\tilde{e}(R)$ *and* $\hat{e}(S)$ *are right s-unital rings and the* $\hat{e}(S)$-$\tilde{e}(R)$*-bimodule* $e(U)$ *is invertible.*

Proof. By Theorems 4.2 and 3.12, we have a Morita context $\{{}_SU_R, {}_RU_S^{-1}, (\ ,\ ), [\ ,\ ]\}$ such that both $(\ ,\ )$ and $[\ ,\ ]$ are isomorphisms.

(1) By Lemma 4.1, the mapping $\phi \mapsto 1_{U^{-1}} \otimes \phi$ from $\mathrm{End}({}_SU)$ to $\mathrm{End}({}_RU^{-1} \underset{S}{\otimes} U)$ is a ring isomorphism. Let $\tilde{\phi} = (\ ,\ )(1_{U^{-1}} \otimes \phi)(\ ,\ )^{-1}$. Then $\tilde{}$ is a ring isomorphism from $\mathrm{End}({}_SU)$ to $\mathrm{End}({}_RR)$ and $\tilde{\phi}((v, u)) = (v, \phi(u))$ $(u \in U, v \in U^{-1})$. This implies that $u\tilde{\phi}(r) = \phi(ur)$ $(u \in U, r \in R)$.

Clearly, $\widetilde{\mathrm{End}({}_SU_R)} \subseteq \mathrm{End}({}_RR_R)$. Let ψ be an arbitrary element of $\mathrm{End}({}_RR_R)$. Then there is $\phi \in \mathrm{End}({}_SU)$ with $\tilde{\phi} = \psi$. For any $u \in U$, choose $a \in R$ with $u = ua$. Then, for any $r \in R$, $\phi(ur) = \phi(uar) = u\psi(ar) = u\psi(a)r = \phi(ua)r = \phi(u)r$, and hence $\phi \in \mathrm{End}({}_SU_R)$.

(2) Using the same method as in (1), we can prove the assertion except the uniqueness. The uniqueness follows from the fact that ${}_SU$ is faithful (Theorem 4.2).

(3) Since $\tilde{e}$ is an idempotent of $\mathrm{End}({}_RR_R)$, $\tilde{e}(R)$ is a direct summand of R as ideal, and hence $\tilde{e}(R)$ is a right s-unital ring. Similarly, $\hat{e}(S)$ is a direct summand of S as ideal. From (1) and (2), we have $e(U) = U\tilde{e}(R) = \hat{e}(S)U$. Now, considering ${}_RU_S^{-1}$ instead of ${}_SU_R$ in (2), we obtain a ring isomorphism $\hat{\hat{}} : \mathrm{End}({}_RU_S^{-1}) \xrightarrow{\sim} \mathrm{End}({}_RR_R)$ such that $r\psi(v) = \hat{\hat{\psi}}(r)v$ ($\psi \in \mathrm{End}({}_RU_S^{-1})$, $r \in R$, $v \in U^{-1}$). Choose $f \in \mathrm{End}({}_RU_S^{-1})$ with $\hat{\hat{f}} = \tilde{e}$. Then $e(U) \underset{\tilde{e}(R)}{\otimes} f(U^{-1}) = e(U) \underset{R}{\otimes} f(U^{-1})$. Since ${}_Se(U)_R$ is a direct summand of ${}_SU_R$ and ${}_Rf(U^{-1})_S$ is a direct summand of ${}_RU_S^{-1}$, we have ${}_Se(U) \underset{R}{\otimes} f(U^{-1})_S \simeq {}_S[e(U), f(U^{-1})]_S = \hat{e}(S)$. Similarly ${}_Rf(U^{-1}) \underset{\hat{e}(S)}{\otimes} e(U)_R \simeq {}_R\tilde{e}(R)_R$. This completes the proof.

Remark. If U_R is faithful then a ring isomorphism $\tilde{}$ in Proposition 4.4.(1) is uniquely determined, and it is easy to see that $\tilde{\phi}(r)vs = rv\hat{\phi}(s)$ ($\phi \in \mathrm{End}({}_SU_R)$, $v \in U^{-1}$, $r \in R$, $s \in S$).

For any S-R-bimodule M, we denote by $L({}_SM_R)$ the lattice of S-R-submodules N of M with $N = SNR$.

Proposition 4.5. *Suppose that an* S-R*-bimodule* U *is invertible.*

(1) *There exists uniquely a lattice isomorphism* $\tilde{}$: $L({}_SU_Z) \xrightarrow{\simeq} L({}_RR_Z)$ *with* $W = U\tilde{W}$ $(W \in L({}_SU_Z))$, *and* $\widetilde{L({}_SU_R)} = L({}_RR_R)$.

(2) *There exists uniquely a lattice isomorphism* $\hat{}$: $L({}_ZU_R) \xrightarrow{\simeq} L({}_ZS_S)$ *with* $W = \hat{W}U$ $(W \in L({}_ZU_R))$, *and* $\widehat{L({}_SU_R)} = L({}_SS_S)$.

(3) $\tilde{W}U^{-1} = U^{-1}\hat{W}$ $(W \in L({}_SU_R))$.

(4) *For any* $W \in L({}_SU_R)$, *the* $S/\hat{W}$-$R/\tilde{W}$-*bimodule* U/W *is invertible.*

Proof. By Theorems 4.2 and 3.12, we have a Morita context $\mathcal{M} = \{{}_SU_R, {}_RU_S^{-1}, (\ ,\ ), [\ ,\ ]\}$ such that both $(\ ,\ )$ and $[\ ,\ ]$ are isomorphisms.

(1) For $W \in L({}_SU_Z)$, we put $\tilde{W} = (U^{-1}, W)$. Then, $U\tilde{W} = U(U^{-1}, W) = [U, U^{-1}]W = W$, and so $R\tilde{W} = (U^{-1}, U)\tilde{W} = (U^{-1}, U\tilde{W}) = (U^{-1}, W) = \tilde{W}$, namely $\tilde{W} \in L({}_RR_Z)$. It is easy to see that $I \mapsto UI$ is the inverse of $\tilde{}$. Therefore, $\tilde{}$ is a lattice isomorphism. The rest of the proof is obvious.

(2) For $W \in L({}_ZU_R)$, we put $\hat{W} = [W, U^{-1}]$. Then the proof of (2) proceeds in the same way as in (1).

(3) For any $W \in L({}_SU_R)$, we have $\tilde{W}U^{-1} = (U^{-1}, W)U^{-1} = U^{-1}[W, U^{-1}] = U^{-1}\hat{W}$.

(4) Let W be an arbitrary element of $L({}_SU_R)$. Put $\bar{R} = R/\tilde{W}$, $\bar{S} = S/\hat{W}$, $\bar{U} = U/W$, and $\bar{V} = U^{-1}/\tilde{W}U^{-1}$. Then $\mathcal{M}$ induces a Morita context $\{{}_{\bar{S}}\bar{U}_{\bar{R}}, {}_{\bar{R}}\bar{V}_{\bar{S}}, \overline{(\ ,\ )}, \overline{[\ ,\ ]}\}$, where $\overline{(\ ,\ )}$ and $\overline{[\ ,\ ]}$ are epimorphisms. Then, by the same argument as in the proof of 1) ⇒ 2) of Proposition 3.11, both $\overline{(\ ,\ )}$ and $\overline{[\ ,\ ]}$ are isomorphisms. This completes the proof.

The next is immediate from Propositions 4.4 and 4.5.

Corollary 4.6. *Suppose that R and S are right Morita equivalent.*

(1) *There exists a ring isomorphism* $\tilde{}$: $\mathrm{End}({}_RR_R) \xrightarrow{\simeq} \mathrm{End}({}_SS_S)$ *such that, for each idempotent e of* $\mathrm{End}({}_RR_R)$, $e(R)$ *and* $\tilde{e}(S)$ *are right Morita equivalent.*

(2) *There exists a lattice isomorphism* $\hat{}$: $L({}_RR_R) \xrightarrow{\simeq} L({}_SS_S)$ *such that, for each* $I \in L({}_RR_R)$, R/I *and* $S/\hat{I}$ *are right Morita equivalent.*

If R has an idempotent e with $R = ReR$ then eR is a small (finitely generated) projective generator of $\mathbf{Mod}_R$, and hence R is right Morita equivalent to eRe (see, e.g., Proposition 2.6). Concerning the converse of this fact, we have the following

Proposition 4.7. *Let A be a subset of R. If R is a right A-s-unital ring and R is right Morita equivalent to a ring S with identity element, then there exists $a \in A$ with $R = RaR$.*

Proof. Since S is a small projective generator of $\mathbf{Mod}_S$, $\mathbf{Mod}_R$ also has a small projective generator U_R. By Proposition 3.4, U_R has a finite set of generators $\{u_1, \ldots, u_n\}$. Then there exists $a \in A$ with $u_i = u_i a$ for all i, by Corollary 1.9. Therefore $U = UaR$. Since, ${}_RR_R \simeq {}_R\mathrm{Hom}_R(U, R) \otimes_{\mathrm{End}(U_R)} U_R$ by Proposition 3.7, we get $R = RaR$.

5. **Quotient rings of right s-unital rings.** Recently, by making use of Gabriel's method, M. Parvathi and P. R. Adhikari [7] constructed quotient rings of rings with local units, and proved that quotient rings of two Morita equivalent rings with local units are again Morita equivalent. We shall generalize this result to right s-unital rings. In this section too, R and S will represent right s-unital rings. For any ring T and any subset $A \subseteq T$, we write A_L for the set of all left multiplications effected by elements in A.

A non-empty set $\mathfrak{F}$ of right ideals of R is called a *right Gabriel topology* on R, if the following conditions are satisfied :

(1) If $I \in \mathfrak{F}$ and $r \in R$ then $r^{-1}I = \{a \in R \mid ra \in I\} \in \mathfrak{F}$.

(2) If I is a right ideal and there exists $J \in \mathfrak{F}$ such that $r^{-1}I \in \mathfrak{F}$ for every $r \in J$, then $I \in \mathfrak{F}$.

It is clear that any intersection of right Gabriel topologies is a right Gabriel topology, and so the right Gabriel topologies on R form a complete lattice $\mathbf{Top}(R)$. The proof of the next lemma is quite similar to that of the same given in [5] when R has an identity element.

Lemma 5.1. *Every right Gabriel topology $\mathfrak{F}$ on R determines a localizing subcategory $\mathbf{L}(\mathfrak{F})$ of $\mathbf{Mod}_R$ consisting of all s-unital right R-modules M such that for each $u \in M$ there exists $I \in \mathfrak{F}$ with $uI = 0$. Conversely, every localizing subcategory $\mathfrak{L}$ of $\mathbf{Mod}_R$ determines a right Gabriel topology $\mathbf{T}(\mathfrak{L}) = \{I_R \subseteq R_R \mid R/I \in \mathfrak{L}\}$. Moreover, $\mathbf{TL}(\mathfrak{F}) = \mathfrak{F}$, $\mathbf{LT}(\mathfrak{L}) = \mathfrak{L}$, and all the localizing subcategories of $\mathbf{Mod}_R$ form a complete lattice $\mathbf{Loc}(R)$ which is isomorphic to $\mathbf{Top}(R)$.*

Now, we shall construct a quotient ring of an s-unital ring R with respect to a right Gabriel topology $\mathfrak{F}$ on R. By Lemma 5.1, $\mathfrak{F}$ determines a localizing subcategory $\mathfrak{L}$ of $\mathbf{Mod}_R$. Put $\mathbf{C} = \mathbf{Mod}_R/\mathfrak{L}$, the quotient category, and let Q: $\mathbf{Mod}_R \to \mathbf{C}$ be the canonical functor. Then $Q(R_L)$ is a subring of $\mathrm{Hom}_{\mathbf{C}}(Q(R), Q(R))$. Define $R_{\mathfrak{F}} = Q(R_L)\mathrm{Hom}_{\mathbf{C}}(Q(R), Q(R))Q(R_L)$. We shall call $R_{\mathfrak{F}}$ the *quotient ring of R with respect to* $\mathfrak{F}$. If R is a right (resp. left) A-s-unital ring for a subset A of R, then $R_{\mathfrak{F}}$ is a right (resp. left) $Q(A_L)$-s-unital ring.

We are now in a position to generalize the main result of [7] as follows :

Theorem 5.2. *If R and S are right Morita equivalent, then there exists a lattice isomorphism* $\tilde{}: \mathbf{Top}(R) \xrightarrow{\simeq} \mathbf{Top}(S)$ *such that $R_{\mathfrak{F}}$ and $S_{\tilde{\mathfrak{F}}}$ are right Morita equivalent* ($\mathfrak{F} \in \mathbf{Top}(R)$).

Proof. Let G: $\mathbf{Mod}_S \to \mathbf{Mod}_R$ be an equivalence. Then G induces a lattice isomorphism $\mathbf{Loc}(S) \xrightarrow{\simeq} \mathbf{Loc}(R)$. Combining this with Lemma 5.1, we obtain a lattice isomorphism $\tilde{}: \mathbf{Top}(R) \xrightarrow{\simeq} \mathbf{Top}(S)$. Now, let $\mathfrak{F}$ be a right Gabriel topology on R, and $\mathfrak{L}$ (resp. $\tilde{\mathfrak{L}}$) the localizing subcategory corresponding to $\mathfrak{F}$ (resp. $\tilde{\mathfrak{F}}$) under the correspondence given in Lemma 5.1. Consider the quotient categories $\mathbf{C} = \mathbf{Mod}_R/\mathfrak{L}$ with the canonical functor P : $\mathbf{Mod}_R \to \mathbf{C}$, and $\mathbf{D} = \mathbf{Mod}_S/\tilde{\mathfrak{L}}$ with the canonical functor Q: $\mathbf{Mod}_S \to \mathbf{D}$. Then, $G(\tilde{\mathfrak{L}}) = \mathfrak{L}$, and there exists uniquely a functor $\bar{G}$: $\mathbf{D} \to \mathbf{C}$ such that $PG = \bar{G}Q$:

$$\begin{array}{ccc} \mathbf{Mod}_S & \xrightarrow{G} & \mathbf{Mod}_R \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle P} \\ \mathbf{D} & \xrightarrow[\bar{G}]{} & \mathbf{C} \end{array}$$

As is easily seen, $\bar{G}$ is an equivalence, and therefore $\bar{G}$ induces a ring isomorphism $\mathrm{Hom}_{\mathbf{D}}(Q(S), Q(S)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{C}}(\bar{G}Q(S), \bar{G}Q(S))$. Hence, we have $S_{\tilde{\mathfrak{F}}} \simeq \bar{G}(S_{\tilde{\mathfrak{F}}})$, and also $\bar{G}(S_{\tilde{\mathfrak{F}}}) = \bar{G}Q(S_L)\mathrm{Hom}_{\mathbf{C}}(\bar{G}Q(S), \bar{G}Q(S))\bar{G}Q(S_L)$ $= PG(S_L)\mathrm{Hom}_{\mathbf{C}}(PG(S), PG(S))PG(S_L)$. Put $U = PG(S_L)\mathrm{Hom}_{\mathbf{C}}(P(R), PG(S))P(R_L)$ and $V = P(R_L)\mathrm{Hom}_{\mathbf{C}}(PG(S), P(R))PG(S_L)$. Then U is a $\bar{G}(S_{\tilde{\mathfrak{F}}})$-$R_{\mathfrak{F}}$-bimodule with $U = UR_{\mathfrak{F}}$, and V is an $R_{\mathfrak{F}}$-$\bar{G}(S_{\tilde{\mathfrak{F}}})$-bimodule with $V = V\bar{G}(S_{\tilde{\mathfrak{F}}})$. As was shown in the proof of Theorem 2.4, $(G(S)$, $\mathrm{Hom}_R(G(S), -)G(S_L))$ is a subprogenerator of $\mathbf{Mod}_R$. By Proposition 3.7.2) and Lemma 1.3, $R_L = \mathrm{Hom}_R(G(S), R)G(S_L)\mathrm{Hom}_R(R, G(S))R_L$, and so

$P(R_L) \subseteq \mathrm{Hom}_C(PG(S), P(R))PG(S_L)\mathrm{Hom}_C(P(R), PG(S))P(R_L)$. Then we can see that $VU = R_{\mathfrak{F}}$, and similarly $UV = \bar{G}(S_{\bar{\mathfrak{F}}})$. Hence, we have a Morita context $\{{}_{\bar{G}(S_{\bar{\mathfrak{F}}})}U_{R_{\mathfrak{F}}}, {}_{R_{\mathfrak{F}}}V_{\bar{G}(S_{\bar{\mathfrak{F}}})}, (,), [,]\}$ such that (,) and [,] are epimorphisms. As noted in the proof of Proposition 4.5.(4), (,) and [,] are isomorphisms. Thus $R_{\mathfrak{F}}$ and $\bar{G}(S_{\bar{\mathfrak{F}}})$ are right Morita equivalent (Corollary 4.3). This completes the proof.

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