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A RING TOPOLOGY BASED ON SUBMODULES OVER AN ASANO ORDER OF A RING

Dedicated to Prof. Hisao Tominaga on his 60th birthday

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1. Introduction. Let R be a ring with unity quantity and $\mathfrak{O}$ a regular (bounded) Asano order of R. Throughout the term $\mathfrak{O}$-*submodule* means a two-sided $\mathfrak{O}$-submodule of R which contains at least one regular element of R. A ring topology $\mathbf{T}$ on R is called here a *ring* $\mathfrak{O}$-*topology* if a set of open $\mathfrak{O}$-submodules of R forms a fundamental system of *nbd* (neighborhood) of zero for $\mathbf{T}$. The aim of this short note is to describe explicitly the weakest ring $\mathfrak{O}$-topology for which any given $\mathfrak{O}$-submodule is open, and utilize it for characterization of some special ring topologies.

2. Factorization of $\mathfrak{O}$-Submodules. Let $\mathfrak{G} = \{\mathfrak{a}, \mathfrak{b}, \dots\}$ be the set of all two-sided $\mathfrak{O}$-ideals in R, and $\mathfrak{P} = \{\mathfrak{p}, \mathfrak{q}, \dots\}$ the set of all prime ideals of $\mathfrak{O}$. Then since $\mathfrak{O}$ is Asano, every $\mathfrak{a}$ in $\mathfrak{G}$ has a unique factorization in the form

$$\mathfrak{a} = \prod_{\mathfrak{p} \in \mathfrak{P}} \mathfrak{p}^{\nu(\mathfrak{p}, \mathfrak{a})}$$

where $\nu(\mathfrak{p}, \mathfrak{a}) \in \mathbf{Z}$ (the integers) and $\nu(\mathfrak{p}, \mathfrak{a}) = 0$ for almost all $\mathfrak{p} \in \mathfrak{P}$ ([1], [3]). Let M be an $\mathfrak{O}$-submodule of R. We define

$$\nu(\mathfrak{p}, M) := \inf\{\nu(\mathfrak{p}, \mathfrak{a})\ ;\ \mathfrak{a} \subseteq M, \mathfrak{a} \in \mathfrak{G}\},$$

and prepare the following notations :

$$\begin{aligned} &\mathrm{P}_+(M) := \{\mathfrak{p} \in \mathfrak{P}\ ;\ \nu(\mathfrak{p}, M) > 0\} \\ &\mathrm{P}_-(M) := \{\mathfrak{p} \in \mathfrak{P}\ ;\ 0 > \nu(\mathfrak{p}, M) > -\infty\} \\ &\mathrm{P}_{-\infty}(M) := \{\mathfrak{p} \in \mathfrak{P}\ ;\ \nu(\mathfrak{p}, M) = -\infty\} \\ &\mathrm{P}(M) := \mathfrak{P} \backslash \mathrm{P}_{-\infty}(M). \end{aligned}$$

Then we have a unique factorization of M as follows ([5], [6]) :

$$M = \prod_{\mathfrak{p} \in \mathrm{P}_+(M)} \mathfrak{p}^{\nu(\mathfrak{p}, M)} \cdot \sum_{\mathfrak{p} \in \mathrm{P}_-(M)} \mathfrak{p}^{\nu(\mathfrak{p}, M)} \cdot \mathfrak{O}_{\mathrm{P}(M)}$$

where $\mathfrak{O}_{\mathrm{P}(M)}$ is the $\mathrm{P}(M)$-component of $\mathfrak{O}$ ([1]). Conversely, let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be any three subsets of $\mathfrak{P}$ such that they are mutually disjoint and $\mathbf{A}$ is finite. Next we choose an arbitrary positive integer $\alpha(\mathfrak{p})$ for each $\mathfrak{p} \in \mathbf{A}$ (when $\mathbf{A}$ is not vacuous), and an arbitrary negative integer $\beta(\mathfrak{p})$ for each $\mathfrak{p} \in \mathbf{B}$ (when $\mathbf{B}$ is not vacuous). Then

$$N := \prod_{\mathfrak{p}\in \mathbf{A}} \mathfrak{p}^{\alpha(\mathfrak{p})} \cdot \sum_{\mathfrak{p}\in \mathbf{B}} \mathfrak{p}^{\beta(\mathfrak{p})} \cdot \mathfrak{O}_{\mathfrak{p}\setminus \mathbf{C}}$$

is an $\mathfrak{O}$-submodule of R, and we have $\mathbf{A} = \mathbf{P}_{+}(N)$, $\mathbf{B} = \mathbf{P}_{-}(N)$, $\mathbf{C} = \mathbf{P}_{-\infty}(N)$; $\alpha(\mathfrak{p}) = \nu(\mathfrak{p}, N)$ for $\mathfrak{p} \in \mathbf{A}$ and $\beta(\mathfrak{p}) = \nu(\mathfrak{p}, N)$ for $\mathfrak{p} \in \mathbf{B}$ ([6]).

For a real number r, $s(r)$ will denote the integer n such that $r \leq n < r+1$. We now consider an $\mathfrak{O}$-submodule

$$(M\,;\,n) := \sum_{\mathfrak{p}\in \mathbf{P}_{-}(M)} \mathfrak{p}^{s(2^{-n}\nu(\mathfrak{p},M))} \cdot \mathfrak{O}_{\mathbf{P}(M)}$$

for any $\mathfrak{O}$-submodule M and any positive integer n. Then evidently $(M\,;\,n)$ contains $\mathfrak{O}_{\mathbf{P}(M)}$ and 1) $(M\,;\,n) \supseteq (M\,;\,n+1)$, 2) $(M\,;\,n) \supseteq (M\,;\,n+1)\cdot(M\,;\,n+1)$ and 3) $((M\,;\,m)\,;\,n) = (M\,;\,m+n)$ for $m, n \in \mathbf{Z}_0$ (the non-negative integers).

3. A Ring $\mathfrak{O}$-Topology. Let $\mathfrak{N}$ be a set of $\mathfrak{O}$-submodules of R with the following three conditions :

(1°) $M_1, M_2 \in \mathfrak{N} \Rightarrow \exists N \in \mathfrak{N}$ *such that* $N \subseteq M_1 \cap M_2$.

(2°) $\mathfrak{a} \in \mathfrak{G}$, $M \in \mathfrak{N} \Rightarrow \exists N \in \mathfrak{N}$ *such that* $\mathfrak{a}N \subseteq M$.

(3°) $M \in \mathfrak{N} \Rightarrow \exists N \in \mathfrak{N}$ *such that* $NN \subseteq M$.

Since regularity (boundedness) is assumed for $\mathfrak{O}$, by using SATZ 1.5 in [1] we can see that the condition (2°) is equivalent to the following each condition :

(2_l) $a \in R$, $M \in \mathfrak{N} \Rightarrow \exists N \in \mathfrak{N}$ *such that* $aN \subseteq M$.

(2_r) $a \in R$, $M \in \mathfrak{N} \Rightarrow \exists N \in \mathfrak{N}$ *such that* $Na \subseteq M$.

Thus $\mathfrak{N}$ is a fundamental system of *nbd* of zero for a ring topology on R ([4], [7]). $\mathbf{T}(\mathfrak{N})$ will denote the topology determined by the above $\mathfrak{N}$.

Theorem. *Let M be an $\mathfrak{O}$-submodule of R, and $\mathfrak{N}(M)$ the set of all $\mathfrak{O}$-submodules N's satisfying*

(1) $\mathbf{P}(N) = \mathbf{P}(M)$,

(2) *there is* $n \in \mathbf{Z}_0$ *such that* $\nu(\mathfrak{p}, N) = s(2^{-n}\nu(\mathfrak{p}, M)) + 1$ *for almost all* $\mathfrak{p} \in \mathbf{P}(M)$.

Then $\mathfrak{N}(M)$ has the three conditions (1°), (2°), (3°) *and* $\mathbf{T}(M) := \mathbf{T}(\mathfrak{N}(M))$ *is the weakest ring $\mathfrak{O}$-topology among the set of all ring $\mathfrak{O}$-topologies on R, for which M is open.*

Proof. We can show easily that an $\mathfrak{O}$-submodule N of R is a member of $\mathfrak{N}(M)$ if and only if N is representable as $N = \mathfrak{a}\cdot(M\,;\,n)$ for $\mathfrak{a} \in \mathfrak{G}$ and $n \in \mathbf{Z}_0$. In particular we have $M \in \mathfrak{N}(M)$ by the factorization of M in Section 2. Now let $N = \mathfrak{a}\cdot(M\,;\,n)$ and $N' = \mathfrak{b}\cdot(M\,;\,m)$ be any two members of $\mathfrak{N}(M)$. Suppose that $n \geq m$. Then we have (1) $(\mathfrak{a} \cap \mathfrak{b})\cdot(M\,;\,n) \subseteq \mathfrak{a}\cdot(M\,;\,n) \cap \mathfrak{b}\cdot(M\,;\,m) = N \cap N'$, (2) $\mathfrak{c}(\mathfrak{c}^{-1}\mathfrak{a}\cdot(M\,;\,n)) = N$ for $\mathfrak{c}$ in $\mathfrak{G}$, and (3) $N = \mathfrak{a}\cdot(M\,;\,n) \supseteq (\mathfrak{a} \cap \mathfrak{O})\cdot(M\,;\,n) \supseteq (\mathfrak{a} \cap \mathfrak{O})\cdot(M\,;\,n+1)\cdot(\mathfrak{a} \cap \mathfrak{O})\cdot(M\,;\,n+1)$, since the multiplication of $\mathfrak{O}$-modules is commutative. Accordingly $\mathbf{T}(M)$ is a ring $\mathfrak{O}$-topology for which M is open. Next we let $\mathfrak{N}$ be an arbitrary set of $\mathfrak{O}$-submodules which satisfies the conditions (1°), (2°), (3°) and M is open for $\mathbf{T}(\mathfrak{N})$. Then we may assume that M is a member of $\mathfrak{N}$. Taking $N_n \in \mathfrak{N}$ such that $N_n^{2^n} \subseteq M$, we have $\nu(\mathfrak{p}, N_n) = 2^n\nu(\mathfrak{p}, N_n) \geq \nu(\mathfrak{p}, M)$. Hence $\nu(\mathfrak{p}, N_n) \geq s(2^{-n}\nu(\mathfrak{p}, M))+1$ for all $\mathfrak{p} \in \mathfrak{P}$. This implies $N_n \subseteq (M\,;\,n)$. Take a member $\mathfrak{a}\cdot(M\,;\,n)$ of $\mathfrak{N}(M)$. Then choosing L in $\mathfrak{N}$ with $\mathfrak{a}^{-1}L \subseteq N_n$ we obtain $L \subseteq \mathfrak{a}N_n \subseteq \mathfrak{a}\cdot(M\,;\,n)$. $\mathbf{T}(M)$ is therefore weaker than $\mathbf{T}(\mathfrak{N})$ as desired.

4. Special Ring $\mathfrak{O}$-Topologies. (a) Let $\mathbf{T}$ be a ring topology on R. A subset X of R is *bounded* for $\mathbf{T}$, if for any *nbd* U of zero there is an *nbd* V of zero such that $xv \in U$ and $vx \in U$ for every $x \in X$ and for every $v \in V$. $\mathbf{T}$ is called *locally bounded* if there is a bounded *nbd* of zero for T. Then we can prove that the following four conditions are equivalent.

1) $\mathbf{T}(M)$ *is locally bounded.*
2) $(M\,;\,n)$ *is the* $\mathrm{P}(M)$*-component of* $\mathfrak{O}$ *for some* $n \in \mathbf{Z}_0$.
3) $\mathbf{T}(M) = \mathbf{T}(\mathfrak{O}_{\mathrm{P}(M)})$.
4) $\{\nu(\mathfrak{p}, M)\,;\,\mathfrak{p} \in \mathfrak{P}\}$ *is bounded on* $\mathrm{P}(M)$.

Then we can prove the following

Statement 1. *A ring $\mathfrak{O}$-topology $\mathbf{T}$ on R is locally bounded if and only if $\mathbf{T} = \mathbf{T}(\mathfrak{O}_{\mathrm{P}})$ for a subset $\mathbf{P}$ of $\mathfrak{P}$.*

(b) In this and next paragraphs we assume that $\mathfrak{O}$ is a Dedekind domain and R is the quotient field of $\mathfrak{O}$. Unity quantity of $\mathfrak{O}$ will be denoted by e. Now let $\mathbf{T}(\mathfrak{N})$ be a field $\mathfrak{O}$-topology, that is, $\mathfrak{N}$ has the above conditions (1°), (2°), (3°) and

(4°) $M \in \mathfrak{N} \Rightarrow \exists N \in \mathfrak{N}$ *such that* $(N+e)^{-1} \subseteq M+e$.

Then we can prove that $\mathbf{T}(\mathfrak{O}_{\mathfrak{p}})$ is the $\mathfrak{p}$-adic topology for each $\mathfrak{p} \in \mathfrak{P}$, and generalize Theorem 3 in [2] as follows :

Statement 2. $\mathrm{T}(\mathfrak{N})$ *is a field* $\mathfrak{O}$*-topology on* R *if and only if* $\mathrm{T}(\mathfrak{N})$ *is a supremum of a family of* $\mathfrak{p}$*-adic topologies.*

(c) A ring topology $\mathbf{T}$ on a field is called a *V-topology*, if for any *nbd* U of zero $(U^c)^{-1}$ is bounded for $\mathbf{T}$, where U^c is the complement of U in the field ([4]). Let $\mathrm{T}(\mathfrak{N})$ be a ring $\mathfrak{O}$-topology on R. Then we can show that $\mathrm{T}(\mathfrak{N})$ is a V-topology if and only if there is an ideal $\mathfrak{a}$ such that $\mathfrak{a}\cdot(M^c)^{-1} \subseteq N$ for any two $\mathfrak{O}$-submodules M and N of R. Now let $\mathbf{T}$ be a non-trivial ring $\mathfrak{O}$-topology on R. If we suppose that $\mathbf{T}$ is a V-topology, then $\mathbf{T} = \mathrm{T}(\mathfrak{O}_\mathfrak{p})$ for some $\mathfrak{p} \in \mathfrak{P}$. For, there is a subset $\mathbf{P}$ of $\mathfrak{P}$ such that $\mathbf{T} = \mathrm{T}(\mathfrak{O}_\mathrm{P})$ by Statement 1. If $\mathbf{P}$ contains two different members $\mathfrak{p}$ and $\mathfrak{q}$, then $\mathfrak{O}_\mathfrak{p} \cap \mathfrak{O}_\mathfrak{q} \subseteq \mathfrak{N}(\mathfrak{O}_\mathrm{P})$. However for any ideal $\mathfrak{a}$ we have $\mathfrak{a}((\mathfrak{O}_\mathfrak{p} \cap \mathfrak{O}_\mathfrak{q})^c)^{-1} \subseteq \mathfrak{O}_\mathfrak{p} \cap \mathfrak{O}_\mathfrak{q}$. This is a contradiction. Thus we obtain the following

Statement 3. *A non-trivial ring* $\mathfrak{O}$*-topology* $\mathbf{T}$ *on* R *is a V-topology if and only if* $\mathbf{T}$ *is a* $\mathfrak{p}$*-adic topology for some* $\mathfrak{p} \in \mathfrak{P}$.

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