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AN ALGEBRAIC PROOF OF A THEOREM OF WARFIELD ON ALGEBRAICALLY COMPACT MODULES

Dedicated to Professor Hisao Tominaga on his 60th birthday

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Let R be a ring with unit element. A (unital) left R-module Q is called *pure-injective* if, for any left R-module A and a pure submodule B of A, every homomorphism $B \to Q$ can be extended to a homomorphism $A \to Q$, or equivalently, if, for any left R-module M which contains Q as a pure submodule, Q is a direct summand of M. Warfield proved in [4, Theorem 2] that the notion of pure-injective R-modules coincides with the notion of algebraically compact R-modules, where a left R-module Q is called *algebraically compact* if, given a row-finite $I \times J$-matrix $[a_{ij}]$ over R and a vector $[q_i]$ in Q^I, the system of linear equations $\sum_j a_{ij}x_j = q_i$ $(i \in I)$ is solvable in Q whenever it is finitely solvable in Q. Warfield gave an elegant proof to the theorem by using the theory of compact Abelian groups. Since however both the notions are purely algebraic, it is desirable to prove the theorem without using the topological concept of compactness. In the following, we shall give an algebraic proof to the theorem. Our proof seems somewhat lengthy, but the point is to prove that, for any right R-module M, $M^* = \mathrm{Hom}_Z(M, U)$ is always an algebraically compact left R-module, where U is the factor group of the additive group $\boldsymbol{Q}$ of rational numbers modulo its subgroup $\boldsymbol{Z}$ of integers.

Proposition 1. *Let Q be a left R-module. Then the following conditions are equivalent:*

(1) *Q is algebraically compact;*

(2) *If A is a left R-module, B a submodule of A and $h: B \to Q$ a homomorphism such that, for any finitely generated submodule B_0 of B, the restriction of h to B_0 can be extended to a homomorphism $A \to Q$, then h itself can be extended to a homomorphism $A \to Q$;*

(3) *The condition (2) holds for all free left R-modules A;*

(4) *If A, B are left R-modules, $\eta: B \to A$ a homomorphism and $h: B \to Q$ a homomorphism such that, for any finitely generated submodule B_0 of B, there exists a homomorphism $f_0: A \to Q$ for which the restrictions of $f_0 \circ \eta$ and*

h to B_0 coincide, then there exists a homomorphism f: $A \to Q$ such that $f \circ \eta = h$.

Proof. (4) ⇒ (2) and (2) ⇒ (3) are clear. (2) ⇒ (4) : Assume (2), and let A, B, η, h be modules and homomorphisms satisfying the assumption in (4). Let K be the kernel of η, and c an arbitrary element of K. Then there exists a homomorphism $g: A \to Q$ such that the restrictions of $g \circ \eta$ and h to the cyclic submodule Rc coincide, i. e., $g(\eta(c)) = h(c)$. But since $\eta(c) = 0$, it follows $h(c) = 0$. Thus K is contained in the kernel of h, and so h can be regarded as a homomorphism $B/K \to Q$. On the other hand, B/K can be regarded as a submodule of A by identifying the coset $b+K$ with $\eta(b)$ for each $b \in B$. Let $(B_0+K)/K$ be any finitely generated submodule of B/K, where B_0 is a suitable finitely generated submodule of B. Then there exists a homomorphism $f_0 : A \to Q$ such that the restrictions of $f_0 \circ \eta$ and h to B_0 coincide, or what is the same thing, f_0 is an extension of the restriction of $h: B/K \to Q$ to $(B_0+K)/K$. Therefore, by our assumption, h can be extended to a homomorphism f: $A \to Q$, which means that $f \circ \eta = h$.

(3) ⇒ (2) : Assume (3). Let A be an arbitrary left R-module, B a submodule of A and $h: B \to Q$ a homomorphism satisfying the assumption in (2). Let F be a free left R-module having an epimorphism $\pi: F \to A$. Let G be the inverse image $\pi^{-1}(B)$ of B and ρ the restriction of π to G. Then ρ is an epimorphism $G \to B$. Let G_0 be a finitely generated submodule of G. Then $B_0 = \rho(G_0)\ (= \pi(G_0))$ is a finitely generated submodule of B. Let h_0 be the restriction of h to B_0. Then h_0 can be extended to a homomorphism $g_0 : A \to Q$. Let ρ_0 be the restriction of ρ to G_0. Then $h_0 \circ \rho_0 : G_0 \to Q$ is the restriction of $h \circ \rho : G \to Q$ to G_0, while $g_0 \circ \pi: F \to Q$ is an extension of $h_0 \circ \rho_0$. Therefore, by assumption, $h \circ \rho$ can be extended to a homomorphism f: $F \to Q$. If K denotes the kernel of π then $K \subset G$ and so we have $f(K) = h(\rho(K)) = h(0) = 0$. Since π is an epimorphism, this implies the existence of a homomorphism $g: A \to Q$ such that $g \circ \pi = f$. Since ρ is an epimorphism, this implies that g is an extension of h.

(1) ⇒ (3) : Let F be a free left R-module. We may assume that $F = R^{(J)}$ for some set J. Let $[x_j]$ be any (column) vector in Q^J. Then, by associating each (row) vector $(r_j) \in F$ with $(r_j)[x_j] = \sum r_j x_j \in Q$, we have a homomorphism $F \to Q$. Conversely, let f: $F \to Q$ be any homomorphism. Then, as is well-known, there is a unique vector $[x_j] \in Q^J$ such that $f(r_j) = (r_j)[x_j]$ for all $(r_j) \in F$. Let G be a submodule of F, and let $g: G \to Q$ be a homomorphism such that its restriction to any finitely generated submodule of G is

extended to a homomorphism $F \to Q$. Let $\{\mu_i \mid i \in I\}$ be a system of generators of G, and denote by a_{ij} the j-th entry of μ_i. Let $\mu = [a_{ij}]$ be the row-finite $I \times J$-matrix over R whose (i, j)-entry is a_{ij}, or what is the same thing, whose i-th row is μ_i. Put further $q_i = g(\mu_i)$ for each $i \in I$. Now assume (1). Let I_0 be any finite subset of I, and let G_0 be the finitely generated submodule of G generated by $\{\mu_i \mid i \in I_0\}$. Then the restriction of g to G_0 is extended to a homomorphism $f_0 : F \to Q$. Let $[x_j^0] \in Q^J$ be the vector corresponding to f_0. Then it satisfies $\mu_i[x_j^0] = \sum a_{ij}x_j^0 = q_i$ for all $i \in I_0$. Since Q is algebraically compact, there exists a vector $[x_j] \in Q^J$ such that $\mu_i[x_j] = \sum a_{ij}x_j^0 = q_i$ for all $i \in I$, or equivalently, $\mu[x_j] = [q_i]$. Let $f: F \to Q$ be the homomorphism corresponding to $[x_j]$. Then we have $f(\mu_i) = q_i$ for all $i \in I$. This means that f is an extension of g, since G is generated by $\{\mu_i \mid i \in I\}$ and $g(\mu_i) = q_i$ for all $i \in I$.

(3)$\Rightarrow$(1) : Let $\mu = [a_{ij}]$ be a row-finite $I \times J$-matrix over R and $[q_i]$ a vector in Q^I such that the equation $\mu[x_j] = [q_i]$, i.e., the system of linear equations $\sum a_{ij}x_j = q_i$ for $i \in I$, is finitely solvable in Q. Consider the free left R-module $F = R^{(J)}$ and its submodule G generated by $\{\mu_i \mid i \in I\}$, where μ_i is the i-th row of μ. Let (r_i) be a vector in $R^{(I)}$ such that $\sum_{i \in I} r_i\mu_i = (r_i)\mu = 0$. There exists a finite subset I' of I such that $r_i = 0$ whenever $i \notin I'$ (since $(r_i) \in R^{(I)}$). Then the last equality actually means that $\sum_{i \in I'} r_i\mu_i = 0$. There exists however a vector $[x_j] \in Q^J$ such that $\mu_i[x_j] = q_i$ for all $i \in I'$. Then we have $\sum_{i \in I} r_iq_i = \sum_{i \in I'} r_iq_i = \sum_{i \in I'} r_i\mu_i[x_j] = 0$. This fact implies that there exists a unique homomorphism $g: G \to Q$ such that $g(\mu_i) = q_i$ for all $i \in I$. Let I_0 be any finite subset of I, and let G_0 be the finitely generated submodule of G generated by $\{\mu_i \mid i \in I_0\}$. Then there exists a vector $[x_j^0] \in Q^J$ such that $\mu_i[x_j^0] = q_i$ for all $i \in I_0$. Let $f_0 : F \to Q$ be the homomorphism corresponding to $[x_j^0]$. Then we have $f_0(\mu_i) = q_i$ for all $i \in I_0$. This shows that f_0 is an extension of the restriction of g to G_0. If we notice that every finitely generated submodule of G is contained in G_0 for a suitable (finite subset) I_0, we know that the restriction of g to any finitely generated submodule of G is extendable to a homomorphism $F \to Q$. Now assume (3). Then g can be extended to a homomorphism $f: F \to Q$. Let $[x_j] \in Q^J$ be the vector corresponding to f. Then it satisfies $\mu_i[x_j] = q_i$ for all $i \in I$. Thus Q is algebraically compact.

Proposition 2. *Let S be a ring and let Q be an algebraically compact left S-module. Let M be a two-sided S-R-module. Then* $\mathrm{Hom}_S(M, Q)$ *is an* *algebraically compact left R-module.*

Proof. Let A be a left R-module. Then there is a well-known natural isomorphism $\sigma(A) : \mathrm{Hom}_R(A, \mathrm{Hom}_S(M, Q)) \to \mathrm{Hom}_S(M \otimes_R A, Q)$ such that if $f \in \mathrm{Hom}_S(A, \mathrm{Hom}_S(M, Q))$ and $\varphi = \sigma(A)f \in \mathrm{Hom}_S(M \otimes_R A, Q)$ then they satisfy $(f(a))(x) = \varphi(x \otimes a)$ for all $a \in A$ and $x \in M$([2, Theorem 2.8]). Let B be a submodule of A. Let κ be the inclusion map $B \to A$ and let $\eta = M \otimes \kappa : M \otimes_R B \to M \otimes_R A$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_R(A, \mathrm{Hom}_S(M, Q)) & \xrightarrow{\sigma(A)} & \mathrm{Hom}_S(M \otimes_R A, Q) \\ \Big\downarrow {\scriptstyle \mathrm{Hom}(\kappa, \mathrm{Hom}_S(M, Q))} & & \Big\downarrow {\scriptstyle \mathrm{Hom}(\eta, Q)} \\ \mathrm{Hom}_R(B, \mathrm{Hom}_S(M, Q)) & \xrightarrow{\sigma(B)} & \mathrm{Hom}_S(M \otimes_R B, Q), \end{array}$$

where $\mathrm{Hom}(\kappa, \mathrm{Hom}_S(M, Q))$ is nothing but the restriction map. Let $h \in \mathrm{Hom}_R(B, \mathrm{Hom}_S(M, Q))$ be such that its restriction to any finitely generated submodule of B is extendable to a homomorphism $A \to \mathrm{Hom}_S(M, Q)$ (i. e., to an element of $\mathrm{Hom}_R(A\ \mathrm{Hom}_S(M, Q))$. Let $\psi = \sigma(B)h \in \mathrm{Hom}_S(M \otimes_R B, Q)$. Let E be a finitely generated submodule of the left S-module $M \otimes_R B$ generated by, say, $e_1, e_2, \ldots, e_n$. If we choose a suitable finite number of elements $x_1, x_2, \ldots, x_m$ of M, each e_j $(j = 1, 2, \ldots, n)$ is expressed as $e_j = \sum_{i=1}^{m} x_i \otimes' b_{ij}$ with $b_{ij} \in B$, where $\otimes'$ means the tensor product in $M \otimes_R B$. Let B_0 be the submodule of B generated by mn elements $\{b_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$. Then the restriction of h to B_0 is extended to a homomorphism $f_0 \in \mathrm{Hom}_R(A, \mathrm{Hom}_S(M, Q))$. Let

$$\varphi_0 = \sigma(A)f_0 \in \mathrm{Hom}_S(M \otimes_R A, Q).$$

Then we have

$$\psi(e_j) = \sum_{i=1}^{m} \psi(x_i \otimes' b_{ij}) = \sum_{i=1}^{m} (h(b_{ij}))(x_i) = \sum_{i=1}^{m} (f_0(b_{ij}))(x_i) =$$

$$\sum_{i=1}^{m} \varphi_0(x_i \otimes b_{ij}) = \varphi_0(\sum_{i=1}^{m} x_i \otimes b_{ij}) = \varphi_0(\eta \sum_{i=1}^{m} x_i \otimes' b_{ij}) = \varphi_0(\eta(e_j))$$

for $j = 1, 2, \cdots, n$.

This means that the restrictions of both ψ and $\varphi_0 \circ \eta$ to E coincide. Since Q is an algebraically compact left S-module, there exists a $\varphi \in \mathrm{Hom}_S(M \otimes_R A,$

Q) such that $\varphi \circ \eta = \psi$ by Proposition 1. Let $f \in \mathrm{Hom}_R(A, \mathrm{Hom}_S(M, Q))$ be the homomorphism corresponding to φ, i.e., $\sigma(A)f = \varphi$. Then we have $(f(b))(x) = \varphi(x \otimes b) = \psi(x \otimes' b) = (h(b))(x)$ for all $b \in B$ and $x \in M$. This implies that f is an extension of h. Thus ${}_R\mathrm{Hom}_S(M, Q)$ is algebraically compact again by Proposition 1.

Let U be the factor group of the additive group $\boldsymbol{Q}$ of rational numbers modulo its subgroup $\boldsymbol{Z}$ of integers. For any (additive) Abelian group A, we define $A^* = \mathrm{Hom}(A, U)$ and call it the dual of A. If $\varphi: A \to B$ is a homomorphism then we define $\varphi^* = \mathrm{Hom}(\varphi, U): B^* \to A^*$ and call it the dual of φ. Since U is an injective cogenerator as a $\boldsymbol{Z}$-module, φ is a monomorphism or an epimorphism if and only if φ^* is an epimorphism or a monomorphism respectively ([2, Lemma 3.34]). Also, that U is a cogenerator implies that by associating each $a \in A$ with the mapping $\chi \to \chi(a)$, $\chi \in A^*$, we have a monomorphism $\lambda_A : A \to A^{**}$, so that we may regard A as a subgroup of A^{**} by identifying each $a \in A$ with $\lambda_A(a)$. If A is a left or right R-module then A^* becomes a right or left R-module respectively. If $\varphi: A \to B$ is an R-module homomorphism then $\varphi^*: B^* \to A^*$ is also an R-module homomorphism. Moreover, if A is an R-module then the canonical embedding $\lambda_A : A \to A^{**}$ is also an R-module monomorphism.

The following proposition is more or less known, but we shall give a proof for completeness:

Proposition 3. *Let A be a left R-module and B a submodule of A, and let $\kappa : B \to A$ be the inclusion map. Then the following conditions are equivalent:*

(1) *B is pure in A;*

(2) *The epimorphism $\kappa^* : A^* \to B^*$ splits;*

(3) *$B^* \otimes \kappa : B^* \otimes_R B \to B^* \otimes_R A$ is a monomorphism;*

(4) $\mathrm{Hom}(\kappa, M^*) : \mathrm{Hom}_R(A, M^*) \to \mathrm{Hom}_R(B, M^*)$ *is an epimorphism for all right R-modules M;*

(5) *There exists a homomorphism $A \to B^{**}$ which fixes B element-wise, when B is regarded as a submodule of B^{**} canonically.*

Proof. Let M be a right R-module. Then we have a natural isomorphism $\sigma(A) : \mathrm{Hom}_R(A, \mathrm{Hom}(M, U)) \to (M \otimes_R A)^* = \mathrm{Hom}(M \otimes_R A, U)$ as was considered in the proof of Proposition 2 (by replacing the situation $({}_RA, {}_SM_R, {}_SQ)$ by $({}_RA, {}_ZM_R, {}_ZU)$). This isomorphism makes the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Hom}_R(A, M^*) & \xrightarrow{\ \sigma(A)\ } & (M \otimes_R A)^* \\ \Big\downarrow {\scriptstyle \mathrm{Hom}(\varkappa,\, M^*)} & & \Big\downarrow {\scriptstyle (M \otimes \varkappa)^*} \\ \mathrm{Hom}_R(B, M^*) & \xrightarrow{\ \sigma(B)\ } & (M \otimes_R B)^*. \end{array}$$

Similarly, by considering the situation $(M_R, {}_RA_Z, U_Z)$, we have another natural isomorphism $\tau(A) : \mathrm{Hom}_R(M, A^*) \to (M \otimes_R A)^*$, which makes the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Hom}_R(M, A^*) & \xrightarrow{\ \tau(A)\ } & (M \otimes_R A)^* \\ \Big\downarrow {\scriptstyle \mathrm{Hom}(M,\, \varkappa^*)} & & \Big\downarrow {\scriptstyle (M \otimes \varkappa)^*} \\ \mathrm{Hom}_R(M, B^*) & \xrightarrow{\ \tau(B)\ } & (M \otimes_R B)^*. \end{array}$$

Now $M \otimes \varkappa : M \otimes_R B \to M \otimes_R A$ is a monomorphism if and only if $(M \otimes \varkappa)^*$ is an epimorphism. But the commutativty of the first and the second diagrams implies that the conditions that $(M \otimes \varkappa)^*$ is an epimorphism, that $\mathrm{Hom}(\varkappa, M^*)$ is an epimorphism and that $\mathrm{Hom}(M, \varkappa^*)$ is an epimorphism are equivalent. In particular, the condition (3) and the condition that $\mathrm{Hom}(B^*, \varkappa^*)$: $\mathrm{Hom}_R(B^*, A^*) \to \mathrm{Hom}_R(B^*, B^*)$ is an epimorphism are equivalent. But the last condition is the same as the condition (2). On the other hand, the condition (2) is also equivalent to the condition that $\mathrm{Hom}(M, \varkappa^*)$ is an epimorphism for all right R-modules M, and therefore (2) is equivalent to the condition (1) as well as to the condition (4).

Finally, combining the above two diagrams, we consider the following commutative diagram for $M = B^*$:

$$\begin{array}{ccc} \mathrm{Hom}_R(A, B^{**}) & \xrightarrow{\ \tau(A)^{-1}\,\sigma(A)\ } & \mathrm{Hom}_R(B^*, A^*) \\ \Big\downarrow {\scriptstyle \mathrm{Hom}(\varkappa,\, B^{**})} & & \Big\downarrow {\scriptstyle \mathrm{Hom}(B^*,\, \varkappa^*)} \\ \mathrm{Hom}_R(B, B^{**}) & \xrightarrow{\ \tau(B)^{-1}\,\sigma(B)\ } & \mathrm{Hom}_R(B^*, B^*). \end{array}$$

Let λ_B be the canonical embedding $B \to B^{**}$. Let $\psi = \sigma(B)\lambda_B \in (B^* \otimes_R B)^*$ and let $\varphi = \tau^{-1}(B)\,\psi \in \mathrm{Hom}_R(B^*, B^*)$, i.e., $\psi = \tau(B)\varphi$. Then we have $(\lambda_B(b))(\omega) = \psi(\omega \otimes b) = (\varphi(\omega))(b)$ for every $\omega \in B^*$ and $b \in B$. But clearly $(\lambda_B(b))(\omega) = \omega(b)$ and so we have $\omega(b) = (\varphi(\omega))(b)$, which implies that $\omega = \varphi(\omega)$ for all $\omega \in B^*$, i.e., $\varphi = 1$, the identity map of B^*. Thus we know in the lower row of the above diagram there corresponds to $\lambda_B \in \mathrm{Hom}_R(B, B^{**})$ the identity map $1 \in \mathrm{Hom}_R(B^*, B^*)$, and therefore λ_B is in the image of $\mathrm{Hom}(\varkappa, B^{**})$ if and only if 1 is in the image of $\mathrm{Hom}(B^*, \varkappa^*)$, which means nothing but that (5) and (2) are equivalent.

Corollary 4. *For every right R-module M, its dual module M^* is a pure-injective left R-module.*

This is an immediate consequence of the implication (1) ⇒(4) in Proposition 3.

Corollary 5. *For every left R-module B, B is pure in its double dual B^{**}.*

This follows from the implication (6) ⇒(1) in Proposition 3 applied to $A = B^{**}$.

Remark 1. The equivalence (1) and (2) in Proposition 3 is a theorem of Stenström. Indeed, this theorem and above Corollaries 4 and 5 are given in [3] as Exercises 40, 42, 41 (p. 48) respectively with hinted proofs.

Remark 2. Proposition 3 and its Corollaries 4, 5 remains true even if U is, as a $\boldsymbol{Z}$-module, assumed to be any injective cogenerator, i.e., U is a divisible Abelian group containing $\boldsymbol{Q}/\boldsymbol{Z}$, as can easily be seen.

We now prove the following theorem of Warfield:

Theorem 6. *Let Q be a left R-module. Then the following conditions are equivalent:*

(1) *Q is pure-injective;*
(2) *Q is algebraically compact;*
(3) *Q is a direct summand of the dual M^* of some right R-module M;*
(4) *Q is a direct summand of Q^{**}.*

Proof. (4) ⇒(3) is clear.
(1) ⇒(4) : Assume (1). Since Q is pure in Q^{**} by Corollary 5, Q must be a direct summand of Q^{**}.

(3) ⇒(1) : Assume (3). Since M^* is pure-injective by Corollary 4, its direct summand Q is also pure-injective.

(3) ⇒(2) : Since U is an injective $\boldsymbol{Z}$-module, U is an algebraically compact $\boldsymbol{Z}$-module by (the implication (2) ⇒(1) in) Proposition 1. Therefore, for any right R-module M, the left R-module $M^* = \mathrm{Hom}(M, U)$ is algebraically compact by Proposition 2. It is easy to see that every direct summand of an algebraically compact module is algebraically compact too. Thus Q is algebraically compact.

(2) ⇒ (4) : Assume (2). Let $\mu = [a_{ij}]$ be a row-finite $I \times J$-matrix over R and $[q_i] \in Q^I$ a vector such that the system of linear equations $\sum_j a_{ij} x_j = q_i$ $(i \in I)$ has a solution $[x_j]$ in $(Q^{**})^J$. Let I_0 be any finite subset of I. Since μ is row-finite and Q is pure in Q^{**}, there exists a vector $[y_j^0] \in Q^J$ such that $\sum_j a_{ij} y_j^0 = q_i$ for $i \in I_0$. Therefore, by the algebraic compactness of Q, there exists $[y_j] \in Q^J$ such that $\sum_j a_{ij} y_j = q_i$ for $i \in I$. Thus we know that Q is μ-pure in Q^{**} for all row-finite matrix μ over R. By [1, Proposition 1], this implies that the natural epimorphism $Q^{**} \to Q^{**}/Q$ is M-pure for all left R-modules M, which means that the epimorphism splits, i.e., Q is a direct summand of Q^{**}. This completes the proof of our theorem.

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