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COZZENS DOMAINS ARE HEREDITARY

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A ring R is said to be a *right V-ring* (after Villamayor) provided that it satisfies the equivalent conditions :

(V.1) rad $M = 0$ for all M in mod-R.

(V.2) Each right ideal is the intersection of maximal right ideals.

(V.3) Each simple right R-module is injective.

For proof consult [3], p. 356 or [6].

Condition (V.3) was introduced by Kaplansky who showed that (V.3) in commutative R is equivalent to R von Neumann regular.

A *generalized right Cozzens domain* is a right Noetherian right V-domain R for which every indecomposable injective right R-module is either simple or else embeds in the right quotient field of R.

A module M of mod-R is a *B-object* (after Bass; see 18.3(b) of [5]) provided that every submodule $S \neq M$ is contained in a maximal submodule. A ring R is a *right B-ring* if every M in mod-R is a B-object (see [5], pp. 330–331 and 155 for further background and related results).

In [1] Cozzens constructed V-domains not fields in providing a negative answer to Bass' question of whether or not a right B-ring with no infinite sets of orthogonal idempotents was necessarily right perfect, equivalently satisfied the d.c.c. on principal left ideals. (Bass proved that the converse held.)

The domain R constructed by Cozzens were right and left hereditary, and right and left Noetherian, in fact, they were principal right (left) ideal rings. Furthermore, R has a unique simple right R-module and a unique indecomposable injective right R-module, namely the right quotient field K of R.

We prove that any right Ore domain with these two properties is necessarily right hereditary. More generally, we have :

Theorem. *Every generalized right Cozzens domain is hereditary.*

Proof. The proof requires a result of [4] which states that a right Noetherian right V-ring is right hereditary provided that each indecomposable injective right R-module E is restricted semisimple, that is, E/S is semisimple, whence injective, for each submodule $S \neq 0$. Since K is the only non-simple indecomposable injective, we check K/M, for a submodule $M \neq 0$.

Now, by the Matlis' structure theory of injective modules over right Noetherian rings, the injective hul $E(K/M)$ is a direct sum of nonzero indecomposable injectives

$$E(K/M) = \oplus E_i$$

where $0 \neq E_i = E(U_i/M)$ for some uniform submodule U_i/M of K/M. Now, there are just 2-possibilities for E_i, namely simple, or $E_i \approx K$. But if $E_i \approx K$, then U_i/M embeds in K, hence there is an R-morphism $f: U_i \to K$ with ker $f = M$. Since K is injective, then there exists a mapping f^*: $K \to K$ extending f. However, M is essential in K, so it follows that for any $k \in K$, there exists $r \neq 0$ in R with $kr = u \in U_i$. Then,

$$f^*(u) = f^*(k)r = 0$$

whence $f^*(k) = 0$, proving that $f^* = 0$. But then, this implies that $f = 0$, whence $U_i = M$, contrary to the choice of U_i/M.

Thus, E_i is simple for all i, so $E(K/M)$ is semisimple, and therefore so is K/M. This proves what we wanted.

Another proof can be given using the results of [6], and the calculations above.

We remark that there exist generalized Cozzens domains with more than one isomorphy class of simple injectives. (See [2] and [7].)

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