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PRODUCTS OF GALOIS OBJECTS AND THE PICARD INVARIANT MAP

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Throughout this paper, R is a commutative ring and H is a commutative, cocommutative Hopf R-algebra which is a finitely generated projective R-module with dual $H^* = \mathrm{Hom}_R(H, R)$. Let $\mathrm{Gal}(H)$ be the abelian group of R-algebra, H-module isomorphism classes of Galois H-objects, and $\mathrm{Pic}(H^*)$ the group of H^*-module isomorphism classes of rank one projective H^*-modules. In [9], A. Nakajima studied the map η' from $\mathrm{Gal}(H)$ to $\mathrm{Pic}(H^*)$ induced by viewing a Galois H-object as an H^*-module, and showed that if $H \cong H^*$ as H^*-modules, then the map η' is a homomorphism. In this note we show that the map η from $\mathrm{Gal}(H)$ to $\mathrm{Pic}(H^*)$ defined by sending the class of S to the class of $S^* = \mathrm{Hom}_R(S, R)$ is a homomorphism. This result is implicit from the cohomological description of $\mathrm{Gal}(H)$ found in [12] and [5], but here we give a direct proof. Nakajima's result is a special case of this result, as we show. We also give another proof of Nakajima's result along the lines of the proof of Garfinkel and Orzech [7] for H the group ring of a finite abelian group. In order to obtain these results, it is appropriate to show that various definitions of the product of Galois extensions in the group $\mathrm{Gal}(H)$ given by Chase [2], Beattie [1] and Nakajima [9], and, when $H^* = RG$, G a finite abelian group, by Harrison [8], are the same : this appears in the first section of the paper.

All rings have unity, and unadorned tensor products are over R. Given the Hopf algebra H, we consider only H-objects S which are associative, commutative R-algebras with identity and which are finitely generated, projective R-modules. If S is a Galois H-object or a rank one projective H^*-module, the class of S in $\mathrm{Gal}(H)$ or in $\mathrm{Pic}(H^*)$ will be denoted by $[S]$.

For the Hopf algebra H, the multiplication, unit map, comultiplication, counit and antipode will be denoted by μ, i, Δ, ε and λ, respectively. If S is an H-object, the structure map is $\alpha_S : S \to S \otimes H$. We use the Sweedler notation :

$$\Delta(h) = \textstyle\sum_{(h)} h_{(1)} \otimes h_{(2)} \text{ for } h \text{ in } H;$$
$$\alpha_S(x) = \textstyle\sum_{(x)} x_{(0)} \otimes x_{(1)},\ x, x_{(0)} \in S,\ x_{(1)} \in H.$$

Our basic reference for Hopf algebras is Sweedler [11] and for Galois objects, Chase and Sweedler [2].

1. Products of Galois extensions. The set Gal(H) of R-algebra, H^*-module isomorphism classes of Galois H-objects is an abelian group via $[S]\cdot[T]=[S\cdot T]$ for S, T Galois H-objects. There are several ways in which the product $S\cdot T$ of two Galois H-objects has been defined.

Chase's definition [2] is

$$S\cdot T=\{u=\sum x_i\otimes t_i\otimes h_i \text{ in } S\otimes T\otimes H \mid 1\otimes 1\otimes(1\otimes\Delta)\Delta(u)=(1\otimes\tau\otimes 1\otimes 1)(\alpha_S\otimes\alpha_T\otimes 1)(u)\},\ (\tau=\text{twist map})$$

an H-object via $\alpha_{S\cdot T}$ where

$$\alpha_{S\cdot T}:\ S\cdot T\to S\cdot T\otimes H \text{ is by } \alpha_{S\cdot T}=1\otimes\Delta.$$

Beattie [1] defined the product as

$$S\cdot T=\{u=\sum x_i\otimes t_i \text{ in } S\otimes T \mid \sum x_{i,(0)}\otimes t_i\otimes x_{i,(1)}=\sum x_i\otimes t_{i,(0)}\otimes t_{i,(1)}\}$$
$$=\{u \text{ in } S\otimes T \mid (1\otimes\tau)(\alpha_S\otimes 1)(u)=(1\otimes\alpha_T)(u)\},$$

an H-object via $\alpha_{S\cdot T}$ where

$$\alpha_{S\cdot T}:\ S\cdot T\to S\cdot T\otimes H \text{ is by } \alpha_{S\cdot T}=1\otimes\alpha_T.$$

Nakajima's two products [9] are

$$S\cdot T=(S\otimes T)^{\text{hker}\mu}$$

where hkerμ is the Hopf algebra kernel of $\mu: H^*\otimes H^*\to H^*$;
and

$$S\cdot T=\text{Hom}^{\mu}_{H^*\otimes H^*}(H^*,\ S\otimes T),$$

which is isomorphic to $(S\otimes T)^{\text{hker}\mu}$ by [9], 2.7.

Here hker$\mu=\{\omega\in H^*\otimes H^*\mid(1\otimes\mu)\Delta(\omega)=\omega\otimes 1 \text{ in } H^*\otimes H^*\otimes H^*\}$, a Hopf R-algebra.

The Harrison product [8] when $H^*=RG$, G a finite abelian group, is given by $S\cdot T=(S\otimes T)^{DG}$ where $DG=\{(\sigma,\sigma^{-1})\mid\sigma\in G\}$. An obvious analogue of DG for general H^* is $\gamma(H^*)\subseteq H^*\otimes H^*$, where $\gamma: H^*\to H^*\otimes H^*$ is defined by $\gamma(x)=(1\otimes\lambda)\Delta(x)$. For H commutative and cocommutative γ is a $1-1$ Hopf algebra homomorphism. Thus the Harrison product generalizes to

$$S\cdot T=(S\otimes T)^{\gamma(H^*)}.$$

But this is the same as Nakajima's first product, for we have:

Lemma 1. $\gamma(H^*) = \mathrm{hker}\mu$.

Proof. That $\gamma(H^*) \subseteq \mathrm{hker}\mu$ is an easy computation. For the opposite inclusion, if $\sum \omega \otimes x \in \mathrm{hker}\mu$, then

$$\sum \omega \otimes x \otimes 1 = \sum \omega_{(1)} \otimes x_{(1)} \otimes \omega_{(2)} x_{(2)}.$$

Apply $(1 \otimes \mu)(1 \otimes 1 \otimes \lambda)$ to both sides, to get

$$\sum \omega \otimes x = \sum \omega_{(1)} \otimes \omega_{(2)}^{\lambda} \varepsilon(x)$$

or

$$\sum \omega \otimes x = \gamma(1 \otimes \varepsilon)(\sum \omega \otimes x) = \gamma(\sum \omega \varepsilon(x)).$$

Proposition 2. *The products of Chase, Beattie and Nakajima are isomorphic.*

Proof. First we show that the products of Chase and Beattie coincide.

We have the map $1 \otimes \alpha_T : S \otimes T \to S \otimes T \otimes H$ with left inverse $1 \otimes 1 \otimes \varepsilon$. We restrict $1 \otimes \alpha_T$ and $1 \otimes 1 \otimes \varepsilon$ to $S \cdot T$ (Beattie) and $S \cdot T$ (Chase), respectively. Then $1 \otimes \alpha$ maps $S \cdot T$ (Beattie) to $S \cdot T$ (Chase). For let $\sum x \otimes t$ be in $S \cdot T$ (Beattie), then

$$(2.1) \quad \sum_{(t)} x \otimes t_{(0)} \otimes t_{(1)} = \sum_{(x)} x_{(0)} \otimes t \otimes x_{(1)}.$$

Applying $(1 \otimes 1 \otimes 1 \otimes \Delta)(1 \otimes 1 \otimes \tau)(1 \otimes \alpha_T \otimes 1)$ to both sides of (2.1) ($\tau =$ switch map) shows that $(1 \otimes \alpha_T)(\sum x \otimes t)$ lies in $S \cdot T$ (Chase).

Also, $1 \otimes 1 \otimes \varepsilon$ maps $S \cdot T$ (Chase) to $S \cdot T$ (Beattie). For let $\sum x \otimes t \otimes h$ be in $S \cdot T$ (Chase), then

$$(2.2) \quad \sum x_{(0)} \otimes t_{(0)} \otimes x_{(1)} \otimes t_{(1)} \otimes h = \sum x \otimes t \otimes h_{(1)} \otimes h_{(2)} \otimes h_{(3)}:$$

applying $1 \otimes 1 \otimes 1 \otimes \varepsilon \otimes \varepsilon$ and $1 \otimes 1 \otimes \varepsilon \otimes 1 \otimes \varepsilon$ to (2.2) shows quickly that $(1 \otimes 1 \otimes \varepsilon)(\sum x \otimes t \otimes h)$ satisifies (2.1).

The identity $\sum \varepsilon(h) x \otimes t_{(0)} \otimes t_{(1)} = \sum x \otimes t \otimes h$ obtained by applying $1 \otimes 1 \otimes \varepsilon \otimes 1 \otimes \varepsilon$ to (2.2) permits one to see quickly that $(1 \otimes 1 \otimes \varepsilon)$ and $(1 \otimes \alpha_T)$ are inverse isomorphisms.

By Lemma 1 we may identify Nakajima's products with $S \cdot T = (S \otimes T)^{\gamma(H^*)}$. We show $S \cdot T$ (Beattie) $= (S \otimes T)^{\gamma(H^*)}$.

In the remainder of the proof, $S \cdot T$ denotes $S \cdot T$ (Beattie).
Thus

$$S \cdot T = \{\sum x \otimes t \mid \sum x_{(0)} \otimes t \otimes x_{(1)} = \sum x \otimes t_{(0)} \otimes t_{(1)}\}.$$

If $\sum x\otimes t\in S\cdot T$, then for any $f\in H^*$,

$$(\textstyle\sum_{(f)} 1\otimes f_{(2)}^{\lambda}\otimes\langle f_{(1)},\ \rangle)(\sum_{(s)} x_{(0)}\otimes t\otimes x_{(1)}) =$$
$$(\textstyle\sum_{(f)} 1\otimes f_{(2)}^{\lambda}\otimes\langle f_{(1)},\ \rangle)(\sum_{(t)} x\otimes t_{(0)}\otimes t_{(1)}),$$

But $f\cdot x=\sum x_{(0)}\langle f, x_{(1)}\rangle$ for any $x\in S$, $f\in H^*$. So we get

$$\textstyle\sum_{(f),(x)} x_{(0)}\langle f_{(1)}, x_{(1)}\rangle\otimes f_{(2)}^{\lambda}\cdot t=\sum_{(f)} x\otimes f_{(2)}^{\lambda}(\sum_{(t)} t_{(0)}\langle f_{(1)}, t_{(1)}\rangle)$$

or

$$\textstyle\sum_{(f)} f_{(1)}\cdot x\otimes f_{(2)}^{\lambda}\cdot t = x\otimes\sum_{(f)} f_{(2)}^{\lambda}\cdot f_{(1)}\cdot t = x\otimes\varepsilon(f)t.$$

Hence $S\cdot T\subseteq(S\otimes T)^{\gamma(H^*)}$.

This inclusion is an R-algebra, H^*-module map where H^* acts via the action on S. Now $(S\otimes T)^{\gamma(H^*)}$ is an H-object. If ϕ is any integral of H^*, then

$$\phi((S\otimes T)^{\gamma(H^*)})\subseteq(S\otimes T)^{\gamma(H^*)H^*}=R.$$

So if I is the space of integrals of H^*, then $I((S\otimes T)^{\gamma(H^*)})\subseteq R$.

By [2], Corollary 9.7, the multiplication map

$$S\cdot T\otimes I((S\otimes T)^{\gamma(H^*)})\to(S\otimes T)^{\gamma(H^*)}$$

is an isomorphism. But since $I((S\otimes T)^{\gamma(H^*)})\subseteq R$, it follows that $S\cdot T\supseteq(S\otimes T)^{\gamma(H^*)}$, completing the proof.

2. The Picard invariant map.

Now we prove

Theorem 3. *The map* η : $\mathrm{Gal}(H)\to\mathrm{Pic}(H^*)$, *defined by* $\eta[S]=[S^*]$ *for a Galois H-object S, is a homomorphism.*

Proof. We must show that for S, T Galois H-objects, $(S\cdot T)^*\cong S^*\otimes_{H^*}T^*$ as H^*-modules. We use Nakajima's second product.

For G an R-algebra and W, X G-modules we have the adjoint associativity isomorphism

(3.1) $\alpha:\mathrm{Hom}_R(X\otimes_G W, R)\cong\mathrm{Hom}_G(W,\mathrm{Hom}_R(X, R))$

by $\alpha(f)(h)(x)=f(x\otimes h)$ for $f\in\mathrm{Hom}_R(X\otimes_G W, R)$, $h\in W$, $x\in X$. If W is an H^*-module, then with the usual induced H^*-module structures on the Homs, α is an H^*-module map. Dualizing (3.1) and setting $G=H^*\otimes H^*$, $W=H^*$, $X=(S\otimes T)^*\cong S^*\otimes T^*$, we obtain

$$(\mathrm{Hom}_{H^*\otimes H^*}(H^*, S\otimes T))^*\cong(S^*\otimes T^*)\otimes_{H^*\otimes H^*}H^*\cong S^*\otimes_{H^*}T^*,$$

the second isomorphism by $(u\otimes v)\otimes h \to u\otimes hv$. It is straightforward to verify that these isomorphisms are as H^*-modules. That completes the proof.

Note that the kernel of $\eta = \{[S] \mid S^* \cong H^* \text{ as } H^*\text{-modules}\}$. If $f: S^* \to H^*$ is an H^*-module isomorphism, then f induces an H^*-module isomorphism

$$f^*: H \cong H^{**} \to S^{**} \cong S$$

by $f^*(\varphi) = \varphi\cdot f$ for $\varphi \in H^{**}$. Thus

$$\ker\eta = \{[S] \mid S \cong H \text{ as } H^*\text{-modules}\} = \{[S] \mid S \text{ is } H^*\text{-isomorphic to the trivial Galois } H\text{-object, } H\}.$$

This reinforces the observation of several authors [12], [6], [13], [4] that the condition $S \cong H$ as H^*-modules is the natural generalization of the normal basis condition for Galois extensions with Galois group G.

3. Nakajima's map. Nakajima [9] proves that under the hypothesis $H \cong H^*$ as H^*-modules, then the map η': $\mathrm{Gal}(H) \to \mathrm{Pic}(H^*)$ by $\eta'[S] = [S]$ is a homomorphism. The hypothesis $H \cong H^*$ is necessary if η' is to take the identity element $[H]$ of $\mathrm{Gal}(H)$ to the identity element $[H^*]$ of $\mathrm{Pic}(H^*)$. One can recover Nakajima's result from Theorem 3. For by [10], $H \cong H^*$ as H^*-modules if and only if the space of integrals I of H^* is a free R-module. But we have proved in [3] that for any Galois H-object S, $S^* \cong S\otimes I$ as H^*-modules. Thus when $H \cong H^*$, the maps η and η' coincide.

When $H^* = RG$, Garfinkel and Orzech [7] have given a proof that η' is a homomorphism which makes use of the trace element $\sum_{\sigma\in G}\sigma$ of RG. Based on the idea that $\sum_{\sigma\in G}\sigma$ generates the space of integrals of RG, here is a proof of Nakajima's result which follows the Garfinkel-Orzech proof.

Theorem 4. *Suppose the space of integrals of H^*, I, $= R\phi$, a free R-module of rank one. Then the map*

$$\eta': \mathrm{Gal}(H) \to \mathrm{Pic}(H^*),$$
$$\eta'[S] = [S], \textit{ is a homomorphism.}$$

Proof. Let S, T be Galois H-objects. We must show that as H^*-modules, $(S\otimes T)^{\gamma(H^*)} \cong S\otimes_{H^*} T$.

Let e be an element of S with $\phi e = 1$. Define $j: (S\otimes T)^{\gamma(H^*)} \to S\otimes_{H^*} T$ by $j(u) = \rho(u(e\otimes 1))$, where $\rho: S\otimes T \to S\otimes_{H^*} T$ is the canonical map. Then j is an H^*-module map, since the H^* action on $(S\otimes T)^{\gamma(H^*)}$ is the restriction of the H^* action on S. To show j is an isomorphism, we find an inverse.

Let $\varkappa : S\otimes T \to (S\otimes T)^{\gamma(H^*)}$ by

$$\varkappa(a\otimes b) = \sum_{(\varphi)} \phi_{(1)}a\otimes \phi^{\lambda}_{(2)} b.$$

Then for any f in H^*,

$$\begin{aligned}&(\sum f_{(1)}\otimes f^{\lambda}_{(2)})(\varkappa(a\otimes b)) =\\ &(1\otimes \lambda)\Delta(f\phi)(a\otimes b) = \sum \varepsilon(f)(1\otimes \lambda)\Delta(\phi)(a\otimes b).\end{aligned}$$

So $\varkappa$ has its image in $(S\otimes T)^{\gamma(H^*)}$. Let φ: $S\otimes_{H^*} T \to (S\otimes T)^{\gamma(H^*)}$ by $\varphi(\rho(u)) = \varkappa(u)$. First, φ is well-defined. For $\rho(u) = \rho(v)$ if and only if $u-v = \sum (a_i\otimes f_ib_i - f_ia_i\otimes b_i)$ for $f_i \in H^*$, $a_i \in S$, $b_i \in T$.
Now

$$\begin{aligned}\varkappa(a\otimes fb) &= (1\otimes \lambda)\Delta\phi(a\otimes fb)\\ &= (\sum \phi_{(1)}\otimes \phi^{\lambda}_{(2)}f)(a\otimes b)\\ &= (\sum \phi_{(1)}\otimes \phi^{\lambda}_{(2)}\varepsilon(f_{(1)})f_{(2)})(a\otimes b)\\ &= \sum (1\otimes \lambda)\Delta(\phi\varepsilon(f_{(1)})((1\otimes f_{(2)})(a\otimes b)\\ &= \sum (1\otimes \lambda)\Delta(\phi f_{(1)})(1\otimes f_{(2)})(a\otimes b)\\ &= (\sum \phi_{(1)}f_{(1)}\otimes \phi^{\lambda}_{(2)}f^{\lambda}_{(2)}f_{(3)})(a\otimes b)\\ &= (\sum \phi_{(1)}f_{(1)}\varepsilon(f_{(2)})\otimes \phi^{\lambda}_{(2)})(a\otimes b)\\ &= (\sum\phi_{(1)}f\otimes \phi^{\lambda}_{(2)})(a\otimes b)\\ &= \varkappa(fa\otimes b).\end{aligned}$$

Thus if $\rho(u) = \rho(v)$, then $\varkappa(u) = \varkappa(v)$, and φ is well-defined.

Now for u in $(S\otimes T)^{\gamma(H^*)}$, $u = \sum u_1\otimes u_2$,

$$\begin{aligned}\varphi j(u) &= \varphi\rho(u(e\otimes 1)) = \varkappa(u(e\otimes 1))\\ &= \sum \phi_{(1)}(u_1 e)\otimes \phi^{\lambda}_{(2)}u_2\\ &= \sum \phi_{(1)}u_1\cdot \phi_{(2)}e\otimes \phi^{\lambda}_{(3)}u_2\\ &= \sum (\phi_{(1)}u_1\otimes \phi^{\lambda}_{(2)}u_2)(\phi_{(3)}e\otimes 1)\\ &= \sum \varepsilon(\phi_{(1)})u\cdot(\phi_{(2)}e\otimes 1)\\ &= u(\sum \varepsilon(\phi_{(1)})\phi_{(2)}e\otimes 1)\\ &= u(\phi e\otimes 1) = u.\end{aligned}$$

Thus $\varphi j(u) = u$ for u in $(S\otimes T)^{\gamma(H^*)}$.

For v in $S\otimes T$, $v = \sum v_1\otimes v_2$,

$$\begin{aligned}j\varphi(\rho(v)) &= j(\varkappa(v))\\ &= \rho(\varkappa(v)(e\otimes 1))\\ &= \rho((\sum \phi_{(1)}v_1\otimes \phi^{\lambda}_{(2)}v_2)(e\otimes 1))\end{aligned}$$

$$= \rho(\sum (\phi_{(1)}v_1)e \otimes \phi^\lambda_{(2)}v_2)$$
$$= \sum \phi^\lambda_{(2)}((\phi_{(1)}v_1)e) \otimes v_2 \text{ (where } \otimes = \otimes_{H^*})$$
$$= \sum \phi^\lambda_{(2)}\phi_{(1)}v_1 \cdot \phi^\lambda_{(3)}e \otimes v_2$$
$$= \sum \varepsilon(\phi_{(1)})v_1(\phi^\lambda_{(2)}e) \otimes v_2$$
$$= v(\phi^\lambda e \otimes 1)\ ;$$

Since ϕ is an integral of H^*, so is ϕ^λ, hence $\phi^\lambda e$ is in R, and

$$j\varphi(\rho(v)) = (\phi^\lambda e)\rho(v).$$

Then for v in $S \otimes T$

$$\varphi(\rho(v)) = \varphi j(\varphi(\rho(v))) = \varphi(j\varphi(\rho(v))) = \varphi((\phi^\lambda e)\rho(v)) = \phi^\lambda(e)\varphi(\rho(v)).$$

This is true for all v in $S \otimes T$, so $\phi^\lambda(e) = 1$, and φ and j are inverse isomorphisms. That completes the proof.

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