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THE GROUP OF GALOIS H-DIMODULE ALGEBRAS

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Let R be a commutative ring with identity, and let H be a commutative, cocommutative, finite Hopf R-algebra with antipode. Using the notion of H-dimodule algebra, given by Long in [3], Nakajima defines in [4] Galois H-dimodule algebras generalizing graded Galois algebras and H-objects of Galois in the sense of Chase-Sweedler. In the same paper, he shows that the set of isomorphism classes of Galois H-dimodule algebras admits a structure of monoid provided that the Hopf algebra is a free R-module of finite type.

The purpose of the present paper is to show that for any commutative. cocommutative, finite Hopf R-algebra with antipode the above mentioned monoid is a group which contains the Galois group of H (Chase-Sweedler [2], Beattie [1]), as a subgroup.

0. Preliminaries. In what follows, R is assumed to be a commutative ring with identity, and H a commutative, cocommutative, finite Hopf algebra with antipode. The counit, comultiplication and antipode of H will be denoted by $\varepsilon : H \longrightarrow R$, $\delta : H \longrightarrow H \oplus H$ and $\lambda : H \longrightarrow H$, respectively, where $\otimes$ stands for $\otimes_R$.

For A, any R-algebra, we denote the unit and multiplication with $\eta_A : R \longrightarrow A$ and $\mu_A : A \otimes A \longrightarrow A$, respectively.

$\phi_A : H \otimes A \longrightarrow A$ will denote a structure of H-module and $\chi_B : B \longrightarrow B \otimes B$ of H-comodule.

An R-algebra A is called H-module (respectively, H-comodule) algebra if η_A and μ_A are H-module (respectively, H-comodule) morphisms.

An H-module algebra A which is also H-comodule algebra and satisfies $\chi_A \cdot \phi_A = (\phi_A \otimes A) \cdot (H \otimes \chi_A) : H \otimes A \longrightarrow A \otimes H$, is called H-dimodule algebra.

For any H-module algebra A, and H-comodule algebra B, the smash product $A \# B$, is a R-algebra: the R-module support is $A \otimes B$, and the multiplication is given by

$$(a \# b) \cdot (c \# d) = \sum_{(b)} a \cdot b_{(1)}(c) \# b_{(0)} \cdot d$$

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1. Galois H-dimodule algebras.

Definition 1.1. *An H-dimodule algebra A is called a Galois H-dimodule algebra if the following conditions are satisfied:*

i) *A is a faithfully flat R-module*

ii) $\gamma_A = (\mu_A \otimes H)\cdot(A \otimes \chi_A) : A \otimes A \longrightarrow A \otimes H$ *is an isomorphism.*

Proposition 1.2. *For A and B any two Galois H-dimodule algebras, the following statements are true:*

a) *A' is a Galois H-dimodule algebra, where A' has the same H-module algebra structure of A and has $\chi_{A'} = (A \otimes \lambda)\cdot \chi_A$ as H-comodule structure.*

b) *A^{op} is a Galois H-dimodule algebra.*

c) *$\bar{A}$ is a Galois H-dimodule algebra, where $\bar{A}$ has the same H-dimodule structure of A, and its algebra structure is given by $\mu_{\bar{A}} = \mu_A \cdot \tau \cdot (A \otimes \phi_A)\cdot (\chi_{A'} \otimes A)$, τ being the twisting isomorphism.*

d) *The smash products, $A \# B$, is an Galois $H \otimes H$-dimodule algebra. The structure of $H \otimes H$-module is given by*

$$(h_1 \otimes h_2)(a \# b) = h_1(a) \# h_2(b)$$

and the one of $H \otimes H$-comodule by

$$\chi_{A\#B}(a \# b) = \sum_{(a)(b)} a_{(0)} \# b_{(0)} \otimes a_{(1)} \otimes b_{(1)}.$$

e) *H is a Galois H-dimodule algedra with the trivial structure of H-module, $\phi_H = \varepsilon \otimes H$, and with its comultiplication as H-comodule structure.*

Proof. b) After [3, Prop 1.7 and Prop 2.10], enough to show that $\gamma_{A^{op}}$ is an isomorphism. But this follows from

$$\begin{aligned}
&(\gamma_A \otimes H)\cdot(A \otimes \gamma_{A^{op}}) \\
&\text{(definitions of } \gamma_A,\ \gamma_{A^{op}}, \text{ and } A \text{ is } H\text{-comodule algebra)} \\
&= (\mu_A \otimes H \otimes H)\cdot(A \otimes \mu_A \otimes \mu_H \otimes H)\cdot(A \otimes A \otimes \chi_A \otimes \delta)\cdot \\
&\quad (A \otimes \tau \otimes H)\cdot(A \otimes A \otimes \chi_A) \\
&= (\mu_A \otimes H \otimes H)\cdot(A \otimes A \otimes \mu_H \otimes H)\cdot(\mu_A \otimes \chi_A \otimes \delta)\cdot \\
&\quad (A \otimes A \otimes \tau)\cdot(A \otimes \chi_A \otimes A)\cdot(A \otimes \tau) \\
&\quad \text{(definitions of } \gamma_A \text{ and } \gamma_H) \\
&= (A \otimes \gamma_H)\cdot(\gamma_A \otimes H)\cdot(A \otimes \tau)\cdot(\gamma_A \otimes A)\cdot(A \otimes \tau).
\end{aligned}$$

c) $\gamma_{\bar{A}} = (\mu_{\bar{A}} \otimes H)\cdot(A \otimes \chi_A) = \gamma_{A^{op}}\cdot(A \otimes \phi_A)\cdot(\chi_{A'} \otimes A)$.

From, $\gamma_{A^{op}}$ and $(A \otimes \phi_A)\cdot(\chi_{A'} \otimes A)$ isomorphims it follows $\gamma_{\bar{A}}$ isomorphism.

The rest of the proof is straightforward though tedious.

2. The group $Gal_{\#}(R, H)$. Two Galois H-dimodule algebras are said to be isomorphic if they are so as R-algebras, as H-modules and as H-comodules. The set of isomorphism classes of Galois H-dimodule algebras is denoted by $Gal_{\#}(R, H)$.

From now on, A and B denote two arbitrary Galois H-dimodule algebras.

Definition 2.1. *The product $A\cdot B$ is defined by*

$$A\cdot B=\{\sum_{(i)} a_i \# b_i \varepsilon A \# B \mid \sum_{(i)} a_{i_{(0)}} \# b_i \otimes a_{i_{(1)}} = \sum_{(i)} a_i \# b_{i_{(0)}} \otimes b_{i_{(1)}}\}$$

and therefore we have the equalizer

$$A\cdot B \xrightarrow{i_{AB}} A \# B \underset{(A\otimes\tau)(\chi_A\otimes B)}{\overset{A\otimes\chi_B}{\rightrightarrows}} A \# B\otimes H$$

Proposition 2.2. *$A\cdot B$ is a Galois H-dimodule algebra.*

Proof. The algebra structure is induced by the one of $A \# B$.

The H-module structure $\phi_{A.B}$ is given by the factorization, through the equalizer i_{AB}, of the morphism

$$H\otimes A\cdot B \xrightarrow{\delta\otimes i_{AB}} H\otimes H\otimes A\otimes B$$
$$\xrightarrow{H\otimes\tau\otimes B} H\otimes A\otimes H\otimes B \xrightarrow{\phi_A\otimes\phi_B} A\otimes B$$

i. e.,

$$h(\sum_{(i)} a_i \# b_i) = \sum_{(h)(i)} h_{(0)}(a_i) \# h_{(1)}(b_i).$$

The H-comodule structure is given by the map

$$\chi_{A\cdot B} : A\cdot B \longrightarrow A\cdot B \otimes H, \text{ defined by}$$

$$\chi_{A\cdot B}(\sum_{(i)} a_i \# b_i) = \sum_{(i)} a_{i_{(0)}} \# b_i \otimes a_{i_{(1)}} = \sum_{(i)} a_i \# b_{i_{(0)}} \otimes b_{i_{(1)}}$$

which is the only one satisfying

$$(i_{AB}\otimes H)\cdot \chi_{A\cdot B} = (A\otimes\tau)\cdot(\chi_A\otimes B)\cdot i_{AB} = (A\otimes\chi_B)\cdot i_{AB}.$$

It readily follows that $A\cdot B$ is an H-dimodule algebra.

Now, $A \# B$ is a faithfully flat R-module ; therefore

$$A \# B \otimes A \cdot B \xrightarrow{A \# B \otimes i_{AB}} A \# B \otimes A \# B \underset{A \# B \otimes ((A \otimes \tau)(\chi_A \otimes B))}{\overset{A \# B \otimes A \otimes \chi_B}{\rightrightarrows}} A \# B \otimes A \# B \otimes H$$

$$A \# B \otimes B \xrightarrow{A \# B \otimes \delta} A \# B \otimes H \otimes H \underset{A \# B \otimes ((H \otimes \tau)(\delta \otimes H))}{\overset{A \# B \otimes H \otimes \delta}{\rightrightarrows}} A \# B \otimes H \otimes H \otimes H$$

are diagrams of equalizer.

The isomorphism $\gamma_{A \# B} : (A \# B) \otimes (A \# B) \longrightarrow A \# B \otimes H \otimes H$ factors through $A \# B \otimes \delta$ yielding, this way, morphisms h and h', inverse to each other, making the diagram

$$\begin{array}{ccc}
(A \# B) \otimes A \cdot B & \xrightarrow{A \# B \otimes i_{AB}} & (A \# B) \otimes (A \# B) \\
h \downarrow \uparrow h' & & \gamma_{A \# B} \downarrow \uparrow \gamma^{-1}_{A \# B} \\
(A \# B) \otimes H & \xrightarrow{A \# B \otimes \delta} & (A \# B) \otimes H \otimes H
\end{array}$$

to commute.

It follows, then, that

$$\begin{aligned}
&((A \# B) \otimes \delta \otimes H) \cdot (h \otimes H) \cdot ((A \# B \otimes \gamma_{A \cdot B}) \\
&= ((A \# B) \otimes \delta \otimes H) \cdot ((A \# B) \otimes (\gamma_H \cdot \tau)) \cdot \\
&\quad (h \otimes H) \cdot ((A \# B) \otimes \tau) \cdot (h \otimes A \cdot B)
\end{aligned}$$

and therefore $\gamma_{A \cdot B}$ is an isomorphism, because $(A \# B) \otimes \delta \otimes H$ is an equalizer and the involved morphisms are isomorphisms.

Corollary 2.3. *The product of Galois H-dimodule algebras induces an operation on $Gal_{\#}(R, H)$ whose identity element is the class of H.*

Proposition 2.4. *$Gal_{\#}(R, H)$ is a group.*

Proof. We will show that $|A|^{-1} = |\bar{A}'|$.

Indeed, since A is a Galois H-object [2],

$$H \xrightarrow{\eta_A \otimes H} A \otimes H \text{ is the equalizer of } A \otimes H \underset{A \otimes \eta_H \otimes H}{\overset{\chi_A \otimes H}{\rightrightarrows}} A \otimes H \otimes H$$

On the other hand, the morphism

$$A\cdot\bar{A}' \xrightarrow{i_{A\bar{A}'}} A\#\bar{A}' \xrightarrow{\gamma_{A'}} A\otimes H$$

equalizes this pair because

$(\chi_A\otimes H)\cdot\gamma_{A'}\cdot i_{A\bar{A}'} = (\chi_A\otimes H)\cdot(\mu_A\otimes H)\cdot(A\otimes\chi_{A'})\cdot i_{A\bar{A}'}$
(χ_A is a morphism of algebras)
$= (\mu_A\otimes\mu_H\otimes H)\cdot(A\otimes\tau\otimes H\otimes H)\cdot(\chi_A\otimes\chi_A\otimes H)\cdot(A\otimes\chi_{A'})\cdot i_{A\bar{A}'}$
(A is an H-comodule)
$= (\mu_A\otimes\mu_H\otimes H)\cdot(A\otimes\tau\otimes H\otimes\lambda)\cdot(A\otimes H\otimes A\otimes\delta)\cdot(\chi_A\otimes\chi_A)\cdot i_{A\bar{A}'}$
(definition 2.1 and $\tau^2=1$)
$= (\mu_A\otimes\mu_H\otimes H)\cdot(A\otimes\tau\otimes H\otimes\lambda)\cdot(A\otimes H\otimes A\otimes\delta)\cdot$
$(A\otimes H\otimes\chi_A)\cdot(A\otimes\tau)\cdot(A\otimes\chi_{A'})\cdot i_{A\bar{A}'}$
(A is an H-comodule)
$= (\mu_A\otimes\mu_H\otimes H)\cdot(A\otimes A\otimes\tau\otimes\lambda)\cdot(A\otimes A\otimes H\otimes\tau)\cdot$
$(A\otimes A\otimes H\otimes H\otimes\lambda)\cdot(A\otimes A\otimes H\otimes\delta)\cdot(A\otimes A\otimes\delta)\cdot(A\otimes\chi_A)\cdot i_{A\bar{A}'}$
(cocommutativity of H)
$= (A\otimes\eta_H\otimes H)\cdot(\mu_A\otimes\lambda)\cdot(A\otimes\chi_A)\cdot i_{A\bar{A}'} = (A\otimes\eta_H\otimes H)\cdot\gamma_{A'}\cdot i_{A\bar{A}'}$.

Therefore, there exists a morphism $f\colon A\cdot\bar{A}'\longrightarrow H$ so that $(\eta_A\otimes H)\cdot f = \gamma_{A'}\cdot i_{A\bar{A}'}$ (i.e. calling q to $f(a\#\bar{a})$, $1_A\# q = \sum_{(\bar{a})} a\bar{a}_{(0)}\#\lambda(\bar{a}_{(1)})$)

This f is an H-module morphism, i.e., the diagram

$$\begin{array}{ccc} H\otimes A\cdot\bar{A}' & \xrightarrow{H\otimes f} & H\otimes H \\ \downarrow{\scriptstyle\phi_{A\cdot\bar{A}'}} & & \downarrow{\scriptstyle\varepsilon\otimes H} \\ A\cdot\bar{A}' & \xrightarrow{f} & H \end{array}$$

commutes. Indeed.

Since $\eta_A\otimes H$ is an equalizer,

$(\eta_A\otimes H)\cdot f\cdot\phi_{A\cdot\bar{A}'} = \gamma_{A'}\cdot i_{A\bar{A}'}\cdot\phi_{A\cdot\bar{A}'}$
(definition of $\phi_{A\cdot\bar{A}'}$)
$= (\mu_A\otimes H)\cdot(A\otimes\chi_{A'})\cdot(\phi_A\otimes\phi_A)\cdot(H\otimes\tau\otimes A)\cdot(\delta\otimes A\otimes A)\cdot(H\otimes i_{A\bar{A}'})$
(A' is an H-dimodule)
$= (\mu_A\otimes H)\cdot(\phi_A\otimes\phi_A\otimes H)\cdot(H\otimes\tau\otimes\chi_{A'})\cdot(\delta\otimes A\otimes A)\cdot(H\otimes i_{A\bar{A}'})$
(A is an H-module algebra)

$= (\phi_A \otimes H)\cdot(H\otimes \mu_A \otimes H)\cdot(H\otimes A\otimes \chi_{A'})\cdot(H\otimes i_{A\bar{A}'})$
$= (\phi_A \otimes H)\cdot(H\otimes \eta_A \otimes f)$
($\eta: R \longrightarrow A$ is an H-module morphism)
$= (\eta_A \otimes H)\cdot(\varepsilon \otimes H)\cdot(H\otimes f)$.

f is a morphism of H-comodules or, what is the same, the diagram

$$\begin{array}{ccc} A\cdot\bar{A}' & \xrightarrow{\chi_{A\cdot A'}} & A\cdot\bar{A}'\otimes H \\ \downarrow f & & \downarrow f\otimes H \\ H & \xrightarrow{\delta} & H\otimes H \end{array}$$

commutes. Indeed.

Since $\eta_A \otimes H\otimes H$ is an equalizer,

$(\eta_A \otimes H\otimes H)\cdot(f\otimes H)\cdot\chi_{A\cdot\bar{A}'} = (\eta_A \otimes f\otimes H)\cdot\chi_{A\cdot\bar{A}'}$
(definition of f)
$= (\gamma_{A'} \otimes H)\cdot(i_{A\bar{A}'} \otimes H)\cdot\chi_{A\cdot\bar{A}'}$
(definition of $\chi_{A\cdot\bar{A}'}$)
$= (\gamma_{A'} \otimes H)\cdot(A\otimes \chi_{A'})\cdot i_{A\bar{A}'}$
(definition of $\gamma_{A'}$)
$= (\mu_A \otimes H\otimes H)\cdot(A\otimes \chi_{A'} \otimes H)\cdot(A\otimes \chi_{A'})\cdot i_{A\bar{A}'}$
(A' is an H-comodule)
$= (\mu_A \otimes H\otimes H)\cdot(A\otimes A\otimes \delta)\cdot(A\otimes \chi_{A'})\cdot i_{A\bar{A}'}$
$= (A\otimes \delta)\cdot(\mu_A \otimes H)\cdot(A\otimes \chi_{A'})\cdot i_{A\bar{A}'}$
(definition of $\gamma_{A'}$)
$= (A\otimes \delta)\cdot\gamma_{A'}\cdot i_{A\bar{A}'} = (A\otimes \delta)\cdot(\eta_A \otimes f) = (\eta_A \otimes H\otimes H)\cdot\delta\cdot f$.

Analogously f is a morphism of R-algebras and by ([1], Lemma 1.1., p. 688) is an isomorphism.

Corollary 2.5. *The map*

$$\begin{array}{ccc} Gal(R, H) & \longrightarrow & Gal_{*}(R, H) \\ |A| & \longrightarrow & |A| \end{array}$$

obtained by regarding any Galois H-object as a Galois H-dimodule algebra with the trivial action([1],[2]) *is a monomorphism of groups, in general not surjective*([4], Remark 2.5, p. 174).

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