Math. J. Okayama Univ. 28 (1986), 7–13

ON SOME COHEN-MACAULAY SUBSETS OF A PARTIALLY ORDERED ABELIAN GROUP

To the memory of Professor Gishiro Maruyama

ANDRZEJ NOWICKI, KAZUO KISHIMOTO and TAKASI NAGAHARA

This paper is about some Cohen-Macaulay subsets of a partially ordered abelian group which are useful in the study of Galois extensions of higher derivation type (cf. Remark and [4]).

Let $N=\{1,\dots,n\}$, and $Z=\{\dots -2, -1, 0, 1, 2, \dots\}$. Now, if $G=(f_1)\times\cdots\times(f_n)$ is an abelian group which is the direct product of infinite cyclic groups (f_i) generated by f_i then G becomes a partially ordered group by

$$(\#):\quad \prod_{i=1}^{n} f_i^{s_i} \geqq \prod_{i=1}^{n} f_i^{t_i} \quad\Leftrightarrow\quad \sum_{j=k}^{n} s_j \geqq \sum_{j=k}^{n} t_j \text{ for all } k\in N.$$

This partially ordered group G will be denoted by $(G, \#)$. Clearly $(G, \#)$ can be regarded as the partially ordered additive group $(Z^n, \#) = Z_1\times\cdots\times Z_n$, where $Z_i = Z$ for all $i\in N$. As it is seen later on, $(Z^n, \#)$ is a modular lattice.

For $u_i, v_i\in Z$ with $u_i\leqq v_i$ $(i\in N)$, we set

$$\Delta = \prod_{i=1}^{n}[u_i, v_i] = \{(a_1,\dots,a_n)\ ;\ u_i\leqq a_i\leqq v_i,\ a_i\in Z\}$$

which is a subposet of $(Z^n, \#)$.

Our purpose of this note is to prove that Δ is a modular lattice under the ordering in $(Z^n, \#)$ (Theorem 7), and if, in particular, $u_i < v_i$ for all $i\in N$ then Δ is a modular sublattice of $(Z^n, \#)$ (Theorem 6).

In what follows, we shall use the following conventions:

Let A be a poset with order $\geqq$ and Δ a subposet of A. Then, for a, $b\in A$ and $c, d\in\Delta$,

$a > b$ if and only if $a\geqq b$ and $a\neq b$.

$a\gg b$ (resp. $c\gg_\Delta d$) if and only if $a>b$ (resp. $c>d$) and there are not elements e in A (resp. e' in Δ) such that $a>e>b$ (resp. $c>e'>d$).

For $a=(a_1,\dots,a_n)\in(Z^n, \#)$, this is sometimes abbreviated to $a=(a_i)$, and for any $k\in N$, $\sum_{j=k}^{n} a_j$ is denoted by $t_k(a)$.

Let $(Z^n, *)$ be a vector group with order $\geqq$ defined by

$$(*):\quad (a_i)\geqq(b_i) \quad\Leftrightarrow\quad a_i\geqq b_i \text{ for all } i\in N \text{ (cf. [3]).}$$

Then, one will easily see that $(Z^n, *)$ is a modular lattice. We consider here the mapping

$$\phi:\quad (Z^n, \#) \longrightarrow (Z^n, *)$$

defined by $\phi(a) = (t_i(a))$. Clearly $\phi(a+b) = \phi(a) + \phi(b)$ for $a, b \in (Z^n, \#)$. By our definition, ϕ is injective. Moreover, since, for $(x_i) \in (Z^n, *)$,

$$\phi((x_1 - x_2, x_2 - x_3, \dots, x_{n-1} - x_n, x_n)) = (x_1, \dots, x_n),$$

ϕ is surjective. Hence ϕ is a group isomorphism which preserves orders, and so, $(Z^n, \#)$ is a modular lattice.

Now, let $a = (a_i)$, $b = (b_i) \in (Z^n, \#)$, and $a > b$. Then $\{c \in (Z^n, \#);\ a \geqq c \geqq b\}$ is a finite set whose cardinal number is

$$\Pi_{i=1}^n (t_i(a) - t_i(b) + 1) = \Pi_{i=1}^n (\textstyle\sum_{j=i}^n (a_j - b_j) + 1).$$

By $f(a)$, we denote $\sum_{i=1}^n t_i(a)$. Then one will easily see that $f(a) = \sum_{i=1}^n ia_i$.

Additionally, let $a \gg b$, and $\phi(a) = (x_1, \dots, x_n)$. Then $\phi(a) \gg \phi(b)$, and whence

$$\phi(b) = (x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_n)$$

for some $1 \geqq i \geqq n$. Hence, it follows that there holds either

$$b = (a_1, \dots, a_{i-2}, a_{i-1}+1, a_i - 1, a_{i+1}, \dots, a_n)\ (2 \leqq i \leqq n) \text{ or}$$
$$b = (a_1 - 1, a_2, \dots, a_n).$$

Moreover, we see that $f(b) = \sum_{i=1}^n t_i(b) = \sum_{i=1}^n x_i - 1 = f(a) - 1$.

Our study starts with the following

Lemma 1. *Let $a = (a_i)$, $b = (b_i) \in (Z^n, \#)$, and $a > b$. If*

$$a = a^{(0)} \gg a^{(1)} \gg \cdots \gg a^{(p)} = b \qquad (a^{(s)} \in (Z^n, \#))$$

then $p = \sum_{i=1}^n i(a_i - b_i)$, whence the length p is uniquely determined by $a > b$.

Proof.

$$\begin{aligned}\textstyle\sum_{i=1}^n i(a_i - b_i) &= \textstyle\sum_{i=1}^n ia_i - \sum_{i=1}^n ib_i = f(a) - f(b)\\ &= \textstyle\sum_{i=0}^{p-1} (f(a^{(i)}) - f(a^{(i+1)})) = p.\end{aligned}$$

The above p will be denoted by $|a > b|$.

Lemma 2. *Let $\Delta = \Pi_{i=1}^n [u_i, v_i]$ where $u_i, v_i \in Z$ and $u_i < v_i$ ($i \in N$).* If $a, b \in \Delta$ and $a \gg_\Delta b$ then $a \gg b$.

Proof. Let $a=(a_i)$ and $b=(b_i)$. One will easily see that our assertion is true for $(Z, \#)$. Hence we assume that our lemma holds for $(Z^{n-1}, \#)$. For any $r=(r_1, \dots, r_n) \in (Z^n, \#)$, we set $C_2(r)=(r_2, \dots, r_n)$, and $C_2(\Delta)=\{C_2(r) ; r \in \Delta\}$. Then $C_2(\Delta)$ can be regarded as a subset of $(Z^{n-1}, \#)$, and $C_2(a)$, $C_2(b) \in C_2(\Delta)$. Clearly $C_2(a) \geqq C_2(b)$. In case $C_2(a)=C_2(b)$, one will easily see that $b=(a_1-1, a_2, \dots, a_n)$ where $a_1>u_1$, and so, $a \gg b$. Hence, let $C_2(a)>C_2(b)$. Then, there exists an element e' in $C_2(\Delta)$ such that

$$C_2(a) \gg_{C_2(\Delta)} e' \geqq C_2(b)$$

By the induction assumption, we have $C_2(a) \gg e'$. Hence e' coincides with either

$$f'=(a_2, \dots, a_{k-2}, a_{k-1}+1, a_k-1, a_{k+1}, \dots, a_n)$$

where $3 \leqq k \leqq n$, $a_{k-1}<v_{k-1}$ and $a_k>u_k$ (in case $k=3$, f' is taken as $(a_2+1, a_3-1, a_4, \dots, a_n)$), or

$$g'=(a_2-1, a_3, \dots, a_n)$$

where $a_2>u_2$. In case $e'=f'$, we set

$$f=(a_1, a_2, \dots, a_{k-2}, a_{k-1}+1, a_k-1, a_{k+1}, \dots, a_n).$$

Then $f \in \Delta$, $t_1(f)=t_1(a) \geqq t_1(b)$, $C_2(f)=e' \geqq C_2(b)$, and whence $a>f \geqq b$ in Δ. Hence $f=b$, and so, $a \gg b$. In case $e'=g'$, we set

$$g=(a_1+1, a_2-1, a_3, \dots, a_n).$$

If $a_1<v_1$ then there exists a chain $a>g \geqq b$ in Δ, whence $g=b$, and so, $a \gg b$. If $a_1=v_1$ then there is a chain

$$a>(a_1-1, a_2, \dots, a_n)>(a_1, a_2-1, a_3, \dots, a_n) \geqq b \text{ in } \Delta$$

which is a contradiction. This completes the proof.

By virtue of the results of Lemma 1 and Lemma 2, we obtain the following

Theorem 3. Let $\Delta=\prod_{i=1}^n [u_i, v_i]$ where $u_i, v_i \in Z$ and $u_i<v_i$ $(i \in N)$. *Then, if* $a=(a_i)$, $b=(b_i) \in \Delta$, $a>b$, *and*

$$a=a^{(0)} \gg_\Delta a^{(1)} \gg_\Delta \cdots \gg_\Delta a^{(q)}=b \qquad (a^{(t)} \in \Delta)$$

then $q=|a>b|=\sum_{i=1}^n i(a_i-b_i)$, *and* $a^{(t)} \gg a^{(t+1)}$ *for* $t=0,1,\dots,q-1$.

Lemma 4. *Let* $\Delta=\prod_{i=1}^n [u_i, v_i]$ *where* $u_i, v_i \in Z$ *and* $u_i<v_i$ $(i \in N)$. *Let* $b=(b_i)$, $c, d \in \Delta$, $c \neq d$, $c \gg b$, $d \gg b$, *and* $e=c \cup d$ *in the lattice* $(Z^n, \#)$. *Then*

$$e \in \Delta,\ e \gg c \gg b,\ e \gg d \gg b,$$

and for $r \in \Delta$ *with* $r \geqq c$ *and* $r \geqq d$,

$$r \geqq e \text{ and } |r \geqq e| = |r > b| - 2.$$

Proof. We have the two cases (1) and (2):

(1) $c = (b_1+1, b_2, \ldots, b_n)$ and
$d = (b_1, \ldots, b_{j-2}, b_{j-1}-1, b_j+1, b_{j+1}, \ldots, b_n)$ $(2 \leqq j \leqq n)$;

(2) $c = (b_1, \ldots, b_{i-2}, b_{i-1}-1, b_i+1, b_{i+1}, \ldots, b_n)$ $(2 \leqq i \leqq n)$ and
$d = (b_1, \ldots, b_{j-2}, b_{j-1}-1, b_j+1, b_{j+1}, \ldots, b_n)$ $(i \leqq j-1 < n)$.

In case (1) and $j-1 > 1$, we set
$$e' = (b_1+1, b_2, \ldots, b_{j-2}, b_{j-1}-1, b_{j+1}+1, b_{j+2}, \ldots, b_n).$$
In case (1) and $j-1 = 1$, we set
$$e' = (b_1, b_2+1, b_3, \ldots, b_n).$$
In case (2) and $j-1 > i$, we set
$$e' = (b_1, \ldots, b_{i-2}, b_{i-1}-1, b_i+1, b_{i+1}, \ldots \ldots, b_{j-2}, b_{j-1}-1, b_j+1, b_{j+1}, \ldots, b_n).$$
In case (2) and $j-1 = i$, we set
$$e' = (b_1, \ldots, b_{i-2}, b_{i-1}-1, b_i, b_{i+1}+1, b_{i+2}, \ldots, b_n).$$
Then we have

$$e' \in \Delta,\ e' \gg c \gg b, \text{ and } e' \gg d \gg b.$$

Since $(Z^n, \#)$ is a lattice, e' coincides with $c \cup d$ in $(Z^n, \#)$, that is, $e' =$ e. Moreover, for $r \in \Delta$ with $r \geqq c$ and $r \geqq d$, we see that

$$r \geqq e' \text{ and } |r \geqq e'| = |r > b| - 2.$$

Lemma 5. *Let* $\Delta = \prod_{i=1}^{n} [u_i, v_i]$ *where* $u_i, v_i \in Z$ *and* $u_i < v_i$ $(i \in N)$. *Let* $c, d \in \Delta$, $e = c \cup d$, *and* $f = c \cap d$ *in the lattice* $(Z^n, \#)$. *Then* $e, f \in \Delta$.

Proof. Clearly $u = (u_i) \in \Delta$, $c \geqq u$, and $d \geqq u$. Hance, we have $e \geqq u$ and a finite length $|e \geqq u|$. Let $\mathrm{m}(c, d)$ be the smallest integer in $\{|e \geqq w| \,;\, c \geqq w, d \geqq w, w \in \Delta\}$. If $\mathrm{m}(c, d) = 0$ then $c = d = e$. By the induction with respect to $\mathrm{m}(c, d)$, we shall prove that $e \in \Delta$. Hence, let $\mathrm{m}(c, d) = t \geqq 1$, and assume that $c' \cup d' \in \Delta$ for $c', d' \in \Delta$ with $\mathrm{m}(c', d') < t$. If either $c \geqq d$ or $d \geqq c$ then our assertion holds trivially. Hence, let $c \ngeqq d$ and $d \ngeqq c$. Let w_0 be an element of Δ such that $|e \geqq w_0| = \mathrm{m}(c, d)$, and consider the following chains (note Theorem 3):

$$e \geqq c = c^{(0)} \gg c^{(1)} \gg \cdots \gg c^{(p)} = w_0 \ (c^{(r)} \in \Delta),$$
$$e \geqq d = d^{(0)} \gg d^{(1)} \gg \cdots \gg d^{(q)} = w_0 \ (d^{(s)} \in \Delta).$$

Clearly $c^{(p-1)} \neq d^{(q-1)}$. To prove $e \in \Delta$, we shall distinguish the following cases :

(1) $c \gg w_0$, $d \gg w_0$;

(2) $c \gg w_0$, $d \geqq d^{(q-2)}$ ($q \geqq 2$) ;

(3) $c \geqq c^{(p-2)}$, $d \geqq d^{(q-2)}$ ($p \geqq 2$, $q \geqq 2$).

In case (1), we have $e = c \cup d \in \Delta$ by Lemma 4.

In case (2), we set $w_1 = c \cup d^{(q-1)}$. Then, we have $w_1 \in \Delta$ by Lemma 4, and $w_1 \cup d = e$. Moreover, $w_1 \geqq d^{(q-1)}$, $d \geqq d^{(q-1)}$, and so, $\mathrm{m}(w_1, d) < t$. Hence, we have $w_1 \cup d \in \Delta$ by the induction assumption, that is, $e \in \Delta$.

In case (3), we set $w_1 = c^{(p-1)} \cup d^{(q-1)}$. Then, we have $w_1 \in \Delta$ by Lemma 4, and $(c \cup w_1) \cup (w_1 \cup d) = e$. Since $\mathrm{m}(c, w_1) < t$ and $\mathrm{m}(w_1, d) < t$, we have $c \cup w_1 \in \Delta$ and $w_1 \cup d \in \Delta$. Moreover, since $\mathrm{m}(c \cup w_1, w_1 \cup d) < t$, it follows that $(c \cup w_1) \cup (w_1 \cup d) \in \Delta$, that is, $e \in \Delta$. By the duality of the lattice $(Z^n, \#)$, we have $f \in \Delta$. This completes the proof.

Now, by virtue of the result of Lemma 5, we obtain the following theorem which is our main result.

Theorem 6. *Let $\Delta = \prod_{i=1}^{n} [u_i, v_i]$ where $u_i, v_i \in Z$ and $u_i < v_i$ ($i \in N$). Then Δ is a modular sublattice of $(Z^n, \#)$.*

By the general theory of modular lattices, it has been known that for any modular lattice with both chain conditions and for any chain $a \geqq b$ in its lattice, the composition chains $a \gg \cdots \gg b$ has a unique length (cf. Lemma 1, Theorem 3, Theorem 6, [1], [2], and [5]). Now, we shall prove the following

Theorem 7. *Let $\Delta = \prod_{i=1}^{n} [u_i, v_i]$ where $u_i, v_i \in Z$ and $u_i \leqq v_i$ ($i \in N$). Then Δ is a modular lattice under the ordering in $(Z^n, \#)$, and whence it is a Cohen-Macaulay poset. If $a = (a_i)$, $b = (b_i) \in \Delta$, $a > b$ and*

$$a = a^{(0)} \gg_\Delta a^{(1)} \gg_\Delta \cdots \gg_\Delta a^{(q)} = b \ (a^{(t)} \in \Delta)$$

then $q = |a > b| - \sum_{i=1}^{n} i(0)(a_i - b_i) = \sum_{i=1}^{n} (i - i(0))(a_i - b_i)$ where $i(0)$ is the cardinal number of the set $\{ j \in N;\ u_j = v_j,\ j < i \}$.

Proof. Let $\{\varepsilon(1), \ldots, \varepsilon(m)\} = \{ i \in N;\ u_i < v_i \}$ where $\varepsilon(1) < \cdots < \varepsilon(m)$. Then, there is an ordered isomorphism

$$\psi:\quad \Delta = \prod_{i=1}^{n} [u_i, v_i] \longrightarrow \prod_{j=1}^{m} [u_{\varepsilon(j)}, v_{\varepsilon(j)}] \ (\subset (Z^m, \#))$$

such that $\psi((d_1, \ldots, d_n)) = (d_{\varepsilon(1)}, \ldots, d_{\varepsilon(m)})$. By Theorem 6, $\psi(\Delta)$ is a modular sublattice of $(Z^m_\cdot, \#)$. Hence Δ is a modular lattice under the ordering in $(Z^n, \#)$. Since $u_{\varepsilon(j)} < v_{\varepsilon(j)}$ for $j = 1, \dots, m$ and $\psi(a^{(t)}) \gg_{\psi(\Delta)} \psi(a^{(t+1)})$ in $\psi(\Delta)$ for $t = 0, 1, \dots, q-1$, we have $q = \sum_{j=1}^m j(a_{\varepsilon(j)} - b_{\varepsilon(j)})$ by Theorem 3. Hence, it follows from Lemma 1 that

$$\begin{aligned} |a > b| &= \textstyle\sum_{i=1}^n i(a_i - b_i) = \sum_{j=1}^m (j + \varepsilon(j)(0))(a_{\varepsilon(j)} - a_{\varepsilon(j)}) \\ &= \textstyle\sum_{j=1}^m j(a_{\varepsilon(j)} - b_{\varepsilon(j)}) + \sum_{j=1}^m \varepsilon(j)(0)(a_{\varepsilon(j)} - b_{\varepsilon(j)}) \\ &= q + \textstyle\sum_{i=1}^n i(0)(a_i - b_i). \end{aligned}$$

Remark. Let B be a ring with an identity 1 and A a subring of B with common identity 1 of B. In [4], a sequence of additive A-endomorphisms $\{f_0 = 1, f_1, \dots, f_n\}$ of B is said to be a relative sequence of homomorphisms with Ψ if it satisfies the following conditions : for every $j \in N = \{1, 2, \dots, n\}$,

(1) $f_j f_k = f_k f_j$ and $f_k(1) = 0$ for all $k \in N$.
(2) There exists $\Psi_j = \{g(f_j, f_i) ; 0 \leqq i \leqq j\} \subset \mathrm{End}(B_A)$ such that
(i) $f_j(xy) = \sum_{i=0}^j g(f_j, f_i)(x) f_i(y)$ for $x, y \in B$,
(ii) $g(f_j, f_0) = f_j$,
(iii) $g(f_j, f_j)$ is a ring isomorphism.

As is easily seen, for an A-automorphism σ of B, if we put $D = \sigma - 1$, then $\Phi = \{D^0 = 1, D, \dots, D^n\}$ becomes a relative sequence of homomorphisms with Ψ such that

$$\Psi_j = \{g(D^j, D^i) = \binom{j}{i} \sigma^i D^{j-i} ; 0 \leqq i \leqq j\}.$$

A subset $\Phi = \{d_0 = 1, d_1, \dots, d_n\}$ of $\mathrm{End}(B_A)$ is said to be an A-higher derivation of B if $d_j(xy) = \sum_{i=0}^j d_{j-i}(x) d_i(y)$ for $x, y \in B$. Then Φ becomes a relative sequence of homomorphisms with Ψ such that $\Psi_j = \{g(d_j, d_i) = d_{j-i} ; 0 \leqq i \leqq j\}$.

Now, for a relative sequence of homomorphisms $\Phi = \{f_0 = 1, f_2, \dots, f_n\}$ with Ψ, we consider the multiplication subsemigroup L of $\mathrm{End}(B_A)$ which is generated by Φ, and assume the following conditions on L:

(1) There exists a positive integer q such that $(f_k)^q = 0$ and $(f_k)^s \neq 0$ for all $k \in N$ and $0 \leqq s \leqq q-1$.

(2) $\prod_{i=1}^n f_i^{s_i} \neq 0$ if $0 \leqq s_i \leqq q-1$ for all $i \in N$.

(3) If $\Omega = \prod_{i=1}^n f_i^{s_i}$ and $\Lambda = \prod_{i=1}^n f_i^{r_i}$ $(0 \leqq s_i, r_i \leqq q-1)$, then $\Omega = \Lambda$ if and only if $s_i = r_i$ for all $i \in N$.

Then $L = U \cup \{0\}$ where $U = \{\prod_{i=1}^n f_i^{s_i} ; 0 \leqq s_i \leqq q-1\}$ becomes a commutative finite multiplicative subsemigroup of $\mathrm{End}(B_A)$, and U becomes a modular lattice as is shown in Theorem 6.

Further, in this case, we can see that

$$(\mathrm{i})^* : \quad \Omega(xy) = \textstyle\sum_{\Gamma \leq \Omega} g(\Omega, \Gamma)(x)\Gamma(y)$$

for $x, y \in B$ where $g(\Omega, \Gamma)$ is obtained as a sum of products of $g(f_{j_s}, f_{i_s})$'s with $\Pi_s f_{j_s} = \Omega$ and $\Pi_s f_{i_s} = \Gamma$. One of the authors made a study on Galois theory of B/A where $A = B^L = \{b \in B;\ \Lambda(b) = 0$ for all $\Lambda(\neq 1) \in U\}$ in [4]. In that paper, (i)* and the uniqueness of $|\Omega > 1|$ play important rôles. One of motivations of this paper comes from this study of Galois theory.

References

[1] G. Birkhoff : Lattice theory, 3rd ed., Amer. Math. Soc., Providence, R. I., 1967.

[2] A. Björner, A. M. Garsia and R. P. Stanley : An introduction to Cohen-Macaulay partially ordered sets, I. Rival, ed., Ordered Sets, Reidel, Dordrecht/Boston, 1982,

[3] A. H. Clifford : Partially ordered abelian groups, Ann. Math. 41 (1940), 465–473.

[4] K. Kishimoto : On relative sequences of homomorphisms and Galois extensions of rings, to appear in Math. J. Okayama Univ. Vol. 30.

[5] R. P. Stanley : Enumerative combinatorics, Vol, 1, Wadsworth & Brooks, 1986.

Institute of Mathematics
N. Copernicus University, 87-100 Toruń, Poland
Department of Mathematics
Shinshu University, Matsumoto 390, Japan
Department of Mathematics
Okayama University, Okayama 700, Japan

(*Received October 1, 1986*)