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SOME INTEGRAL FORMULAS FOR RIEMANNIAN MANIFOLDS

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1. Introduction. Let (M, g) be a compact n-dimensional Riemannian manifold with positive definite metric tensor $g=(g_{ij})$ and ∇ the Levi-Civita connection. By $R=(R_{hij}{}^{k})$, $\rho(R)=(R_{ij})$ and $\tau(R)=R$, we denote the Riemannian curvature tensor, the Ricci curvature tensor and the scalar curvature, respectively.

In his paper [2], T. Sakai obtained the following integral formula :

$$(1.1)\qquad \int_M K\, dM=\int_M (\|(\nabla\tau)(R)\|^2-4\|(\nabla\rho)(R)\|^2+\|\nabla R\|^2)\, dM,$$

where dM is the volume element of (M, g) and

$$(1.2)\qquad \begin{aligned} K=&4R^{hi}R_h{}^aR_{ia}-4R^{hi}R^{jk}R_{hjki}-2R^{ab}R_{ahij}R_b{}^{hij}\\ &-4R^{hijk}R^a{}_{hj}{}^bR_{aikb}-R^{hijk}R_{hiab}R_{jk}{}^{ab}.\end{aligned}$$

In this note, we define a generalized curvature tensor field $T=(T_{hij}{}^{k})$ by

$$(1.3)\qquad \begin{aligned} T_{hijk}=&-\frac{1}{4}(R_h{}^aR_{aijk}+R_i{}^aR_{hajk}+R_j{}^aR_{hiak}+R_k{}^aR_{hija})\\ &+R^a{}_{hi}{}^bR_{jkab}-R^a{}_{hj}{}^bR_{aikb}+R^a{}_{ij}{}^bR_{ahkb}.\end{aligned}$$

where $T_{hijk}=T_{hij}{}^ag_{ak}$. Then, as for T, we shall prove the following theorems.

Theorem A. *Let (M, g) be a compact Riemannian manifold and C the 3-dimensional Weyl's conformal curuature tensor field. Then the following formula holds :*

$$2\int_M (C, T)\, dM=\int_M (\|(\nabla\tau)(R)\|^2-4\|(\nabla\rho)(R)\|^2+\|\nabla R\|^2)\, dM.$$

Theorem B. *Let (M, g) be a locally symmetric Riemannian manifold. Then the generalized curuature tensor field T is identically zero.*

Theorem C. *Let (M, g) be a compact Riemannian manifold. If the Ricci curuature tensor satisfies the Codazzi equation $\nabla_hR_{ij}=\nabla_iR_{hj}$, then we* *have*

$$\int_M (\boldsymbol{R},\ \boldsymbol{T})\, d\boldsymbol{M} \geqq 0.$$

The equality holds on $\boldsymbol{M}$ *if* $(\boldsymbol{M},\ \boldsymbol{g})$ *is a locally symmetric Riemannian manifold.*

2. Notations and definitions. We denote the contravariant components of $\boldsymbol{g}$ by g^{ij} and follow the summation convention for repeated indices. Let $(\ ,\ )$ denote inner product and $\|\ \|$ norm.

The Riemannian curvature tensor $\boldsymbol{R}=(R_{hij}{}^k)$ is defined by $\boldsymbol{R}(\boldsymbol{X},\ \boldsymbol{Y})\boldsymbol{Z}=[\nabla_X,\ \nabla_Y]\boldsymbol{Z}-\nabla_{[X,Y]}\boldsymbol{Z}$ for vector fields $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ and $\boldsymbol{R}(\partial_h,\ \partial_i)\partial_j=R_{hij}{}^k\partial_k$. Let us denote the covariant components of $\boldsymbol{R}$ by $R_{hijk}=\boldsymbol{g}(\boldsymbol{R}(\partial_h,\ \partial_i)\partial_j,\ \partial_k)$. Then the components of the Ricci curvature tensor $\rho(\boldsymbol{R})$ and the scalar curvature $\tau(\boldsymbol{R})$ are given, respectively, by $R_{ij}=g^{ab}R_{aijb}$ and $R=g^{ab}R_{ab}$.

A (1, 3)-tensor field $\boldsymbol{D}=(D_{hij}{}^k)$ is called a generalized curvature tensor field if it satisfies the conditions :

$$(2.\ 1)\qquad \begin{aligned} &(1)\quad D_{hij}{}^k=-D_{ihj}{}^k,\\ &(2)\quad D_{hijk}=-D_{hikj}\qquad (D_{hijk}=D_{hij}{}^a g_{ak}),\\ &(3)\quad D_{hij}{}^k+D_{ijh}{}^k+D_{jhi}{}^k=0 \end{aligned}$$

We denote the Ricci curvature tensor and the scalar curvature associated with $\boldsymbol{D}$ by $\rho(\boldsymbol{D})=(D_{ij})=(g^{ab}D_{aijb})$ and $\tau(\boldsymbol{D})=D=g^{ab}D_{ab}$, respectively. By virtue of the first Bianchi identity, we can find that $\boldsymbol{T}$ has the properties (1) ~ (3) in (2. 1). The Ricci curvature tensor $\rho(\boldsymbol{T})=(T_{ij})$ and the scalar curvature $\tau(\boldsymbol{T})=T$ associated with the tensor field $\boldsymbol{T}$ are given, respectively, by

$$(2.\ 2)\qquad T_{ij}=\frac{1}{2}(R^{ab}R_{aijb}-R_i{}^aR_{aj}),\qquad T=0.$$

If we define the (1, 3)-tensor fields $\boldsymbol{E}=(E_{hij}{}^k)$ and $\boldsymbol{F}=(F_{hij}{}^k)$, in the covariant form, by

$$(2.\ 3)\qquad \begin{aligned} E_{hijk}&=-\frac{1}{4}(R_h{}^aR_{aijk}+R_i{}^aR_{hajk}+R_j{}^aR_{hiak}+R_k{}^aR_{hija}),\\ F_{hijk}&=R^a{}_{hi}{}^bR_{jkab}-R^a{}_{hj}{}^bR_{aikb}+R^a{}_{ij}{}^bR_{ahkb}, \end{aligned}$$

then the tensor fields $\boldsymbol{E}$ and $\boldsymbol{F}$ are also generalized curvature tensor fields. Obviously, $\boldsymbol{T}=\boldsymbol{E}+\boldsymbol{F}$. The Ricci curvature tensors $\rho(\boldsymbol{E})=(E_{ij})$ and $\rho(\boldsymbol{F})=(F_{ij})$ are given, respectively, by

(2. 4)
$$E_{ij} = -\frac{1}{2}(R^{ab}R_{aijb} + R_i{}^aR_{aj}),$$
$$F_{ij} = R^{ab}R_{aijb},$$

and the scalar curvatures $\tau(E) = E$ and $\tau(F) = F$ are given by

(2. 5)
$$F = -E = R^{ab}R_{ab}.$$

3. Proof of the theorems. Transvecting (1. 3) with R^{hijk} and using the first Bianchi identity, we obtain

(3. 1)
$$2(T, R) = -R^{hijk}R_{hiab}R_{jk}{}^{ab} - 4R^{hijk}R_{ahjb}R^a{}_{ik}{}^b - 2R^{ab}R_{aijk}R_b{}^{ijk},$$

and transvecting (2. 2) with R^{ij}, we have

(3. 2)
$$2(\rho(T), \rho(R)) = R^{hi}R^{jk}R_{hjki} - R_i{}^aR_{aj}R^{ij}.$$

Substituting (3. 1) and (3. 2) into (1. 2), we can find that K is written as

(3. 3)
$$K = 2(T, R) - 8(\rho(T), \rho(R)).$$

On the other hand, the covariant components of the 3-dimensional Weyl's conformal curvature tensor field $C = (C_{hij}{}^k)$ are given by

(3. 4)
$$C_{hijk} = R_{hijk} - (g_{hk}R_{ij} - g_{ik}R_{hj} + R_{hk}g_{ij} - R_{ik}g_{hj}) + \frac{R}{2}(g_{hk}g_{ij} - g_{ik}g_{hj})$$

Transvecting (3. 4) with T^{hijk} and using (2. 2), we have

(3. 5)
$$(C, T) = (R, T) - 4(\rho(R), \rho(T)).$$

By (3. 3), K is rewritten as $K = 2(C, T)$. Thus, we obtain Theorem A.

Next, we define a (0, 6)-tensor field $H = (H_{hijk,pq})$ by

(3. 6)
$$H_{hijk,pq} = -(\nabla_p\nabla_qR_{hijk} - \nabla_q\nabla_pR_{hijk}).$$

Then, $H_{hijk,pq}$ is expressed in the form

(3. 7)
$$H_{hijk,pq} = R_{pqh}{}^aR_{aijk} + R_{pqi}{}^aR_{hajk} + R_{pqj}{}^aR_{hiak} + R_{pqk}{}^aR_{hija}.$$

By contracting h with q in (3. 7) and using (2. 3), we have

(3. 8)
$$H^a{}_{ijk,pa} = R_p{}^aR_{aijk} - F_{jkpi}.$$

If (M, g) is a locally symmetric manifold, then the equation $H=O$ holds. By (3. 8), we have $R_h{}^a R_{aijk} = F_{hijk}$. Substituting this equation into the first equation in (2. 3), we have $T = O$. Thus, we obtain theorem B.

Finally, we shall prove Theorem C. The following integral formula is well known (cf. [1]) :

$$2\int_M (\nabla_h R_{ij} - \nabla_i R_{hj})(\nabla^h R^{ij} - \nabla^i R^{hj}) - R^{hijk} H^a{}_{ijk,ha}\, dM = \int_M \| \nabla R \|^2\, dM \tag{3. 9}$$

where dM is the volume element of (M, g). On the other hand, by (3. 8), we have

$$R^{hijk} H^a{}_{ijk,ha} = -R^{hijk} T_{hijk}. \tag{3. 10}$$

Therefore, if the Ricci curvature tensor $\rho(R) = (R_{ij})$ satisfies the condition $\nabla_h R_{ij} = \nabla_i R_{hj}$, then the formula (3. 9) can be reduced to

$$2\int (R,\ T)\ dM = \int_M \| \nabla R \|^2\, dM. \tag{3. 11}$$

Theorem C is an immediate consequence of (3. 11).

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