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ON THE SPACES $U(2n)/\Delta U$ AND THE BOTT MAPS

Dedicated to Professor Nobuo Shimada on his 60th birthday

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Introduction. In [9] the author defined certain maps

$$\varphi_n^o : O(4n)/Sp \to \widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R}; O(4n)) \text{ and } \varphi_n^{sp} : Sp(n)/O \to \widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R}; Sp(n))$$

whose direct limits

$$\varphi_\infty^o : O(\infty)/Sp \to \widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R}; O(\infty)) \text{ and } \varphi_\infty^{sp} : Sp(\infty)/O \to \widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R}; Sp(\infty))$$

are homotopy equivalences, where $\mathbf{P}_2\mathbf{R}$ is the real projective plane and $\widetilde{\mathscr{C}}(X; Y)$ denotes the space of basepoint-preserving continuous maps from X to Y. In the present paper, we consider a "unitary analogue" of the matter treated in [9], and in what follows, we continue to use the notation and conventions of [9].

Let $U(n)$ be the group of $n \times n$ complex unitary matrices. In this paper we define maps

$$\varphi_n^U : U(2n)/\Delta U \to \widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R}; U(2n)) \ (n = 1, 2, \ldots),$$

where $U(2n)/\Delta U$ are certain homogeneous spaces such that (as will be shown by our theorem) the limit $U(\infty)/\Delta U = \varinjlim U(2n)/\Delta U$ is a classifying space for the functor $KU^1(\ ; \mathbf{Z}/2)$. The definition of φ_n^U is given in § 3, and one of our purposes here is to prove (Theorem 3. 5) that the limit map

$$\varphi_\infty^U : U(\infty)/\Delta U \to \widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R}; U(\infty))$$

is a homotopy equivalence, and that the homomorphism

$$(\varphi_n^U)_* : \pi_r(U(2n)/\Delta U) \to \pi_r(\widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R}; U(2n)))$$

induced by φ_n^U is isomorphic for $r \leq 2n$ with $(r, n) \neq (2, 1)$.

Also, in § 5, we define certain maps

$$\lambda_n^{U/\Delta U} : O(4n)/Sp \to U(4n)/\Delta U \text{ and } \mu_n^{U/\Delta U} : Sp(n)/O \to U(2n)/\Delta U,$$

which correspond to the natural embeddings

$$O(4n) \to U(4n) \text{ and } Sp(n) \to U(2n).$$

These maps are "compatible" with the maps φ_n^U, φ_n^O and φ_n^{Sp}, as we shall show (see Proposition 5.3).

1. Preliminaries. As noted in the introduction, we retain the notation of [9]. Thus

$$J_n = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix},$$

I_n being the $n \times n$ identity matrix. Given square matrices A and B, we often write

$$\operatorname{diag}(A, B) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

In particular we put

$$T_n = \operatorname{diag}(I_n, -I_n) = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}.$$

We define the map $m_n^U : U(n) \times U(n) \to U(2n)$ by putting

$$m_n^U(A, B) = P_n \operatorname{diag}(A, B) P_n^{-1} \text{ for } (A, B) \in U(n) \times U(n),$$

where P_n is the $2n \times 2n$ permutation matrix defined in [9; § 1]. Clearly m_n^U is an injective homomorphism, and by this we embed the product group $U(n) \times U(n)$ in $U(2n)$. We consider the associated homogeneous space, and write

$$U(2n)/(U \times U) = U(2n)/m_n^U(U(n) \times U(n)),$$

where $m_n^U(U(n) \times U(n))$ is the image of $U(n) \times U(n)$ by m_n^U. Also, we consider the subgroup $\Delta U(n) = \{(A, A) \mid A \in U(n)\}$ of $U(n) \times U(n)$, and put

$$U(2n)/\Delta U = U(2n)/m_n^U(\Delta U(n)),$$

where $m_n^U(\Delta U(n)) = \{P_n \operatorname{diag}(A, A) P_n^{-1} \mid A \in U(n)\} \subset m_n^U(U(n) \times U(n))$. The canonical surjections

$$U(2n) \to U(2n)/(U \times U) \text{ and } U(2n) \to U(2n)/\Delta U$$

are denoted by $\xi_n^{U/(U \times U)}$ and $\xi_n^{U/\Delta U}$ respectively. Note that $U(2n)/(U \times U)$ is just the space denoted by G_n in [6; § 1] and can be identified with the complex Grassmann manifold of n-dimensional C-vector subspaces of C^{2n}.

We define the limit spaces

$$U(\infty)/(U\times U) = \varinjlim U(2n)/(U\times U) \text{ and } U(\infty)/\Delta U = \varinjlim U(2n)/\Delta U$$

in the usual way.

Further, we define the map $\nu_n^U: U(n) \to U(2n)$ by putting

$$\nu_n^U(A) = P_n \mathrm{diag}(A, I_n) P_n^{-1} \text{ for } A \in U(n),$$

and define $\Delta_n^U: U(n) \to U(n)\times U(n)$ to be the diagonal map. Note that the sequences

$$(1.1) \quad U(n) \xrightarrow{\xi_n^{U/\Delta U}\circ \nu_n^U} U(2n)/\Delta U \longrightarrow U(2n)/(U\times U)$$

and

$$(1.2) \quad U(n) \xrightarrow{m_n^U\circ \Delta_n^U} U(2n) \xrightarrow{\xi_n^{U/\Delta U}} U(2n)/\Delta U$$

are fibration sequences, where the unlabelled map in (1.1) is the canonical surjection associated to the inclusion $\Delta U \subset U\times U$.

2. Bott maps for the unitary case. Here we recall well-known results on the Bott maps for the unitary group. As usual let $\Omega(X)$ denote the loop space of X, and let $\Omega_0(X)$ denote the arcwise-connected component of the trivial loop. Let us consider the maps

$$\omega_n^U: U(2n)/(U\times U) \to \Omega_0(U(2n)),$$
$$\omega_n^{U/(U\times U)}: U(n) \to \Omega(U(2n)/(U\times U))$$

defined as follows :

$$\omega_n^U(\xi_n^{U/(U\times U)}(P_nAP_n^{-1}))(t) = P_n \mathrm{comm}(\exp(\pi t i T_n), A) P_n^{-1}$$

where $A \in U(2n)$, $t \in [0, 1]$;

$$\omega_n^{U/(U\times U)}(A)(t) = \xi_n^{U/(U\times U)}\left(P_n \mathrm{comm}\left(\exp\left(\frac{\pi}{2} tJ_n\right), \mathrm{diag}(A, I_n)\right)P_n^{-1}\right)$$
$$= \xi_n^{U/(U\times U)}\left(P_n \exp\left(\frac{\pi}{2} tJ_n\right)\mathrm{diag}(A, I_n)\exp\left(-\frac{\pi}{2} tJ_n\right)P_n^{-1}\right)$$

where $A \in U(n)$, $t \in [0, 1]$. Here $\mathrm{comm}(A, B) = ABA^{-1}B^{-1}$, and exp is the exponential map. Note that

$$
\begin{aligned}
\exp(tiT_n) &= I_{2n}\cos(t)+iT_n\sin(t),\\
\exp(tJ_n) &= I_{2n}\cos(t)+J_n\sin(t)
\end{aligned}
$$

for every $t \in \mathbf{R}$. Taking the direct limits, we then get maps

$$
\begin{aligned}
\omega_\infty^U &= \varinjlim \omega_n^U : U(\infty)/(U\times U) \to \Omega_0(U(\infty)),\\
\omega_\infty^{U/(U\times U)} &= \varinjlim \omega_n^{U/(U\times U)} : U(\infty) \to \Omega(U(\infty)/(U\times U)),
\end{aligned}
$$

and the Bott periodicity theorem for the unitary group is an immediate consequence of the following :

Theorem 2.1 (see [1], [2], [3], [4], [5], and also [8; § 23]). *The maps ω_∞^U and $\omega_\infty^{U/(U\times U)}$ are homotopy equivalences.*

3. The maps φ_n^U. As in [9], let $\widetilde{\mathscr{C}}(X;\,Y)$ denote the space of basepoint-preserving continuous maps from X to Y. For each n, we define the map

$$
\varphi_n^U : U(2n)/\Delta U \to \widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R};\,U(2n))
$$

by

$$
\varphi_n^U(\xi_n^{U/\Delta U}(P_nAP_n^{-1}))([u_0 : u_1 : u_2]) = P_n\mathrm{comm}(u_0I_{2n}+u_1iT_n+u_2J_n,\,A)P_n^{-1}
$$

where $A \in U(2n)$, $(u_0, u_1, u_2) \in \mathbf{R}^3$ and $u_0^2+u_1^2+u_2^2 = 1$. Let $\widetilde{\mathscr{C}}_0(X;\,Y)$ denote the arcwise-connected component of the basepoint in $\widetilde{\mathscr{C}}(X;\,Y)$, and consider the diagram

$$
\begin{array}{ccccc}
U(n) & \xrightarrow{\ \xi_n^{U/\Delta U}\circ\,\nu_n^U\ } & U(2n)/\Delta U & \longrightarrow & U(2n)/(U\times U)\\
\downarrow{\scriptstyle \omega_n^{U/(U\times U)}} & & \Big| & & \Big|\\
\Omega(U(2n)/(U\times U)) & & & & \\
\Big| & (3.1\mathrm{a}) & \varphi_n^U & (3.1\mathrm{b}) & \omega_n^U\\
\downarrow{\scriptstyle \Omega(\omega_n^U)} & & \Big| & & \downarrow\\
\Omega^2(U(2n)) & & & & \Omega_0(U(2n))\\
\| & & \downarrow & & \|\\
\widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R}/\mathbf{P}_1\mathbf{R};\,U(2n)) & \to & \widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R};\,U(2n)) & \to & \widetilde{\mathscr{C}}_0(\mathbf{P}_1\mathbf{R};\,U(2n))
\end{array}
\tag{3.1}
$$

where the top row is just the fibration sequence (1.1) in § 1 and the bottom row is induced by the cofibration sequence

$$\mathbf{P}_2\mathbf{R}/\mathbf{P}_1\mathbf{R} \leftarrow \mathbf{P}_2\mathbf{R} \leftarrow \mathbf{P}_1\mathbf{R}.$$

Passing to the direct limit and writing $\varphi_\infty^U = \varinjlim \varphi_n^U$, etc., we then get the diagram

$$(3.2)\qquad \begin{array}{ccccc}
U(\infty) & \xrightarrow{\xi_\infty^{U/\Delta U}\circ\,\nu_\infty^U} & U(\infty)/\Delta U & \longrightarrow & U(\infty)/(U\times U) \\
\downarrow{\scriptstyle \omega_\infty^{U/(U\times U)}} & & & & \\
\Omega(U(\infty)/(U\times U)) & \text{(3.2a)} & \downarrow{\scriptstyle \varphi_\infty^U} & \text{(3.2b)} & \downarrow{\scriptstyle \omega_\infty^U} \\
\downarrow{\scriptstyle \Omega(\omega_\infty^U)} & & & & \\
\Omega^2(U(\infty)) & & & & \Omega_0(U(\infty)) \\
\| & & & & \| \\
\widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R}/\mathbf{P}_1\mathbf{R};\, U(\infty)) & \to & \widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R};\, U(\infty)) & \to & \widetilde{\mathscr{C}}_0(\mathbf{P}_1\mathbf{R};\, U(\infty))
\end{array}$$

and, in the next section, we shall prove :

Proposition 3.3. *The diagrams* (3.1) *and* (3.2) *are homotopy-commutative. In particular,* (3.1b) *and* (3.2b) *are strictly commutative.*

On the other hand, we have :

Proposition 3.4. *The homomorphism* $\pi_r(U(2n)/\Delta U) \to \pi_r(U(\infty)/\Delta U)$ *induced by the canonical injection is isomorphic for* $r \leq 2n$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
U(\infty) & \xrightarrow{m_\infty^U\circ\,\Delta_\infty^U} & U(\infty) & \xrightarrow{\xi_\infty^{U/\Delta U}} & U(\infty)/\Delta U \\
\uparrow & & \uparrow & & \uparrow \\
U(n) & \xrightarrow{m_n^U\circ\,\Delta_n^U} & U(2n) & \xrightarrow{\xi_n^{U/\Delta U}} & U(2n)/\Delta U
\end{array}$$

where the bottom row is the fibration sequence (1.2) in § 1 and the top row is the direct limit of (1.2), and where the vertical maps are the canonical injections. Then since the homomorphism

$$\pi_r(U(n)) \to \pi_r(U(\infty))$$

induced by the canonical injection is an isomorphism for $r \leq 2n-1$ and an epimorphism for $r = 2n$, the proposition follows immediately by the five-lemma.

Combining Theorem 2.1 and Proposition 3.3, and noting Proposition 3.4, we obtain the following, which is a "unitary version" of Theorem 3.6 of [9] :

Theorem 3.5. *The map* φ^U_∞ *is a homotopy equivalence, and the homomorphism* $(\varphi^U_n)_* : \pi_r(U(2n)/\Delta U) \to \pi_r(\widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R};\ U(2n)))$ *induced by* φ^U_n *is isomorphic for* $r \leq 2n$ *with* $(r, n) \neq (2, 1)$.

Proof. By the same five-lemma argument as in the proof of Theorem 3.6 of [9], it follows from Theorem 2.1 and Proposition 3.3 that φ^U_∞ induces isomorphisms of homotopy groups in all dimensions. Thus, by J. H. C. Whitehead's theorem (and by [7 ; Theorem 3]), the first part of the theorem follows.

It remains to prove the result about the homomorphism induced by φ^U_n. Let us consider the commutative diagram

$$\begin{array}{ccccc}
\widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R}/\mathbf{P}_1\mathbf{R};\ U(\infty)) & \to & \widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R};\ U(\infty)) & \to & \widetilde{\mathscr{C}}_0(\mathbf{P}_1\mathbf{R};\ U(\infty)) \\
\| & & \uparrow & & \| \\
\Omega^2(U(\infty)) & & & & \Omega_0(U(\infty)) \\
\uparrow & & & & \uparrow \\
\Omega^2(U(2n)) & & & & \Omega_0(U(2n)) \\
\| & & & & \| \\
\widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R}/\mathbf{P}_1\mathbf{R};\ U(2n)) & \to & \widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R};\ U(2n)) & \to & \widetilde{\mathscr{C}}_0(\mathbf{P}_1\mathbf{R};\ U(2n))
\end{array}$$

where the rows are induced by the cofibration sequence

$$\mathbf{P}_2\mathbf{R}/\mathbf{P}_1\mathbf{R} \leftarrow \mathbf{P}_2\mathbf{R} \leftarrow \mathbf{P}_1\mathbf{R}$$

and the vertical arrows are the maps induced by the canonical injection $U(2n) \to U(\infty)$. Then noting that $(U(\infty), U(2n))$ is $4n$-connected, we see by the five-lemma that the canonical homomorphism

$$\pi_r(\widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R};\ U(2n))) \to \pi_r(\widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R};\ U(\infty)))$$

is isomorphic for $r \leq 4n-3$. Next, consider the commutative diagram

$$\begin{array}{ccc}
\pi_r(U(\infty)/\Delta U) & \xrightarrow{(\varphi^U_\infty)_*} & \pi_r(\widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R};\ U(\infty))) \\
\uparrow & & \uparrow \\
\pi_r(U(2n)/\Delta U) & \xrightarrow{(\varphi^U_n)_*} & \pi_r(\widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R};\ U(2n)))
\end{array}$$

where the vertical homomorphisms are induced by the canonical injections. Then noting Proposition 3. 4 (and the fact that $2n \leq 4n-3$ for $n \geq 2$), we obtain the remaining part of the theorem.

Remark. For $(r, n) = (2, 1)$, it is easy to see that the group $\pi_2(U(2)/\Delta U)$ is trivial and $\pi_2(\widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R}; U(2)))$ is a cyclic group of order 2. Thus the homomorphism

$$(\varphi_1^U)_* : \pi_2(U(2)/\Delta U) \to \pi_2(\widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R}; U(2)))$$

is trivial and not epimorphic.

4. Proof of Proposition 3.3. The proof of Proposition 3. 3 is analogous to that of Proposition 3. 5 of [9], but we present this here. First, note that

$$\begin{aligned}&\varphi_n^U(\xi_n^{U/\Delta U}(P_nAP_n^{-1}))([\cos(\pi t) : \sin(\pi t)\cos(\pi s) : \sin(\pi t)\sin(\pi s)])\\ &= P_n\mathrm{comm}\left(\exp\left(\frac{\pi}{2}siT_nJ_n\right)\exp(\pi tiT_n)\exp\left(-\frac{\pi}{2}siT_nJ_n\right), A\right)P_n^{-1}\end{aligned}$$

where $A \in U(2n)$, $s \in [0, 1]$, $t \in [0, 1]$. Then we can easily check the commutativity of (3. 1b) and (3. 2b).

To prove the homotopy-commutativity of (3. 1a) and (3. 2a), let us now put

$$\begin{aligned}&E_n(r, s, t)\\ &= \exp\left(\frac{\pi}{4}riT_n\right)\exp\left(\frac{\pi}{2}siT_nJ_n\right)\exp(\pi tiT_n)\exp\left(-\frac{\pi}{2}siT_nJ_n\right)\exp\left(-\frac{\pi}{4}riT_n\right),\end{aligned}$$

and consider the family of maps

$$\Theta_n^U(r) : U(n) \to \Omega^2(U(2n)) \ (r \in [0, 1])$$

defined by

$$\begin{aligned}&\Theta_n^U(r)(A)(s)(t)\\ &= P_n\exp\left(\frac{\pi}{2}rsJ_n\right)\mathrm{comm}(E_n(r, s, t), \mathrm{diag}(A, I_n))\exp\left(-\frac{\pi}{2}rsJ_n\right)P_n^{-1}\end{aligned}$$

where $A \in U(n)$, $s \in [0, 1]$, $t \in [0, 1]$. Then the diagram

$$
\begin{array}{ccc}
U(n) & \xrightarrow{\xi_n^{U/\Delta U}\circ \nu_n^U} & U(2n)/\Delta U \\
\downarrow \Theta_n^U(0) & & \\
\Omega^2(U(2n)) & & \Big\downarrow \varphi_n^U \\
\| & & \\
\widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R}/\mathbf{P}_1\mathbf{R};\, U(2n)) & \to & \widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R};\, U(2n))
\end{array}
$$

commutes, where as in (3.1a) the bottom map is induced by the canonical surjection $\mathbf{P}_2\mathbf{R} \to \mathbf{P}_2\mathbf{R}/\mathbf{P}_1\mathbf{R}$. On the other hand, noting

$$E_n(1, s, t) = \exp\left(-\frac{\pi}{2} sJ_n\right)\exp(\pi t i T_n)\exp\left(\frac{\pi}{2} sJ_n\right),$$

we see, by a direct calculation, that

$$\Theta_n^U(1) = \Omega(\omega_n^U)\circ \omega_n^{U/(U\times U)}.$$

Thus the homotopy-commutativity of (3.1a) follows, and if we consider the direct limit $\Theta_\infty^U(r) = \varinjlim \Theta_n^U(r)$, we see that (3.2a) is also homotopy-commutative.

5. Compatibility. Let us now consider the canonical embeddings

$$\iota_n^U : O(n) \to U(n) \text{ and } \varkappa_n^U : Sp(n) \to U(2n)$$

defined by putting

$$\iota_n^U(A) = A \text{ for } A \in O(n),$$

and

$$\varkappa_n^U(A) = P_n \mathrm{deq}(A) P_n^{-1} \text{ for } A \in Sp(n),$$

with deq(A) being as in [9; § 1]. Further, let us define the maps

$$\lambda_n^{U/\Delta U} : O(4n)/Sp \to U(4n)/\Delta U \text{ and } \mu_n^{U/\Delta U} : Sp(n)/O \to U(2n)/\Delta U$$

as follows :

$$\begin{aligned}
&\lambda_n^{U/\Delta U}(\xi_n^{O/Sp}(Q_n A Q_n^{-1})) \\
&= \xi_{2n}^{U/\Delta U}\left(Q_n \frac{1}{\sqrt{2}}\begin{bmatrix} I_{2n} & iI_{2n} \\ iJ_n & J_n \end{bmatrix} A \frac{1}{\sqrt{2}}\begin{bmatrix} I_{2n} & iJ_n \\ -iI_{2n} & -J_n \end{bmatrix} Q_n^{-1}\right)
\end{aligned}$$

where $A \in O(4n)$;

$$\mu_n^{U/\Delta U}(\xi_n^{Sp/O}(A)) = \xi_n^{U/\Delta U}(P_n \mathrm{deq}(A) P_n^{-1})$$

where $A \in Sp(n)$. Here, as in [9], $Q_n = P_{2n}\mathrm{diag}(P_n, P_n)$. Then we have diagrams

$$(5.1)\qquad \begin{array}{ccc} O(4n)/Sp & \xrightarrow{\lambda_n^{U/\Delta U}} & U(4n)/\Delta U \\ \downarrow \varphi_n^O & & \downarrow \varphi_{2n}^U \\ \widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R};\, O(4n)) & \to & \widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R};\, U(4n)) \end{array}$$

and

$$(5.2)\qquad \begin{array}{ccc} Sp(n)/O & \xrightarrow{\mu_n^{U/\Delta U}} & U(2n)/\Delta U \\ \downarrow \varphi_n^{Sp} & & \downarrow \varphi_n^U \\ \widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R};\, Sp(n)) & \to & \widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R};\, U(2n)) \end{array}$$

where φ_n^O and φ_n^{Sp} are the maps defined in [9 ; § 3] and where the bottom map of (5.1) (resp. of (5.2)) is induced by the canonical embedding

$$\iota_{4n}^U : O(4n) \to U(4n) \text{ (resp. } \kappa_n^U : Sp(n) \to U(2n)).$$

The following proposition shows that "complexification" and "dequaternionification" are compatible (up to homotopy) with the maps φ_n^U, φ_n^O and φ_n^{Sp}.

Proposition 5.3. *The diagram* (5.1) *is homotopy-commutative, and the diagram* (5.2) *is strictly commutative.*

Proof. The proof of the commutativity of (5.2) is straightforward, and we leave this to the reader. The homotopy-commutativity of (5.1) can be seen as follows. Put

$$Z_n(t) = \mathrm{diag}\left(I_{2n}, \exp\left(\frac{\pi}{2} tJ_n\right)\right)\exp\left(\frac{\pi}{4} tiJ_{2n}T_{2n}\right),$$

so that

$$Z_n(1) = \frac{1}{\sqrt{2}}\begin{bmatrix} I_{2n} & iI_{2n} \\ iJ_n & J_n \end{bmatrix} \text{ and } (Z_n(1))^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} I_{2n} & iJ_n \\ -iI_{2n} & -J_n \end{bmatrix}.$$

Further, consider the family of maps

$$\Phi_n(t) : O(4n)/Sp \to \widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{R};\, U(4n)) \; (t \in [0, 1])$$

defined by

$$\Phi_n(t)(\xi_n^{O/Sp}(Q_nAQ_n^{-1}))([u_0 : u_1 : u_2])$$
$$= Q_nZ_n(t)\mathrm{comm}\left(u_0I_{4n}+u_1J_{2n}+u_2K_n\exp\left(\frac{\pi}{2}tJ_{2n}\right), A\right)(Z_n(t))^{-1}Q_n^{-1}$$

where $A \in O(4n)$, $(u_0, u_1, u_2) \in \mathbf{R}^3$ and $u_0^2+u_1^2+u_2^2 = 1$. Then $\Phi_n(0)$ is just the composite

$$O(4n)/Sp \xrightarrow{\varphi_n^O} \widetilde{\mathscr{C}}(\mathrm{P}_2\mathbf{R}; O(4n)) \rightarrow \widetilde{\mathscr{C}}(\mathrm{P}_2\mathbf{R}; U(4n))$$

(where the second map is the bottom of (5. 1)), while

$$\Phi_n(1) = \varphi_{2n}^U \circ \lambda_n^{U/\Delta U}.$$

So $(\Phi_n(t))_{t\in[0,1]}$ provides the required homotopy.

Appendix: remarks on the maps m_∞^U and ν_∞^U. It is easy to see that for every n the map ν_n^U defined in § 1 is homotopic to the canonical injection

$$A \mapsto \mathrm{diag}(A, I_n) : U(n) \rightarrow U(2n),$$

and that the limit map $\nu_\infty^U = \varinjlim \nu_n^U$ is a homotopy self-equivalence of $U(\infty)$. In fact, one can further show the following : Let G denote either U, SU, O, SO, or Sp (so that $G(n)$ is one of the classical Lie groups $U(n)$, $SU(n)$, $O(n)$, $SO(n)$, or $Sp(n)$), and let us define the maps

$$\nu_n^G : G(n) \rightarrow G(2n)\ (n = 1, 2, \ldots)$$

by putting $\nu_n^G(A) = P_n\mathrm{diag}(A, I_n)P_n^{-1}$ for $A \in G(n)$, and consider the limit map

$$\nu_\infty^G = \varinjlim \nu_n^G : G(\infty) \rightarrow G(\infty).$$

Then

Lemma A.1. *The map ν_∞^G is homotopic to the identity map $1_{G(\infty)}$ of $G(\infty)$.*

This implies that the map $\xi_\infty^{U/\Delta U} \circ \nu_\infty^U$ in the top row of (3. 2) is homotopic to the canonical surjection $\xi_\infty^{U/\Delta U}$ from $U(\infty)$ onto $U(\infty)/\Delta U$.

We leave the proof of Lemma A. 1 to the reader.[1] Note that Lemma A. 1 can be used to show the following well-known fact : For $G = U$, SU, O, SO or Sp, let us define the maps

$$m_n^G : G(n)\times G(n) \to G(2n)\ (n = 1, 2, \ldots)$$

by putting $m_n^G(A, B) = P_n \mathrm{diag}(A, B)P_n^{-1}$ for $(A, B) \in G(n)\times G(n)$, and consider the limit map

$$m_\infty^G = \varinjlim m_n^G : G(\infty)\times G(\infty) \to G(\infty).$$

Let I_∞ be the identity element of the group $G(\infty)$. Then

Proposition A.2 (see for instance [5 ; § 1]). *The map m_∞^G defines a Hopf space structure on $G(\infty)$ with I_∞ being the basepoint. In other words, I_∞ is a homotopy unit under the multiplication m_∞^G.*

Remark. Lemma A. 1 shows that under m_∞^G the element I_∞ acts as a homotopy right unit.

References

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