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ON LATTICE POINTS IN PLANAR DOMAINS

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1. Introduction. Let $\mathscr{D}$ be a compact subset of $\boldsymbol{R}^2$ bounded by a (closed) smooth Jordan curve $\mathscr{C}$ which is defined by $\phi(u, v) = 0$ where ϕ is analytic on $\mathscr{C}$ and grad $\phi \neq (0, 0)$ throughout. For a large real parameter T, denote by $A(T)$ the number of lattice points (of the standard lattice $\boldsymbol{Z}^2$) in the "blown up" domain $T\mathscr{D}$ (points at the boundary of any set involved are to be counted with weight $1/2$, throughout the paper), by V the area of $\mathscr{D}$ and define the "lattice rest" by $P(T) := A(T) - VT^2$. The classical problems of lattice point theory involve the order resp. the asymptotic behaviour of $P(T)$ for special domains $\mathscr{D}$. (See Fricker [3] for an enlightening survey.)

Under the assumption that the curvature $\varkappa$ of $\mathscr{C}$ does not vanish anywhere, one obtains results quite similar to the case of the circle: It is known (due to Van der Corput [13]) that

$$P(T) = O(T^{\theta}) \tag{1}$$

for some exponent $\theta < 2/3$, furthermore [9] that

$$P(T) = \Omega_{-}(T^{1/2}(\log T)^{1/4})$$

and [10]

$$\int_0^T P^2(\sqrt{t})\, dt = O(T^{3/2}).$$

For the case that the curvature $\varkappa$ of $\mathscr{C}$ has only zeros of order $\leqq n-2$ ($n \geqq 3$), Y. Colin de Verdière [2] derived the estimate

$$P(T) = O(T^{1-1/n}),$$

using a modern method based on the theory of singularities.

Let P_l be all points of $\mathscr{C}$ with $\varkappa = 0$ and let $n_l - 2$ denote the order of these zeros ($n_l \geqq 3$), and suppose in addition that $\mathscr{C}$ has rational slope in each of the P_l, then a precise asymptotic formula for the lattice rest is known, with an error term $O(T^{\theta})$, $\theta < 2/3$, like in (1) : According to [8] one has

$$P(T) = \sum_{P_l} \sum_{j=1}^{n_l-1} F_{j,l}(T) T^{1-j/n_l} + O(T^{\theta}), \tag{2}$$

where the functions $F_{j,i}(T)$ are $O(1)$ as $T \to \infty$ and (in general) $\Omega_{\pm}(1)$ and have been given explicitly (by absolutely convergent Fourier series) in [8]. (Observe that, since $n_i \geqq 3$, the main term here is at least of order $T^{2/3}$, hence it actually dominates the error term.)

2. Statement of results. It is the objective of the present paper to extend these investigations to the case that $\mathscr{C}$ contains points Q_i with curvature 0 and irrational slope β_i. It turns out that, under suitable assumptions about the "approximability" of the β_i by rationals, these points Q_i contribute only a comparatively "small" amount to the lattice rest which is not greater than the error term in the case $\varkappa \neq 0$.

Theorem. *Let $\mathscr{D}$ be a compact planar domain bounded by a closed smooth Jordan curve $\mathscr{C}$ which is defined by $\phi(u, v) = 0$ where $\phi(u, v)$ is an analytic function on $\mathscr{C}$ with* grad $\phi \neq (0, 0)$. *Suppose that the curvature $\varkappa$ of $\mathscr{C}$ has zeros exactly in two (finite) sets of points $\{P_i\}$ and $\{Q_i\}$ with the following properties :*

(i) *In the points P_i the slope of $\mathscr{C}$ is rational and $\varkappa$ has a zero of order n_i-2 $(n_i \geqq 3)$.*

(ii) *In the points Q_i the slope β_i of $\mathscr{C}$ is irrational, and there exist positive numbers c and α_i such that*

$$|h\beta_i-p| \geqq ch^{-1-\alpha_i} \tag{3}$$

for any $h \in \boldsymbol{N}$ and $p \in \mathbb{Z}$. Moreover, if m_i-2 is the order of the zero of $\varkappa$ in Q_i $(m_i \geqq 3)$, we suppose that

$$\alpha_i < (3m_i-7)/(m_i-2)(m_i-3). \tag{4}$$

(If $m_i = 3$, α_i may be arbitrarily large.)

Then

$$P(T) = \sum_{P_i}\sum_{j=1}^{n_i-1} F_{j,i}(T)T^{1-j/n_i}+O(T^{\theta})$$

with an exponent $\theta < \frac{2}{3}$, the O-constant depending on $\mathscr{D}$. The functions $F_{j,i}(T)$ are bounded and (in general) $\Omega_{\pm}(1)$ and can be represented as in [8].

Corollary 1. *If the curvature of $\mathscr{C}$ is 0 only in points Q_i with irrational* *slope satisfying the above condition* (ii), *then*

$$P(T) = O(T^{\theta}). \qquad \left(\theta < \frac{2}{3}\right)$$

Remarks. 1. By the celebrated theorem of Thue-Siegel-Roth (see e. g. Cassels [1], p. 104), for algebraic irrationals β_i the numbers α_i in (3) can be chosen arbitrarily small, hence (4) is certainly fulfilled. Therefore the case that $\mathscr{C}$ is an algebraic curve is completely contained in our theorem. This result, in particular, improves and generalizes a work by M. Tarnopolska-Weiss [12].

2. According to the metric theorem of Khintchine (see e. g. [6], p. 74), the same is true for almost all numbers β_i (in the sense of Lebesgue measure). Thus we obtain the following corollary, which improves earlier results of B. Randol [11], M. Tarnopolska-Weiss [12] and Y. Colin de Verdière [2].

Corollary 2. *Let* $\mathscr{D}$ *be as in the theorem* (*without any assumptions about the boundary points with curvature* 0). *Consider the image* $\mathscr{D}_\tau$ *of* $\mathscr{D}$ *under a rotation by an angle* τ (*with the origin as center*). *Then for almost all* τ $(0 \leqq \tau \leqq 2\pi)$ *the lattice rest* $P_\tau(T)$ *of* $T\mathscr{D}_\tau$ *satisfies*

$$P_\tau(T) = O_\tau(T^{\theta})$$

with an exponent $\theta < \frac{2}{3}$ *not depending on* τ.

3. Proof of the theorem. According to the discussion in [8], it suffices to treat the problem for any single point Q_i of $\mathscr{C}$ with $\varkappa = 0$ and irrational slope β_i. (The points P_i with rational slope have been dealt with completely in [8].) By obvious symmetry considerations (cf. also [8]), we only have to establish the following.

Proposition. *Let* $\{(u, g(u)) : a \leqq u \leqq b\}$ *be a piece of the curve* $\mathscr{C}$, *and suppose that* $g''(a) = 0$, $g'(a) = \beta \notin \mathbb{Q}$, *and* $g''(u) \neq 0$, $g(u) > 0$, $g'(u)$ *bounded for* $a < u \leqq b$. *The point* $Q = (a, g(a))$ *may satisfy condition* (ii) *of our theorem. Then for the lattice rest* $P^*(T)$ *of the domain* $aT \leqq x \leqq bT$, $0 \leqq y \leqq G(x) := T\, g\left(\frac{x}{T}\right)$, *we have*

$$P^*(T) = \psi(aT)\,T\,g(a) - \psi(bT)\,T\,g(b) + O(T^{\theta})$$

with $\theta < \frac{2}{3}$, *where* $\psi(z) = 0$ *for* $z \in \mathbf{Z}$, $\psi(z) = z - [z] - \frac{1}{2}$ *otherwise.*

Proof. By Euler's summation formula, we obtain for the corresponding number of lattice points

$$A^*(T) = \sum_{aT \leq k \leq bT}^* (G(k) - \psi(G(k))) = \int_{aT}^{bT} G(x)\, dx - \psi(bT)\, G(bT) +$$

$$+\psi(aT)\, G(aT) + \int_{aT}^{bT} \psi(x) G'(x)\, dx - \sum_{aT \leq k \leq bT}^* \psi(G(k))$$

(summation being extended over all integers of the interval indicated, the terms corresponding to its boundary points (if these are integers) being weighted with the factor $\frac{1}{2}$). Since, by the second mean-value theorem, the second integral can be estimated by $O(1)$, we only have to show that, for some $\theta < \frac{2}{3}$,

$$\sum_{aT \leq k \leq bT}^* \psi(G(k)) = O(T^\theta). \tag{5}$$

To this end we split up the interval of summation : The sum

$$S_0 := \sum_{W \leq k \leq bT}^* \psi(G(k))$$

where $W = aT + T^{1-\varepsilon}$, $\varepsilon > 0$ some sufficiently small constant, is estimated by a deep theorem of Van der Corput.

Lemma 1 (Van der Corput [13], Satz 2). *Suppose that the real-valued function* $G(x)$ *is five times continuously differentiable on a compact interval* I, *and for any three integers* $p, q, r \geqq 0$ *with* $p+q+r = 3$, *for some* $\eta > 0$,

$$|G^{(p+2)}(x)\, G^{(q+2)}(x)\, G^{(r+2)}(x)| \leqq |G''(x)|^{17/3+\eta} \tag{6}$$

on I. *Assume further that* $G''(x)$ *is monotone and* $\neq 0$ *on* I *and, for some* $\eta' > 0$,

$$|G'''(x)| \leqq |G''(x)|^{4/3+\eta'}. \tag{7}$$

Then there exists $\omega > 0$ *(depending only on* η *and* η'*) such that*

$$\sum_I^* \psi(G(k)) \ll \int_I |G''(x)|^{1/3+\omega}\, dx + \max_I (|G''(x)|^{-1/2}). \tag{8}$$

We verify the conditions of this lemma for our function $G(x)$: By the assumptions of the proposition, in some neighbourhood of $u = a$ (putting $x = Tu$), we have

$$g'(u) = \beta + \sum_{j=m-1}^{\infty} c_j (u-a)^j$$

with $c_{m-1} \neq 0$, where $m-2$ is the order of the zero of χ in $(a, g(a))$. Therefore,

$$T^{-1}(u-a)^{m-2} \ll |G''(x)| \ll T^{-1}(u-a)^{m-2}$$

and

$$G^{(j)}(x) \ll T^{1-j}(u-a)^{m-j} \qquad (j \geqq 2)$$

for $a < u \leqq b$. Then (6) is true for sufficiently large T if

$$T^{-1+3\eta}(u-a)^{-8m+7-3\eta(m-2)} = o(1)$$

which, for $u-a \geqq T^{-\varepsilon}$, is obvious for sufficiently small η and ε. In the same way one verifies (7), arriving at

$$T^{-2+3\eta'}(u-a)^{-m-1-3\eta'(m-2)} = o(1).$$

From lemma 1 we therefore infer the estimate

$$S_0 \ll T^{-\frac{1}{3}-\omega} \int_W^{bT} \left(\frac{x}{T} - a\right)^{(\frac{1}{3}+\omega)(m-2)} dx + T^{(1+\varepsilon(m-2))/2} \ll T^{\frac{2}{3}-\omega} \tag{9}$$

with some $\omega > 0$.

Secondly, we estimate the sum over $V \leqq k \leqq W$, where $V = aT + T^{1-\lambda}$, $\lambda = (3m-6)^{-1} - \varepsilon$, by a simpler version of the above lemma which is also due to Van der Corput.

Lemma 1' (Van der Corput [14]). *Suppose that a real-valued function $G(x)$ is twice continuously differentiable on a compact interval I and that $G''(x)$ is monotone and $\neq 0$ on I. Then the estimate (8) holds with $\omega = 0$.*

Applying this lemma and recalling the bounds for $|G(x)|$ given above, we immediately infer that

$$\sum_{V \leqq k \leqq W}^{*} \psi(G(k)) \ll T^{\theta} \qquad \left(\theta < \frac{2}{3}\right) \tag{9'}$$

The remaining sum over $aT \leqq k \leqq aT+T^{1-\lambda}$ is now treated by a completely different method : First of all, we know by the inequalities of Erdös-Turán and of Koksma (see e. g. [7], p. 112 and p. 143) that, for an arbitrary positive integer H,

$$S := \sum_{aT\leq k\leq V}^{*} \psi(G(k)) \ll T^{1-\lambda}H^{-1}+\sum_{h=1}^{H}\frac{1}{h}|S(h)| \tag{10}$$

where (with $e(z) := e^{2\pi i z}$)

$$S(h) := \sum_{aT\leq k\leq V} e(hG(k)).$$

We now consider an arc $\mathscr{C}_1$ of a circle with radius $cT^{1-\lambda}$ joining the points $(aT, G(aT))$ and $(V, G(V))$, choosing the constant c so large that the slope of $\mathscr{C}_1$ is bounded. Writing $\mathscr{C}_1 : y = F(x)$, $aT \leqq x \leqq V$ (and assuming that $F(x) \geqq G(x)$ on this interval), we see that $F'(x) \ll 1$, $|F''(x)| \gg T^{\lambda-1}$, and therefore, by another estimate of Van der Corput (cf. [7], p. 17, theorem 2.7.), for $h \leqq T$,

$$S_1(h) := \sum_{aT\leq k\leq V} e(hF(k)) \ll h^{\frac{1}{2}}T^{\frac{1}{2}}. \tag{11}$$

By Poisson's formula, we get

$$\begin{aligned} S_2(h) :&= \sum_{aT\leq k\leq V} (e(hF(k))-e(hG(k))) \\ &= \sum_{k\in Z}\int_{aT}^{V}(e(hF(x))-e(hG(x)))e(kx)\,\mathrm{d}x+O(1) \\ &= 2\pi i h\sum_{k\in Z}\iint_{\mathscr{B}} e(kx+hy)\,\mathrm{d}x\mathrm{d}y+O(1), \end{aligned} \tag{12}$$

where $\mathscr{B}$ is the plane region bounded by $\mathscr{C}_1$ and the curve $\mathscr{C}_2 : y = G(x)$, $aT \leqq x \leqq V$. Therefore our task will be almost done if we establish the following

Lemma 2. *With the above notations and assumptions,*

$$I(\mathbf{k}) := \int_{\mathscr{B}} e(\mathbf{k}\mathbf{x})\,\mathrm{d}\underline{\mathbf{x}} \ll |\mathbf{k}|^{-\frac{3}{2}}T^{\frac{1}{2}}\gamma^{-\frac{u}{2}}+|\mathbf{k}|^{-2}\gamma^{-1}, \tag{13}$$

where $\mathbf{k} = (k, h) \in (\mathbf{Z}^2-\{(0,0)\})$, $\frac{\pi}{2}-\gamma$ is the angle between the tangent line to the curve $\mathscr{C}_2$ in $(aT, G(aT))$ and the vector $\pm\,\mathbf{k}$ (such that $0 < \gamma <$

$\frac{\pi}{2}$), $\mu := \frac{m-2}{m-1}$ *and* $m-2$ *is the order of the zero of* $\varkappa$ *in* $(aT, G(aT))$, $m \geqq 3$.

Remark. Earlier results on such Fourier transforms of plane regions (which are not quite appropriate for our purpose) have been obtained by Hlawka [5], Herz [4] and Randol [11].

Proof of lemma 2. Putting $\mathrm{v}(\boldsymbol{x}) := (2\pi i|\boldsymbol{k}|)^{-1}e(\boldsymbol{kx})\boldsymbol{k}_0$, where $\boldsymbol{k}_0 = |\boldsymbol{k}|^{-1}\boldsymbol{k}$, we infer from the divergence theorem that

$$I(\boldsymbol{k}) = \int_{\mathscr{B}} \mathrm{div}\, v(\boldsymbol{x})\, \mathrm{d}\underline{x} = \int_{\partial\mathscr{B}} v(\boldsymbol{x})\boldsymbol{n}^*\, \mathrm{d}\sigma = \pm\int_{\mathscr{C}_1} \pm \int_{\mathscr{C}_2} \tag{14}$$

where $\boldsymbol{n}^*$ is the outward normal vector of $\partial\mathscr{B}$ with length unity. We write $\boldsymbol{u} = \boldsymbol{u}(s)$ for the natural parametrization of the curve $T^{-1}\mathscr{C}_2$ (s the arclength with $s = 0$ corresponding to the point $(a, g(a))$), put $\boldsymbol{x} = T\boldsymbol{u}$ and define $f(s) := \boldsymbol{k}_0\boldsymbol{u}(s)$. By Frenet's equations for plane curves,

$$f'(s) = \boldsymbol{k}_0\boldsymbol{t}(s), \qquad f''(s) = \boldsymbol{k}_0\boldsymbol{n}(s)\varkappa(s), \tag{15}$$

where $\boldsymbol{t}(s)$ is the tangent vector and $\boldsymbol{n}(s)$ is the normal vector of $T^{-1}\mathscr{C}_2$ (both of length unity) and $\varkappa(s)$ denotes the curvature. Then we obtain

$$\int_{\mathscr{C}_2} = (2\pi i|\boldsymbol{k}|)^{-1}T\int_0^L e(T|\boldsymbol{k}|f(s))\, \boldsymbol{k}_0\boldsymbol{n}(s)\, \mathrm{d}s \tag{16}$$

where $L \ll T^{-\lambda}$ is the arclength of $T^{-1}\mathscr{C}_1$.

If $|f'(0)| \geqq \frac{1}{2}$, then (for T sufficiently large) $|f'(s)| \gg 1$ on $0 \leqq s \leqq L$, and the second mean-value theorem yields

$$\int_{\mathscr{C}_2} \ll |\boldsymbol{k}|^{-2}$$

which is even stronger than (13).

Therefore we may suppose that $|f'(0)| < \frac{1}{2}$. Then $|\boldsymbol{k}_0\boldsymbol{n}(0)| > \frac{\sqrt{3}}{2}$ and hence $|\boldsymbol{k}_0\boldsymbol{n}(s)| \gg 1$ on $0 \leqq s \leqq L$ (T sufficiently large). Using (15) and recalling that $\varkappa(s)$ has a zero of order $m-2$ at $s = 0$, we thus get

$$s^{m-2} \ll |f''(s)| \ll s^{m-2}$$

for $0 \leqq s \leqq L$. Now denote by s_0 the number where $|f'(s)|$ attains its minimum on $0 \leqq s \leqq L$. We claim that, throughout this interval,

$$|f'(s)| \gg \min(\gamma^{\mu}|s-s_0|, \gamma). \tag{18}$$

By the monotonicity of $f'(s)$ (cf. (17)), there are three possibilities: either $f'(s_0) = 0$ or $s_0 = 0$ or $s_0 = L$.

If $f'(s_0) = 0$, $s_0 \neq 0$, we conclude that

$$|f'(s)| \gg |f'(s)-f'(s_0)| \gg \left|\int_{s_0}^{s} \sigma^{m-2}\, d\sigma\right| \gg |s^{m-1}-s_0^{m-1}| \gg$$
$$\gg |s-s_0| s_0^{m-2} \tag{19}$$

and, on the other hand,

$$\gamma \ll |\sin\gamma| = |f'(0)| \ll |f'(s_0)-f'(0)| \ll \int_0^{s_0} \sigma^{m-2}\, d\sigma \ll s_0^{m-1}, \tag{20}$$

which together yields

$$|f'(s)| \gg \gamma^{\mu}|s-s_0|. \tag{21}$$

If $s_0 = 0$, we simply get

$$|f'(s)| \gg |f'(0)| = |\sin\gamma| \gg \gamma. \tag{22}$$

The same is true for $s_0 = L$, $|f'(L)| \geqq \frac{1}{2}|f'(0)|$, whereas for $s_0 = L$, $|f'(L)| < \frac{1}{2}|f'(0)|$ the estimates (19), (20) and (21) are valid again. This completes the proof of (18).

We now put $\delta := (T|\boldsymbol{k}|r^{\mu})^{-1/2}$ and split up the integral in (16), obtaining

$$\int_{C_2} = (2\pi i|\boldsymbol{k}|)^{-1}T\Big(\int_{|s-s_0|<\delta} + \int_{|s-s_0|\geq\delta}\Big) \ll$$
$$\ll |\boldsymbol{k}|^{-3/2}T^{1/2}\gamma^{-\mu/2} + |\boldsymbol{k}|^{-2}\gamma^{-1}, \tag{23}$$

where we have estimated the first integral trivially and the second one by the second mean-value theorem, making use of (18).

The circle arc $\mathscr{C}_1$ now can be dealt with by a simplified version of the above reasoning. Defining $f(s)$ and s_0 as before (with $\mathscr{C}_1$ instead of $\mathscr{C}_2$) and observing that $T^{-1}\mathscr{C}_1$ has constant curvature of order T^{λ}, we either get (for $|f'(0)| \geqq \frac{1}{2}$)

$$\int_{\varphi_1} \ll |k|^{-2}$$

or (for $|f'(0)| < \frac{1}{2}$)

$$|f'(s)| \geqq |f'(s)-f'(s_0)| \gg \left|\int_{s_0}^{s} \varkappa \mathrm{d}\sigma\right| \gg |s-s_0|. \tag{24}$$

Using the same argument as in (23) (replacing (18) by (24)), we obtain

$$\int_{\varphi_1} \ll |k|^{-3/2}T^{1/2}+|k|^{-2}$$

which, together with (14) and (23), completes the proof of lemma 2.

We now enter this result into (12) and sum over all $k \in \mathbf{Z}$ and over $h = 1,\dots, H$ (according to (10)), $H \leqq T$ at our disposition. Let as before $t(0) = (\tau_1, \tau_2)$ denote the tangent vector of the curve $v = g(u)$ at $u = a$, then (for $h \in \boldsymbol{N}$) we define $k(h) \in \mathbf{Z}$ such that $|k(h)+h\tau_2/\tau_1| < \frac{1}{2}$. We note that $h \ll |k(h)| \ll h$ and that $\tau_2/\tau_1 = \pm\beta \notin \mathbf{Q}$. By the assumptions of our proposition (in particular (3)), we conclude that, for $\boldsymbol{k} = (k(h), h)$,

$$\gamma \gg \sin\gamma = |k(h)\tau_1+h\tau_2|\,|\boldsymbol{k}|^{-1} \gg h^{-1}|k(h)+h\tau_2/\tau_1| \gg h^{-2-\alpha}. \tag{25}$$

Using lemma 2, we get

$$\sum_{h=1}^{H} |I(k(h), h)| \ll T^{1/2}\sum_{h=1}^{H} h^{-3/2+\mu(1+\alpha/2)}+\sum_{h=1}^{H} h^{\alpha} \ll \ll T^{1/2}H^{-1/2+\mu(2+\alpha)/2}+H^{1+\alpha}. \tag{26}$$

For $k \neq k(h)$ we obtain

$$\gamma \gg (k^2+h^2)^{-1/2}|k\tau_1+h\tau_2| \gg (k^2+h^2)^{-1/2}|k-k(h)|$$

and therefore, again by lemma 2, for fixed $h \in \boldsymbol{N}$

$$\sum_{k\neq k(h)} I(k,h) \ll T^{1/2}\sum_{k\neq k(h)}(k^2+h^2)^{(\mu-3)/4}|k-k(h)|^{-\frac{\mu}{2}}+$$
$$+\sum_{k\neq k(h)}(k^2+h^2)^{-\frac{1}{2}}|k-k(h)|^{-1} \ll T^{1/2}\int_{-\infty}^{\infty}(x^2+h^2)^{(\mu-3)/4}|x-k(h)|^{-\frac{\mu}{2}}\mathrm{d}x+$$
$$+h^{-1}+\sum_{0\neq k\neq k(h)}|k|^{-1}|k-k(h)|^{-1} \ll h^{-1/2}T^{1/2}+h^{-1}(1+\log h) \ll h^{-1/2}T^{1/2}$$

(substituting $x = hu$ and recalling that $h \ll |k(h)| \ll h$). Summing over

$h = 1, \ldots, H$ we thus get

$$\sum_{h=1}^{H} \Big| \sum_{k \neq k(h)} I(k, h) \Big| \ll H^{1/2} T^{1/2}. \tag{27}$$

Finally we infer from (11) that

$$\sum_{h=1}^{H} \frac{1}{h} | S_1(h) | \ll H^{1/2} T^{1/2}, \tag{28}$$

collect the results (12), (26), (27) and (28) to enter them into (10), obtaining

$$S \ll T^{1-\lambda} H^{-1} + T^{1/2} (H^{1/2} + H^{-1/2+\mu(1+\alpha/2)}) + H^{1+\alpha}. \tag{29}$$

We now distinguish two cases : $1/2 < -1/2 + \mu(1+\alpha/2)$ and $1/2 \geqq -1/2 + \mu(1+\alpha/2)$. In the first case (which is equivalent to $\alpha > 2/(m-2)$), we choose $H = [T^{(1-2\lambda)/(1+\mu(2+\alpha))}]$, then the first three terms on the right hand side of (29) are

$$\ll T^{1-\lambda-(1-2\lambda)/(1+\mu(2+\alpha))}.$$

To make this exponent $< 2/3$ (for suitable $\varepsilon > 0$), it suffices that

$$1 - \frac{1}{3(m-2)} - \frac{1-2/3(m-2)}{1+\mu(2+\alpha)} < \frac{2}{3}.$$

which is true for $m = 3$, α arbitrary, or for $m \geqq 4$,

$$\alpha < (3m-7)/(m-2)(m-3).$$

To ensure that the last term in (29), namely $H^{1+\alpha}$, is $\ll T^{\theta}$, $\theta < 2/3$, we have to show that

$$(1+\alpha)\left(1 - \frac{2}{3(m-2)}\right)(1+\mu(2+\alpha))^{-1} < \frac{2}{3}$$

or, equivalently (by a short computation),

$$\alpha(m^2 - 3m) < 3m^2 - 11m + 12.$$

This is obvious for $m = 3$, α arbitrary, whereas for $m \geqq 4$, we may replace α by its upper bound $(3m-7)/(m-2)(m-3)$, arriving finally at

$$0 < (m-1)(m-3)(3m-8).$$

If now $\alpha \leqq 2/(m-2)$ (which implies $1/2 > -1/2 + \mu(1+\alpha/2)$), we simply choose $H=[T^{(1-2\lambda)/3}]$, then the first three terms in (29) are $\ll T^{(2-\lambda)/3}$ and the last one is $H^{1+\alpha}\ll T^{\theta}$, $\theta<2/3$, since

$$(1+\alpha)\frac{1}{3}\left(1-\frac{2}{3(m-2)}\right)\leqq\frac{2}{3}\,\frac{3m^2-8m}{6(m-2)^2}<\frac{2}{3}$$

for any $m\geqq 3$. So in any case we obtain from (29) that

$$S\ll T^{\theta}\qquad\left(\theta<\frac{2}{3}\right)$$

which, together with (9) and (9'), completes the proof of (5) and thereby that of our theorem.

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REFERENCES

[1] J. W. S. CASSELS : An introduction to Diophantine approximation, Cambridge 1965.

[2] Y. COLIN DE VERDIÈRE: Nombres de points entiers dans une famille homothétique de domaines de $\mathbb{R}^n$, Ann. Sci. Ecole Norm. Sup. **10**, Sér. 4, (1977), 559–576.

[3] F. FRICKER : Einführung in die Gitterpunktlehre, Basel-Boston-Stuttgart 1982.

[4] C. S. HERZ : Fourier transforms related to convex sets, Annals of Math. **75** (1962), 81–92.

[5] E. HLAWKA : Über Integrale auf konvexen Körpern I, Monatsh. Math. **54** (1951), 1–36.

[6] A. KHINTCHINE : Kettenbrüche, Leipzig 1956.

[7] L. KUIPERS and H. NIEDERREITER : Uniform distribution of sequences, New York-London-Sydney-Toronto 1974.

[8] W. G. NOWAK : Zur Gitterpunktlehre der euklidischen Ebene, Indagationes math. **46** (1984), 209–223.

[9] W. G. NOWAK : An Ω-estimate for the lattice rest of a convex planar domain, Proc. Royal Soc. Edinburgh, Sect. A, **100** (1985), 295–299.

[10] W. G. NOWAK : On the average order of the lattice rest of a convex planar domain, Math. Proc. Cambridge Phil. Soc. **98** (1985), 1–4.

[11] B. RANDOL : On the Fourier transform of the indicator function of a planar set, Trans. Amer. Math. Soc. **139** (1969), 271–278.

[12] M. TARNOPOLSKA-WEISS : On the number of lattice points in planar domains, Proc. Amer. Math. Soc. **69** (1978), 308–311.

[13] J. G. VAN DER CORPUT : Neue zahlentheoretische Abschätzungen, Math. Ann. **89** (1923), 215–254.

[14] J. G. VAN DER CORPUT : Zahlentheoretische Abschätzungen mit Anwendung auf Gitterpunktprobleme. Math. Z. **17** (1923), 250–259.

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