A NOTE ON THE NUMBER
OF PRIME FACTORS OF INTEGERS
IN SHORT INTERVALS

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1. Introduction. Let \( n \) and \( m \) be integers such that \( 3 \leq n < m \). Let \( \omega(m) \) denote the number of distinct prime factors of \( m \). Let \( 1 < b(n) \leq n \) be a sequence of positive integers. Let \( \# \{m: \ldots\} \) denote the number of positive integers \( m \) which satisfy some conditions '…'. Throughout this paper \( p, p_1, p_2, \ldots \) stand for prime numbers, and \( c_1, c_2, \ldots \) stand for positive constants.

We put

\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy.
\]

Then the following result was carried out by Babu [1].

Let \( 1 \leq a(n) \leq (\log \log n)^{1/2} \) be a sequence of real numbers tending to infinity. Then

\[
(1) \quad \frac{1}{b(n)} \# \{m: n < m \leq n + b(n), \omega(m) - \log \log m < x\sqrt{\log \log m}\} \rightarrow \Phi(x) \text{ as } n \to \infty, \text{ provided that } b(n) \geq n^{\frac{a}{(\log \log n)^{-1/3}}}.
\]

In addition to this, he mentioned the following problems which were given by P. Erdös and I. Z. Ruzsa.

(a) What is the largest value of \( f(n) \) such that if \( b(n) < f(n) \) for all \( n \), then (1) fails to hold?

(b) Does (1) hold if \( b(n) = n^{1/(\log \log n)} \)?

In this paper we consider the problem (b) and obtain the following

**Theorem 1.** (1) holds if \( b(n) \geq n^{1/(\log \log n)} \).

This also gives an answer to the problem (b). **Theorem 1** can be deduced from the following

**Theorem 2.** Let \( \alpha < \beta \) be real numbers. Let \( b(n) \geq n^{1/(\log \log n)} \) be
a sequence of positive integers. We put $\mu = \max \{|1|, |\alpha|, |\beta|\}$ and

$$A(n, b(n), \alpha, \beta) = \# \{m; n \leq m \leq n + b(n), \log \log m + \alpha \sqrt{\log \log m} < \omega(m) < \log \log m + \beta \sqrt{\log \log m}\}.$$ 

Then we have

$$(2) \quad \frac{1}{b(n)} A(n, b(n), \alpha, \beta) = \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{-\frac{1}{2} y^2} dy + O\left(\frac{\mu^2 (\log \log n)^{1/2}}{(\log \log n)^{1/4}}\right) + O\left(\mu \sqrt{\log \log n} \frac{\log \log n}{\log \log b(n)} \right).$$

The $O$-terms are uniform in $n$ sufficiently large.

In order to prove this theorem we shall use Selberg's sieve method and the arguments of Erdös [3] and Tanaka [5] (cf. [2]).

2. Sieve Method. In this section we shall give two lemmas which are obtained by Selberg's sieve method.

Lemma 1. Let $z$ be a positive integer. Let $r \geq 2$ with $\log r \leq c_r \log z$, where $c_r$ is a sufficiently small constant. Let $Q$ be an arbitrary non-empty set of primes, none of which exceed $r$. Let $D$ be the set of all positive square-free integers which are divisible only by primes of $Q$, assuming that $1 \in D$. Further, let $a_1, a_2, \ldots, a_z$ be $z$ integers and $d$ be an integer of $D$. Assume that the number of $a_i$ which are divisible by $d$ is equal to $z \theta(d) + R(d)$, where $\theta(d)$ is a multiplicative function defined on $D$ satisfying

$$0 \leq \theta(d) \leq 1 \text{ for } d > 1, \quad |R(d)| \leq c_z d \theta(d)$$

and

$$\theta(p) \leq \frac{c_r}{c_r + p} \text{ for } p \in Q.$$ 

Then the number of $a(i)$ $(1 \leq i \leq z)$ which are not divisible by any prime of $Q$ is

$$z \prod_{p \in Q} \left(1 - \theta(p)\right) 1 + O\left(e^{-c_z \frac{\log z}{\log r}}\right).$$

The $O$-term is uniform in $z$ sufficiently large.

Proof. Kubilius [3], lemma 1.4.
Lemma 2. Let \( b_1(n) \) be a sequence of positive integers tending to infinity. Let \( g \leq \sqrt{b_1(n)} \) be a positive integer, and \( q \cdot (0 \leq q < g) \) be an integer. We put \( n_1 = \lfloor (n-q)/g \rfloor \) and \( n_2 = \lfloor (n+b_1(n)-q)/g \rfloor \), here \( \lfloor x \rfloor \) denotes the largest integer not exceeding \( x \). Let \( r_1 \geq 2 \) with \( \log r_1 \leq c_6 \log (n_2-n_1) \), where \( c_6 \) is a sufficiently small constant. Let \( p_1, p_2, \ldots, p_h \) be prime numbers such that \( p_j \not\equiv g \) and \( p_j \leq r_1 \) for each \( j = 1, 2, \ldots, h \). We put

\[
F(n, b_1(n), q, g : p_1, p_2, \ldots, p_h) = \# \{ m ; n \leq m \leq n+b_1(n), \\
m \equiv q (\text{mod } g), m \not\equiv 0 (\text{mod } p_j), j = 1, 2, \ldots, h \}.
\]

Then we have

\[
F(n, b_1(n), q, g : p_1, p_2, \ldots, p_h) = \frac{b_1(n)}{g} \prod_{j=1}^{h} \left( 1 - \frac{1}{p_j} \right) \left[ 1 + O \left( e^{-c_1 \log \frac{b_1(n)}{\log r_1}} \right) \right].
\]

The \( O \)-term is uniform in \( n \) sufficiently large and \( g \leq \sqrt{b_1(n)} \).

Proof. Let \( m \) be an integer such that \( m \equiv q (\text{mod } g) \) and \( n < m \leq n+b_1(n) \). Then \( m = q+kg \) with \( n_1 < k \leq n_2 \) and \( F(n, b_1(n), q, g : p_1, p_2, \ldots, p_h) \) is equal to the number of \( k \) satisfying the conditions \( m = q+kg \equiv 0 \pmod{p_j} \) \( (1 \leq j \leq h) \). Let \( Q' \) be the set of primes \( p_1, p_2, \ldots, p_h \). Let \( D' \) be the set of all square-free integers which are divisible only by primes of \( Q' \) and we assume that \( 1 \in D' \). For any \( d' \in D' \) we consider the congruence \( q+kg \equiv 0 \pmod{d'} \). Since \( (d', g) = 1 \), this congruence has only one solution \( k \pmod{d'} \). Let \( N_k \) be the number of \( k (n_1 < k \leq n_2) \) satisfying the congruence \( q+kg \equiv 0 \pmod{d'} \) and put \( \theta_k(d') = 1/d' \). Then we have

\[
N_k = (n_2-n_1) \theta_k(d') + R_k(d') \quad \text{for } d' > 1,
\]

\[
| R_k(d') | \leq d' \theta_k(d'), \quad \theta_k(p_j) = \frac{1}{p_j} \leq \frac{2}{p_j+2} \quad (1 \leq j \leq h).
\]

Therefore by lemma 1, we have

\[
F(n, b_1(n), q, g : p_1, p_2, \ldots, p_h)
= (n_2-n_1) \prod_{j=1}^{h} \left( 1 - \frac{1}{p_j} \right) \left[ 1 + O \left( e^{-c_1 \log \frac{n_2-n_1}{\log r_1}} \right) \right].
\]

By the assumption \( g \leq \sqrt{b_1(n)} \), it follows that

\[
n_2-n_1 = b_1(n) + O(1) = \frac{b_1(n)}{g} \left[ 1 + O \left( e^{-c_1 \log \frac{b_1(n)}{\log r_1}} \right) \right].
\]

and
\[
\log (n_2 - n_1) \geq \log \left( \frac{b_1(n)}{g} - 1 \right) > c_2 \log b_1(n).
\]

Thus we have our assertion.

3. The Poisson distribution of \(\omega'(m)\). We denote by \(P = P(n)\) the set of all prime numbers \(p\) which satisfy the inequality

\[
\log n < p < n^{\frac{1}{2\log \log n}}.
\]

Let \(\omega'(m)\) be the number of distinct primes in \(P\) which are divisors of \(m\). In this section we shall show that \(\omega'(m)\) has approximately the Poisson distribution (see lemma 5). Let

\[
y(n) = \sum_{p \leq n} \frac{1}{p},
\]

then we have

\[
y(n) = \log \log n + O(\log \log \log n),
\]

using the well known formula

\[
\sum_{p \leq n} \frac{1}{p} = \log \log n + O(1).
\]

Also, we obtain

\[
\sum_{p \leq n+b(m)} \frac{1}{p} = O(\log \log \log n).
\]

First we consider the difference between \(\omega(m)\) and \(\omega'(m)\).

Lemma 3. Let \(g(n)\) be a sequence of real numbers tending to infinity. Then we have

\[
\# \{ m; n < m \leq n + b(n), \omega(m) - \omega'(m) > g(n) \} = O\left( \frac{b(n) \log \log \log n}{g(n)} \right).
\]

The \(O\)-term is uniform in \(n\) sufficiently large.

Proof.

\[
\sum_{n < m \leq n + b(m)} |\omega(m) - \omega'(m)| = \sum_{n < m \leq n + b(m)} \sum_{p \nmid m} 1
\]
\[ = \sum_{p \leq n^{1/2 \log \log n}} \sum_{n \leq m \leq n + b(n)} 1 \leq \sum_{p \leq n^{1/2 \log \log n}} \frac{b(n)}{p}. \]

Thus the lemma follows from (5).

Secondly we shall prove the following lemma which will be used for the proof of lemma 5.

\textbf{Lemma 4.} Let \( t \) be a positive integer such that \( t < 2 \log \log n \). Let \( L(t) \) be a set of all positive square-free integers which have exactly \( t \) prime factors belonging to the set \( P \). Then we have

\[ \sum_{l \in L(t)} \frac{1}{l} = \frac{\gamma(n)^t}{t!} + O\left( \frac{1}{\log \log n} \right). \]

The \( O \)-term is uniform in \( t \) and \( n \) sufficiently large.

\textbf{Proof.} Let \[ \sum_{p \in P} \frac{1}{p^d} \]
be the sum obtained from

\[ \frac{\gamma(n)^t}{t!} - \sum_{l \in L(t)} \frac{1}{l}. \]

It is clear that

\[ d \leq \frac{t}{n^{1/2 \log \log n^2}} \leq \frac{2 \log \log n}{n^{1/2 \log \log n^2}} = n^{1/4 \log \log n}. \]

Hence we have

\[ \sum_{p \in P} \frac{1}{p^d} \leq \sum_{p \in P} \sum_{d < n^{1/4 \log \log n}} \frac{1}{p^d} \leq \sum_{p > \log n} \frac{1}{p^d} \sum_{d < n^{1/4 \log \log n}} \frac{1}{d} = O\left( \frac{1}{\log \log n} \right). \]

Thus the lemma is proved.

Now, we put

\[ N(n, b(n), t) = \# \{ m ; n < m \leq n + b(n), \omega(m) = t \}. \]

Then we have the following

\textbf{Lemma 5.} Assume that \( b(n) \geq n^{1/4 \log \log n} \). Then we have

\[ N(n, b(n), t) = b(n) \frac{\gamma(n)^t}{t!} e^{-\gamma \eta} + O\left( \frac{b(n)}{\log \log n} \right) + O\left( b(n) e^{-c_{\eta} (\log \log n^2 \log \log n^2)} \right). \]
The $O$-terms are uniform in $t$ and $n$ sufficiently large.

Proof. We put
\[
N_t(n, b(n), t) = \# \{ m; n < m \leq n + b(n), \omega(m) = t, p^2 \nmid m \text{ for all } p \in P \}.
\]
We let $l$ be an element of $L(t)$ and put
\[
H(n, b(n), l) = \# \{ m; n < m \leq n + b(n), l \mid m, p \nmid \frac{m}{l} \text{ for all } p \in P \}.
\]
Then we have
\[
(6) \quad N_t(n, b(n), t) = \sum_{l \in L(t)} H(n, b(n), l).
\]
Let $p_1, p_2, \ldots, p_h$ be all the prime numbers such that $p \nmid l$ and $p \in P$. We put $v = \Phi(l)$ where $\Phi$ is Euler's function. Then we have
\[
(7) \quad H(n, b(n), l) = \# \{ m; n < m \leq n + b(n), l \mid m, (l, m/l) = 1, m \equiv 0 \pmod{p_j}, j = 1, 2, \ldots, h \} \\
= \# \{ m; n < m \leq n + b(n), m \equiv q_i l \pmod{l^2}, i = 1, 2, \ldots, v, m \equiv 0 \pmod{p_j}, j = 1, 2, \ldots, h \} \\
= \sum_{i=1}^{v} F(n, b(n), q_i l, l^2; p_1, p_2, \ldots, p_h),
\]
where $|q_1, q_2, \ldots, q_v|$ is a reduced set of residues mod $l$. By (3) and the definition of $L(t)$ we have
\[
(8) \quad l^2 \leq n^{\frac{1}{\log \log n}} < \left( n^{\frac{1}{\log \log n}} \right)^{\frac{1}{2}} \leq b(n).
\]
Let $r_2 = n^{\frac{1}{\log \log n}}$, and put $n_3 = [(n-q_1)/l^2]$, $n_4 = [(n+b(n)-q_4)/l^2]$. Then we have
\[
\log r_2 = \log \frac{n}{8(\log \log n)^2} \\
= \frac{\log (n_4-n_3)}{\log (n_4-n_3)} \cdot \frac{\log n}{8(\log \log n)^2} \\
\leq \frac{\log (n_4-n_3)}{\log (b(n)/l^2-1)} \cdot \frac{\log n}{8(\log \log n)^2}.
\]
Since $b(n) \geq n^{1/\log \log n}$ and by (8) there exists a sufficiently small positive constant $c_{11}$ such that

$$\log r_2 < c_{11} \log (n_4 - n_3).$$

From (8), (9) we know that Lemma 2 can be applied to estimate $F(n, b(n), q_1 l, l^j ; p_1, p_2, \ldots, p_n)$ in (7). Hence we obtain

$$H(n, b(n), l) = \frac{b(n) \varphi(l)}{l^j} \prod_{j=1}^{h} \left( 1 - \frac{1}{p_j} \right) \left[ 1 + O\left( e^{-c_{11} \frac{\log \log n}{\log r_2}} \right) \right].$$

Inserting (10) into (6) we have

$$N_i(n, b(n), t) = \sum_{i \in \mathbb{N}_0} \frac{b(n) \varphi(l)}{l^j} \prod_{j=1}^{h} \left( 1 - \frac{1}{p_j} \right) \left[ 1 + O\left( e^{-c_{11} \frac{\log \log n}{\log r_2}} \right) \right]$$

$$= b(n) \prod_{p \in P} \left( 1 - \frac{1}{p} \right) \left( \sum_{i \in \mathbb{N}_0} \frac{1}{l} \right) \left[ 1 + O\left( e^{-c_{11} \frac{\log \log \log n}{\log n}} \right) \right].$$

It is clear that

$$\prod_{p \in P} \left( 1 - \frac{1}{p} \right) = \exp \left[ \sum_{p \in P} \log \left( 1 - \frac{1}{p} \right) \right]$$

$$= \exp \left[ - \sum_{p \in P} \frac{1}{p} + O\left( \sum_{p \in P} \frac{1}{p^2} \right) \right]$$

$$= e^{-\gamma} \left[ 1 + O\left( \frac{1}{\log n} \right) \right].$$

Hence we have from this and lemma 4

$$N_i(n, b(n), t)$$

$$= b(n) e^{-\gamma} \frac{\gamma(n)^t}{t!} + O\left( \frac{b(n)}{\log \log n} \right) + O\left( b(n) e^{-c_{11} \frac{\log \log n}{\log \log \log n}} \right).$$

Now the number of positive integers $m \leq n + b(n)$ divisible by $p^t$ for some $p \in P$ is less than $\sum_{p \in P} \left( \frac{b(n)}{p^2} + 1 \right)$. Since

$$b(n) \geq n^{1/\log \log n} > \frac{1}{n^{1/\log \log n^2}} > p^2$$

for $p \in P$, we obtain

$$\sum_{p \in P} \left( \frac{b(n)}{p^2} + 1 \right) \leq 2 \sum_{p \in P} \frac{b(n)}{p^2} = O\left( \frac{b(n)}{\log \log n} \right).$$
From this we have

\[ N(n, b(n), t) = N_0(n, b(n), t) + O\left( \frac{b(n)}{\log \log n} \right), \]

which implies our lemma. The main term of \( N(n, b(n), t)/b(n) \) is the so-called Poisson probability.

4. The normal distribution of \( \omega'(m) \). In this section we shall show that \( \omega'(m) \) has approximately the normal distribution (see lemma 7).

**Lemma 6.** Let \( \alpha < \beta \) be real numbers. Let \( n \) be a sufficiently large integer for which there exists a natural number \( t_1 \) such that

\[ y(n) + \alpha \sqrt{y(n)} < t_1 < y(n) + \beta \sqrt{y(n)}. \]

Let \( \mu = \max(1, |\alpha|, |\beta|) \) and \( t_1 = y(n) + u\sqrt{y(n)} \), where \( u \) \((\alpha < u < \beta)\) is a real number which is determined by \( t_1 \) and \( n \). Then we have

\[ N(n, b(n), t_1) = \frac{1}{\sqrt{2\pi y(n)}} b(n) e^{-\frac{1}{2} u^2} + O\left( \frac{\mu^4 b(n)}{\log \log n} \right) + O\left( b(n) e^{-c, \frac{\log \log n}{\log n}} \right). \]

The \( O \)-terms are uniform in \( n \).

**Proof.** We know that

\[ t_1! = \sqrt{2\pi} t_1^{t_1 + \frac{1}{2}} e^{-t_1} \left[ 1 + O\left( \frac{1}{t_1} \right) \right]. \]

On the other hand we can put

\[ t_1 = y(n) + u\sqrt{y(n)}. \]

Hence we have

\[ t_1! = \sqrt{2\pi y(n)} y(n)^{t_1} \left( 1 + \frac{u}{\sqrt{y(n)}} \right)^{\frac{y(n)}{y(n)} + u\sqrt{y(n)} + 1/2} \]

\[ \times e^{-y(n) - u\sqrt{y(n)}} \left[ 1 + O\left( \frac{1}{y(n) + u\sqrt{y(n)}} \right) \right]. \]

Since \( \log(1 + x) = x - \frac{1}{2} x^2 + O(|x|^3) \) for a real number \( x \) with \(|x| < 1\), we obtain
(12) \( \left(1 + \frac{u}{\sqrt{y(n)}}\right)^{\pi n + u \sqrt{\pi n} + \frac{1}{2}} = \exp \left[u \sqrt{y(n)} + \frac{u^2}{2} + O\left(\frac{\mu^4}{\sqrt{y(n)}}\right)\right]. \)

By (11) and (12) it follows that

(13) \( t_1! = \sqrt{2} \pi y(n) \gamma(n)^{1/2} e^{-\pi n + \frac{u^2}{2} + O\left(\frac{\mu^4}{\sqrt{\pi n}}\right)} \left[1 + O\left(\frac{1}{y(n) + u \sqrt{y(n)}}\right)\right]. \)

By (4) it is clear that \( t_1 < 2 \log \log n \) for a sufficiently large \( n \). Hence by (13) and lemma 5 we have

\[
N(n, b(n), t_1) = \frac{b(n)}{\sqrt{2} \pi y(n)} e^{-\frac{u}{2} u^2} \left[1 + O\left(\frac{\mu^4}{\sqrt{y(n)}}\right)\right] \left[1 + O\left(\frac{1}{\sqrt{y(n)}}\right)\right]
+ O\left(\frac{b(n)}{\log n}\right) + O\left(b(n) e^{-c_1 \left(\frac{\log \log n \cdot \log \log \log n}{\log n}\right)}\right)
\]

\[
= \frac{b(n)}{\sqrt{2} \pi y(n)} e^{-\frac{u}{2} u^2} + O\left(\frac{\mu^4 b(n)}{y(n)}\right) + O\left(\frac{b(n)}{\log n}\right)
+ O\left(b(n) e^{-c_1 \left(\frac{\log \log n \cdot \log \log \log n}{\log n}\right)}\right)
\]

Thus the lemma is proved.

**Lemma 7.** Let

\[
B_1(n, b(n), \alpha, \beta) = \# \{m; n < m < n + b(n), \gamma(n) + \alpha \sqrt{y(n)} < \omega(m) < \gamma(n) + \beta \sqrt{y(n)} \}.
\]

Then we have

\[
B_1(n, b(n), \alpha, \beta) = \frac{b(n)}{\sqrt{2} \pi} \int_\alpha^\beta e^{-\frac{1}{2} u^2} du + O\left(\frac{\mu^4 b(n)}{\sqrt{\log \log n}}\right)
+ O\left(b(n) \sqrt{\log \log n} e^{-c_1 \left(\frac{\log \log n \cdot \log \log \log n}{\log n}\right)}\right).
\]

The \( O \)-terms are uniform in \( n \) sufficiently large.

**Proof.** For a sufficiently large integer \( n \) let \( t_1 = t_0 + 1, t_0 + 2, \ldots, t_0 + s \) be \( s \) natural numbers such that

\[
\gamma(n) + \alpha \sqrt{y(n)} < t_1 < \gamma(n) + \beta \sqrt{y(n)}.
\]
Further, we put \( t_0 + i = y(n) + u_i \sqrt{y(n)} \). It is obvious that
\[
u_{i+1} - u_i = \frac{1}{\sqrt{y(n)}}, \quad s = O(\mu \sqrt{\log \log n}).
\]

From lemma 6 we have
\[
B_i(n, b(n), \alpha, \beta) = \frac{b(n)}{\sqrt{2\pi}} \sum_{i=1}^{s} (u_{i+1} - u_i) e^{-\frac{1}{2} u_i^2} + O \left( \frac{\mu^5 b(n)}{\sqrt{\log \log n}} \right) + O \left( b(n) \mu \sqrt{\log \log n} e^{-c_{14} \frac{(\log \log \log n)}{\log n}} \right).
\]

Using a mean value theorem, we have
\[
\sum_{i=1}^{s} (u_{i+1} - u_i) e^{-\frac{1}{2} u_i^2} = \int_{a}^{s} e^{-\frac{1}{2} u^2} du + O \left( \frac{\mu^2}{\sqrt{y(n)}} \right).
\]

Hence it follows that
\[
B_i(n, b(n), \alpha, \beta) = \frac{b(n)}{\sqrt{2\pi}} \int_{a}^{s} e^{-\frac{1}{2} u^2} du + O \left( \frac{\mu^5 b(n)}{\sqrt{\log \log n}} \right) + O \left( b(n) \mu \sqrt{\log \log n} e^{-c_{14} \frac{(\log \log \log n)}{\log n}} \right).
\]

Thus the lemma is proved.

4. Proofs of Theorem 1 and Theorem 2. We put
\[
B_2(n, b(n), \alpha, \beta) = \# \{ m; \ n < m \leq n + b(n), \ y(n) + a \sqrt{y(n)} < \omega(m) < y(n) + \beta \sqrt{y(n)}, \ \omega(m) - \omega'(m) < g(n) \}
\]
and
\[
A_1(n, b(n), \alpha, \beta) = \# \{ m; \ n < m \leq n + b(n), \ 
\log \log m + a \sqrt{\log \log m} < \omega(m) < \log \log m + \beta \sqrt{\log \log m}, \\
\omega(m) - \omega'(m) < g(n) \}.
\]

Let
\[
w = \frac{g(n) + \mu \log \log n}{\sqrt{y(n)}},
\]
provided that the function \( g(n) > 0 \) is such that \( w > 0 \) becomes sufficiently
small for a large \( n \). For a positive integer \( m \) such that \( \omega(m) - \omega'(m) < g(n) \), the inequality

\[
\log \log m + \alpha \sqrt{\log \log m} < \omega(m) < \log \log m + \beta \sqrt{\log \log m}.
\]

is equivalent to

\[
y(n) + (\alpha + O(w)) \sqrt{n} < \omega(m) < y(n) + (\beta + O(w)) \sqrt{n}.
\]

Hence we have

\[
A(n, b(n), \alpha, \beta) = B(n, b(n), \alpha + O(w), \beta + O(w)).
\]

From Lemma 3, Lemma 7 and (14) we have

\[
A(n, b(n), \alpha, \beta) = A(n, b(n), \alpha, \beta) + O\left( \frac{b(n) \log \log \log n}{g(n)} \right)
\]

\[
= B(n, b(n), \alpha + O(w), \beta + O(w)) + O\left( \frac{b(n) \log \log \log n}{g(n)} \right)
\]

\[
= B(n, b(n), \alpha + O(w), \beta + O(w)) + O\left( \frac{b(n) \log \log \log n}{g(n)} \right)
\]

\[
= b(n) \int_{\alpha + O(w)}^{\beta + O(w)} e^{-\frac{1}{2} u^2} du + O\left( \frac{b(n) \log \log \log n}{g(n)} \right)
\]

\[
= b(n) \int_{\alpha}^{\beta} e^{-\frac{1}{2} u^2} du + O\left( \frac{b(n) \log \log \log n}{g(n)} e^{-c_{14}(\log \log n)^{1/4}} \right)
\]

We have immediately Theorem 2.

Finally we shall prove Theorem 1. Let \( \alpha < \beta \) be real numbers. If

\[
b(n) \geq n^{\epsilon \log \log n},
\]

then it follows from Theorem 2 that

\[
\lim_{n \to \infty} \frac{1}{b(n)} A(n, b(n), \alpha, \beta) = \frac{1}{\sqrt{2 \pi}} \int_{\alpha}^{\beta} e^{-\frac{1}{2} y^2} dy.
\]

Here we use the argument in [5]. For any real \( \varepsilon > 0 \) let \( \alpha(\varepsilon) \) and \( \beta(\varepsilon) \) be real numbers such that
\[
\frac{1}{\sqrt{2\pi}} \int_{\alpha \varepsilon}^{\alpha \varepsilon} e^{-\frac{1}{2} y^2} dy > 1 - \varepsilon.
\]

Since
\[
\frac{1}{b(n)} A(n, b(n), -\infty, \alpha(\varepsilon)) \leq 1 - \frac{1}{b(n)} A(n, b(n), \alpha(\varepsilon), \beta(\varepsilon)),
\]
we obtain from (15)
\[
\limsup_{n \to \infty} \frac{1}{b(n)} A(n, b(n), -\infty, \alpha(\varepsilon))
\leq 1 - \liminf_{n \to \infty} \frac{1}{b(n)} A(n, b(n), \alpha(\varepsilon), \beta(\varepsilon))
\leq 1 - \frac{1}{\sqrt{2\pi}} \int_{\alpha \varepsilon}^{\beta(\varepsilon)} e^{-\frac{1}{2} y^2} dy < \varepsilon.
\]

Then we have
\[
\liminf_{n \to \infty} \frac{1}{b(n)} A(n, b(n), -\infty, x)
\geq \limsup_{n \to \infty} \frac{1}{b(n)} A(n, b(n), \alpha(\varepsilon), x)
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{\alpha \varepsilon}^{x} e^{-\frac{1}{2} y^2} dy > \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2} y^2} dy - \varepsilon
\]
and
\[
\limsup_{n \to \infty} \frac{1}{b(n)} A(n, b(n), -\infty, x)
\leq \limsup_{n \to \infty} \frac{1}{b(n)} A(n, b(n), \alpha(\varepsilon), x)
\]
\[
+ \limsup_{n \to \infty} \frac{1}{b(n)} A(n, b(n), -\infty, \alpha(\varepsilon))
\]
\[
< \frac{1}{\sqrt{2\pi}} \int_{\alpha \varepsilon}^{x} e^{-\frac{1}{2} y^2} dy + \varepsilon < \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2} y^2} dy + \varepsilon.
\]

These complete the proof of Theorem 1.

REFERENCES

A NOTE ON THE NUMBER OF PRIME FACTORS


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