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A NOTE ON THE NUMBER OF PRIME FACTORS OF INTEGERS IN SHORT INTERVALS

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1. Introduction. Let n and m be integers such that $3 \leq n < m$. Let $\omega(m)$ denote the number of distinct prime factors of m. Let $1 < b(n) \leq n$ be a sequence of positive integers. Let $\#\{m; \ldots\}$ denote the number of positive integers m which satisfy some conditions '...'. Throughout this paper $p, p_1, p_2, \ldots$ stand for prime numbers, and $c_1, c_2, \ldots$ stand for positive constants.

We put

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy.$$

Then the following result was carried out by Babu [1].

Let $1 \leq a(n) \leq (\log\log n)^{1/2}$ be a sequence of real numbers tending to infinity. Then

$$(1) \quad \frac{1}{b(n)} \#\{m;\ n < m \leq n+b(n),\ \omega(m) - \log\log m < x\sqrt{\log\log m}\} \longrightarrow \Phi(x) \text{ as } n \to \infty, \text{ provided that } b(n) \geq n^{a(n)(\log\log n)^{-1/2}}.$$

In addition to this, he mentioned the following problems which were given by P. Erdös and I. Z. Ruzsa.

(a) What is the largest value of $f(n)$ such that if $b(n) < f(n)$ for all n, then (1) fails to hold ?
(b) Does (1) hold if $b(n) = n^{1/\sqrt{\log\log n}}$?

In this paper we consider the problem (b) and obtain the following

Theorem 1. (1) *holds if* $b(n) \geq n^{1/(\log\log n)}$.

This also gives an answer to the problem (b). Theorem 1 can be deduced from the following

Theorem 2. *Let $\alpha < \beta$ be real numbers. Let $b(n) \geq n^{1/(\log\log n)}$ be* *a sequence of positive integers. We put* $\mu = \max\{1, |\alpha|, |\beta|\}$ *and*

$$A(n, b(n), \alpha, \beta) = \#\{m;\ n < m \leq n+b(n),\ \log\log m + \alpha\sqrt{\log\log m} < \omega(m) < \log\log m + \beta\sqrt{\log\log m}\}.$$

Then we have

$$(2)\quad \frac{1}{b(n)} A(n, b(n), \alpha, \beta) = \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta e^{-\frac{1}{2}y^2} dy + O\left(\frac{\mu^5(\log\log\log n)^{1/2}}{(\log\log n)^{1/4}}\right) + O\left(\mu\sqrt{\log\log n}\ e^{-c_1\frac{(\log\log n)^2\log b(n)}{\log n}}\right).$$

The O-terms are uniform in n sufficiently large.

In order to prove this theorem we shall use Selberg's sieve method and the arguments of Erdös [3] and Tanaka [5] (cf. [2]).

2. Sieve Method. In this section we shall give two lemmas which are obtained by Selberg's sieve method.

Lemma 1. *Let* z *be a positive integer. Let* $r \geq 2$ *with* $\log r \leq c_2 \log z$, *where* c_2 *is a sufficiently small constant. Let* Q *be an arbitrary non-empty set of primes, none of which exceed* r. *Let* D *be the set of all positive square-free integers which are divisible only by primes of* Q, *assuming that* $1 \in D$. *Further, let* $a_1, a_2, \ldots, a_z$ *be* z *integers and* d *be an integer of* D. *Assume that the number of* a_i *which are divisible by* d *is equal to* $z\theta(d)+R(d)$, *where* $\theta(d)$ *is a multiplicative function defined on* D *satisfying*

$$0 \leq \theta(d) \leq 1 \text{ for } d > 1,\ |R(d)| \leq c_3 d\theta(d)$$

and

$$\theta(p) \leq \frac{c_4}{c_4+p} \quad \text{for } p \in Q.$$

Then the number of $a(i)$ $(1 \leq i \leq z)$ *which are not divisible by any prime of* Q *is*

$$z \prod_{p \in Q} (1-\theta(p))\left\{1 + O\left(e^{-c_5\frac{\log z}{\log r}}\right)\right\}.$$

The O-term is uniform in z sufficiently large.

Proof. Kubilius [3], lemma 1.4.

Lemma 2. *Let $b_1(n)$ be a sequence of positive integers tending to infinity. Let $g \leq \sqrt{b_1(n)}$ be a positive integer, and $q\ (0 \leq q < g)$ be an integer. We put $n_1 = [(n-q)/g]$ and $n_2 = [(n+b_1(n)-q)/g]$, here $[x]$ denotes the largest integer not exceeding x. Let $r_1 \geq 2$ with $\log r_1 \leq c_6 \log(n_2-n_1)$, where c_6 is a sufficiently small constant. Let $p_1, p_2, \ldots, p_h$ be prime numbers such that $p_j \nmid g$ and $p_j \leq r_1$ for each $j = 1, 2, \ldots, h$. We put*

$$F(n, b_1(n), q, g\ ;\ p_1, p_2, \ldots, p_h) = \#\{m;\ n < m \leq n+b_1(n),\ m \equiv q (\mathrm{mod}\, g),\ m \not\equiv 0\ (\mathrm{mod}\, p_j),\ j = 1, 2, \ldots, h\}.$$

Then we have

$$F(n, b_1(n), q, g\ ;\ p_1, p_2, \ldots, p_h) = \frac{b_1(n)}{g} \prod_{j=1}^{h} \left(1-\frac{1}{p_j}\right)\left\{1+O\left(e^{-c_7\frac{\log b_1(n)}{\log r_1}}\right)\right\}.$$

The O-term is uniform in n sufficiently large and $g \leq \sqrt{b_1(n)}$.

Proof. Let m be an integer such that $m \equiv q\ (\mathrm{mod}\, g)$ and $n < m \leq n+b_1(n)$. Then $m = q+kg$ with $n_1 < k \leq n_2$ and $F(n, b_1(n), q, g\ ;\ p_1, p_2, \ldots, p_h)$ is equal to the number of k satisfying the conditions $m = q+kg \not\equiv 0\ (\mathrm{mod}\, p_j)$ $(1 \leq j \leq h)$. Let Q' be the set of primes $p_1, p_2, \ldots, p_h$. Let D' be the set of all square-free integers which are divisible only by primes of Q' and we assume that $1 \in D'$. For any $d' \in D'$ we consider the congruence $q+kg \equiv 0\ (\mathrm{mod}\, d')$. Since $(d', g) = 1$, this congruence has only one solution $k \bmod d'$. Let N_0 be the number of k $(n_1 < k \leq n_2)$ satisfying the congruence $q+kg \equiv 0\ (\mathrm{mod}\, d')$ and put $\theta_1(d') = 1/d'$. Then we have

$$N_0 = (n_2-n_1)\theta_1(d')+R_1(d') \quad \text{for } d' > 1,$$
$$|R_1(d')| \leq d'\theta_1(d'),\ \theta_1(p_j) = \frac{1}{p_j} \leq \frac{2}{p_j+2} \quad (1 \leq j \leq h).$$

Therefore by lemma 1, we have

$$F(n, b_1(n), q, g\ ;\ p_1, p_2, \ldots, p_h) = (n_2-n_1)\prod_{j=1}^{h}\left(1-\frac{1}{p_j}\right)\left\{1+O\left(e^{-c_8\frac{\log(n_2-n_1)}{\log r_1}}\right)\right\}.$$

By the assumption $g \leq \sqrt{b_1(n)}$, it follows that

$$n_2-n_1 = \frac{b_1(n)}{g}+O(1) = \frac{b_1(n)}{g}\left\{1+O\left(e^{-\frac{1}{2}\log b_1(n)}\right)\right\},$$

and

$$\log(n_2-n_1) \geq \log\left(\frac{b_1(n)}{g}-1\right) > c_9 \log b_1(n).$$

Thus we have our assertion.

3. The Poisson distribution of $\omega'(m)$. We denote by $P = P(n)$ the set of all prime numbers p which satisfy the inequality

$$\log n < p < n^{\frac{1}{8(\log\log n)^2}}. \tag{3}$$

Let $\omega'(m)$ be the number of distinct primes in P which are divisors of m. In this section we shall show that $\omega'(m)$ has approximately the Poisson distribution (see lemma 5). Let

$$y(n) = \sum_{p \in P} \frac{1}{p},$$

then we have

$$y(n) = \log\log n + O(\log\log\log n), \tag{4}$$

using the well known formula

$$\sum_{p \leq n} \frac{1}{p} = \log\log n + O(1).$$

Also, we obtain

$$\sum_{\substack{p \notin P \\ p \leq n+b(n)}} \frac{1}{p} = O(\log\log\log n). \tag{5}$$

First we consider the difference between $\omega(m)$ and $\omega'(m)$.

Lemma 3. *Let $g(n)$ be a sequence of real numbers tending to infinity. Then we have*

$$\#\{m;\, n < m \leq n+b(n),\, \omega(m)-\omega'(m) > g(n)\} = O\left(\frac{b(n)\log\log\log n}{g(n)}\right).$$

The O-term is uniform in n sufficiently large.

Proof.

$$\sum_{n<m\leq n+b(n)} |\omega(m)-\omega'(m)| = \sum_{n<m\leq n+b(n)} \sum_{\substack{p \mid m \\ p \notin P}} 1$$

$$= \sum_{\substack{p < n+b(n) \\ p \notin P}} \sum_{\substack{p | m \\ n < m \leq n+b(n)}} 1 \leq \sum_{\substack{p \leq n+b(n) \\ p \notin P}} \frac{b(n)}{p}.$$

Thus the lemma follows from (5).

Secondly we shall prove the following lemma which will be used for the proof of lemma 5.

Lemma 4. *Let t be a positive integer such that $t < 2\log\log n$. Let $L(t)$ be a set of all positive square-free integers which have exactly t prime factors belonging to the set P. Then we have*

$$\sum_{l \in L(t)} \frac{1}{l} = \frac{y(n)^t}{t!} + O\left(\frac{1}{\log\log n}\right)$$

The O-term is uniform in t and n sufficiently large.

Proof. Let $\sum'_{p,d} \frac{1}{p^2 d}$ be the sum obtained from

$$\frac{y(n)^t}{t!} - \sum_{l \in L(t)} \frac{1}{l}.$$

It is clear that

$$d \leq n^{\frac{t}{8(\log\log n)^2}} \leq n^{\frac{2\log\log n}{8(\log\log n)^2}} = n^{1/(4\log\log n)}.$$

Hence we have

$$\sum' \frac{1}{p^2 d} \leq \sum_{p \in P} \sum_{d < n^{1/(4\log\log n)}} \frac{1}{p^2 d} \leq \sum_{p > \log n} \frac{1}{p^2} \sum_{d < n^{1/(4\log\log n)}} \frac{1}{d} = O\left(\frac{1}{\log\log n}\right).$$

Thus the lemma is proved.

Now, we put

$$N(n, b(n), t) = \#\{m;\ n < m \leq n+b(n),\ \omega'(m) = t\}.$$

Then we have the following

Lemma 5. *Assume that $b(n) \geq n^{1/\log\log n}$. Then we have*

$$N(n, b(n), t)$$
$$= b(n)\frac{y(n)^t}{t!} e^{-y(n)} + O\left(\frac{b(n)}{\log\log n}\right) + O\left(b(n) e^{-c_{10}\frac{(\log\log n)^2 \log b(n)}{\log n}}\right).$$

The O-terms are uniform in t and n sufficently large.

Proof. We put

$$N_1(n, b(n), t) = \#\{m;\ n < m \leq n+b(n),\ \omega'(m) = t,\ p^2 \nmid m \text{ for all } p \in P\}.$$

We let l be an element of $L(t)$ and put

$$H(n, b(n), l) = \#\{m;\ n < m \leq n+b(n),\ l \mid m,\ p \nmid \frac{m}{l} \text{ for all } p \in P\}.$$

Then we have

$$N_1(n, b(n), t) = \sum_{l \in L(t)} H(n, b(n), l). \tag{6}$$

Let $p_1, p_2, \ldots, p_h$ be all the prime numbers such that $p \nmid l$ and $p \in P$. We put $v = \varphi(l)$ where φ is Euler's function. Then we have

$$\begin{aligned} H(n, b(n), l) &= \#\{m;\ n < m \leq n+b(n),\ l \mid m,\ (l, m/l) = 1, \\ &\qquad\qquad m \not\equiv 0 \pmod{p_j},\ j = 1, 2, \ldots, h\} \\ &= \#\{m;\ n < m \leq n+b(n),\ m \equiv q_i l \pmod{l^2},\ i = 1, 2, \ldots, v, \\ &\qquad\qquad m \not\equiv 0 \pmod{p_j},\ j = 1, 2, \ldots, h\} \\ &= \sum_{i=1}^{v} F(n, b(n), q_i l, l^2; p_1, p_2, \ldots, p_h), \end{aligned} \tag{7}$$

where $\{q_1, q_2, \ldots, q_v\}$ is a reduced set of residues mod l. By (3) and the definition of $L(t)$ we have

$$l^2 \leq n^{\frac{2t}{8(\log\log n)^2}} < \left(n^{\frac{1}{\log\log n}}\right)^{\frac{1}{2}} \leq \sqrt{b(n)}\,. \tag{8}$$

Let $r_2 = n^{\frac{1}{8(\log\log n)^2}}$, and put $n_3 = [(n-q_i)/l^2]$, $n_4 = [(n+b(n)-q_i)/l^2]$. Then we have

$$\begin{aligned} \log r_2 &= \frac{\log n}{8(\log\log n)^2} \\ &= \frac{\log(n_4-n_3)}{\log(n_4-n_3)} \cdot \frac{\log n}{8(\log\log n)^2} \\ &\leq \frac{\log(n_4-n_3)}{\log(b(n)/l^2-1)} \cdot \frac{\log n}{8(\log\log n)^2}\,. \end{aligned}$$

Since $b(n) \geq n^{1/(\log\log n)}$ and by (8) there exists a sufficiently small positive constant c_{11} such that

$$(9) \qquad \log r_2 < c_{11} \log (n_4 - n_3).$$

From (8), (9) we know that Lemma 2 can be applied to estimate $F(n, b(n), q_t l, l^2; p_1, p_2, \dots, p_h)$ in (7). Hence we obtain

$$(10) \qquad H(n, b(n), l) = \frac{b(n)\varphi(l)}{l^2} \prod_{j=1}^{h} \left(1 - \frac{1}{p_j}\right)\left\{1 + O\left(e^{-c_{12}\frac{\log b(n)}{\log r_2}}\right)\right\}.$$

Inserting (10) into (6) we have

$$\begin{aligned} N_1(n, b(n), t) &= \sum_{l \in L(t)} \frac{b(n)\varphi(l)}{l^2} \prod_{j=1}^{h} \left(1 - \frac{1}{p_j}\right)\left\{1 + O\left(e^{-c_{12}\frac{\log b(n)}{\log r_2}}\right)\right\} \\ &= b(n) \prod_{p \in P} \left(1 - \frac{1}{p}\right)\left(\sum_{l \in L(t)} \frac{1}{l}\right)\left\{1 + O\left(e^{-c_{13}\frac{(\log\log n)^2 \log b(n)}{\log n}}\right)\right\}. \end{aligned}$$

It is clear that

$$\begin{aligned} \prod_{p \in P} \left(1 - \frac{1}{p}\right) &= \exp\left\{\sum_{p \in P} \log\left(1 - \frac{1}{p}\right)\right\} \\ &= \exp\left\{-\sum_{p \in P} \frac{1}{p} + O\left(\sum_{p \in P} \frac{1}{p^2}\right)\right\} \\ &= e^{-y(n)}\left\{1 + O\left(\frac{1}{\log n}\right)\right\}. \end{aligned}$$

Hence we have from this and lemma 4

$$\begin{aligned} &N_1(n, b(n), t) \\ &\quad = b(n) e^{-y(n)} \frac{y(n)^t}{t!} + O\left(\frac{b(n)}{\log\log n}\right) + O\left(b(n) e^{-c_{10}\frac{(\log\log n)^2 \log b(n)}{\log n}}\right). \end{aligned}$$

Now the number of positive integers m $(n < m \leq n + b(n))$ divisible by p^2 for some $p \in P$ is less than $\sum_{p \in P}\left(\frac{b(n)}{p^2} + 1\right)$. Since

$$b(n) \geq n^{1/(\log\log n)} > n^{\frac{1}{4(\log\log n)^2}} > p^2$$

for $p \in P$, we obtain

$$\sum_{p \in P}\left(\frac{b(n)}{p^2} + 1\right) \leq 2 \sum_{p \in P} \frac{b(n)}{p^2} = O\left(\frac{b(n)}{\log\log n}\right).$$

From this we have

$$N(n, b(n), t) = N_1(n, b(n), t) + O\left(\frac{b(n)}{\log\log n}\right),$$

which implies our lemma. The main term of $N(n, b(n), t)/b(n)$ is the so-called Poisson probability.

4. The normal distribution of $\omega'(m)$. In this section we shall show that $\omega'(m)$ has approximately the normal distribution (see lemma 7).

Lemma 6. *Let $\alpha < \beta$ be real numbers. Let n be a sufficiently large integer for which there exists a natural number t_1 such that*

$$y(n) + \alpha\sqrt{y(n)} < t_1 < y(n) + \beta\sqrt{y(n)}.$$

Let $\mu = \max(1, |\alpha|, |\beta|)$ and $t_1 = y(n) + u\sqrt{y(n)}$, where u $(\alpha < u < \beta)$ is a real number which is determined by t_1 and n. Then we have

$$N(n, b(n), t_1)$$
$$= \frac{1}{\sqrt{2\pi y(n)}} b(n) e^{-\frac{1}{2}u^2} + O\left(\frac{\mu^4 b(n)}{\log\log n}\right) + O\left(b(n) e^{-c_{14}\frac{(\log\log n)^2 \log b(n)}{\log n}}\right).$$

The O-terms are uniform in n.

Proof. We know that

$$t_1! = \sqrt{2\pi}\, t_1^{t_1+\frac{1}{2}} e^{-t_1}\left\{1 + O\left(\frac{1}{t_1}\right)\right\}.$$

On the other hand we can put

$$t_1 = y(n) + u\sqrt{y(n)}.$$

Hence we have

$$(11) \qquad t_1! = \sqrt{2\pi y(n)}\, y(n)^{t_1}\left(1 + \frac{u}{\sqrt{y(n)}}\right)^{y(n) + u\sqrt{y(n)} + 1/2} \times$$
$$\times e^{-y(n) - u\sqrt{y(n)}}\left\{1 + O\left(\frac{1}{y(n) + u\sqrt{y(n)}}\right)\right\}.$$

Since $\log(1+x) = x - \frac{1}{2}x^2 + O(|x|^3)$ for a real number x with $|x| < 1$, we obtain

$$(12)\quad \left(1+\frac{u}{\sqrt{y(n)}}\right)^{y(n)+u\sqrt{y(n)}+\frac{1}{2}} = \exp\left\{u\sqrt{y(n)}+\frac{u^2}{2}+O\left(\frac{\mu^4}{\sqrt{y(n)}}\right)\right\}.$$

By (11) and (12) it follows that

$$(13)\quad t_1! = \sqrt{2\pi y(n)}\, y(n)^{t_1} e^{-y(n)+\frac{1}{2}u^2+O\left(\frac{\mu^4}{\sqrt{y(n)}}\right)}\left\{1+O\left(\frac{1}{y(n)+u\sqrt{y(n)}}\right)\right\}.$$

By (4) it is clear that $t_1 < 2\log\log n$ for a sufficiently large n. Hence by (13) and lemma 5 we have

$$\begin{aligned}
N(n, b(n), t_1) &= \frac{b(n)}{\sqrt{2\pi y(n)}} e^{-\frac{1}{2}u^2}\left\{1+O\left(\frac{\mu^4}{\sqrt{y(n)}}\right)\right\}\left\{1+O\left(\frac{1}{\sqrt{y(n)}}\right)\right\} \\
&\quad +O\left(\frac{b(n)}{\log n}\right)+O\left(b(n)e^{-c_{14}\frac{(\log\log n)^2\log b(n)}{\log n}}\right) \\
&= \frac{b(n)}{\sqrt{2\pi y(n)}} e^{-\frac{1}{2}u^2}+O\left(\frac{\mu^4 b(n)}{y(n)}\right)+O\left(\frac{b(n)}{\log n}\right) \\
&\quad +O\left(b(n)e^{-c_{14}\frac{(\log\log n)^2\log b(n)}{\log n}}\right) \\
&= \frac{b(n)}{\sqrt{2\pi y(n)}} e^{-\frac{1}{2}u^4}+O\left(\frac{\mu^4 b(n)}{\log\log n}\right) \\
&\quad +O\left(b(n)e^{-c_{14}\frac{(\log\log n)^2\log b(n)}{\log n}}\right).
\end{aligned}$$

Thus the lemma is proved.

Lemma 7. *Let*

$$\begin{aligned}
B_1(n, b(n), \alpha, \beta) = \#\{m;\ n < m \leq n+b(n), \\
y(n)+\alpha\sqrt{y(n)} < \omega'(m) < y(n)+\beta\sqrt{y(n)}\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
B_1(n, b(n), \alpha, \beta) &= \frac{b(n)}{\sqrt{2\pi}}\int_\alpha^\beta e^{-\frac{1}{2}u^2}du+O\left(\frac{\mu^5 b(n)}{\sqrt{\log\log n}}\right) \\
&\quad +O\left(b(n)\mu\sqrt{\log\log n}\, e^{-c_{15}\frac{(\log\log n)^2\log b(n)}{\log n}}\right).
\end{aligned}$$

The O-terms are uniform in n sufficiently large.

Proof. For a sufficiently large integer n let $t_1 = t_0+1, t_0+2, \ldots, t_0+s$ be s natural numbers such that

$$y(n)+\alpha\sqrt{y(n)} < t_1 < y(n)+\beta\sqrt{y(n)}.$$

Further, we put $t_0+i = y(n)+u_i\sqrt{y(n)}$. It is obvious that

$$u_{i+1}-u_i = \frac{1}{\sqrt{y(n)}}, \quad s = O(\mu\sqrt{\log\log n}).$$

From lemma 6 we have

$$\begin{aligned} B_1(n, b(n), \alpha, \beta) &= \sum_{i=1}^{s} N(n, b(n), t_0+i) \\ &= \frac{b(n)}{\sqrt{2\pi}} \sum_{i=1}^{s} (u_{i+1}-u_i)e^{-\frac{1}{2}u_i^2} + O\left(\frac{\mu^5 b(n)}{\sqrt{\log\log n}}\right) \\ &\quad + O\left(b(n)\mu\sqrt{\log\log n}\, e^{-c_{15}\frac{(\log\log n)^2\log b(n)}{\log n}}\right). \end{aligned}$$

Using a mean value theorem, we have

$$\sum_{i=1}^{s} (u_{i+1}-u_i)e^{-\frac{1}{2}u_i^2} = \int_{\alpha}^{\beta} e^{-\frac{1}{2}u^2}du + O\left(\frac{\mu^2}{\sqrt{y(n)}}\right).$$

Hence it follows that

$$\begin{aligned} B_1(n, b(n), \alpha, \beta) &= \frac{b(n)}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-\frac{1}{2}u^2}du + O\left(\frac{\mu^5 b(n)}{\sqrt{\log\log n}}\right) \\ &\quad + O\left(b(n)\mu\sqrt{\log\log n}\, e^{-c_{15}\frac{(\log\log n)^2\log b(n)}{\log n}}\right). \end{aligned}$$

Thus the lemma is proved.

4. Proofs of Theorem 1 and Theorem 2. We put

$$\begin{aligned} B_2(n, b(n), \alpha, \beta) = \#\{m;\ & n < m \le n+b(n), \\ & y(n)+\alpha\sqrt{y(n)} < \omega'(m) < y(n)+\beta\sqrt{y(n)},\ \omega(m)-\omega'(m) < g(n)\} \end{aligned}$$

and

$$\begin{aligned} A_1(n, b(n), \alpha, \beta) = \#\{m;\ & n < m \le n+b(n), \\ & \log\log m+\alpha\sqrt{\log\log m} < \omega(m) < \log\log m+\beta\sqrt{\log\log m}, \\ & \omega(m)-\omega'(m) < g(n)\}. \end{aligned}$$

Let

$$w = \frac{g(n)+\mu\log\log\log n}{\sqrt{y(n)}},$$

provided that the function $g(n) > 0$ is such that $w > 0$ becomes sufficiently small for a large n. For a positive integer m such that $\omega(m)-\omega'(m) < g(n)$, the inequality

$$\log\log m+\alpha\sqrt{\log\log m} < \omega(m) < \log\log m+\beta\sqrt{\log\log m},$$

is equivalent to

$$y(n)+(\alpha+O(w))\sqrt{y(n)} < \omega'(m) < y(n)+(\beta+O(w))\sqrt{y(n)}.$$

Hence we have

$$A_1(n, b(n), \alpha, \beta) = B_2(n, b(n), \alpha+O(w), \beta+O(w)). \tag{14}$$

From Lemma 3, Lemma 7 and (14) we have

$$\begin{aligned}
&A(n, b(n), \alpha, \beta) \\
&= A_1(n, b(n), \alpha, \beta)+O\left(\frac{b(n)\log\log\log n}{g(n)}\right) \\
&= B_2(n, b(n), \alpha+O(w), \beta+O(w))+O\left(\frac{b(n)\log\log\log n}{g(n)}\right) \\
&= B_1(n, b(n), \alpha+O(w), \beta+O(w))+O\left(\frac{b(n)\log\log\log n}{g(n)}\right) \\
&= \frac{b(n)}{\sqrt{2\pi}}\int_{\alpha+O(w)}^{\beta+O(w)} e^{-\frac{1}{2}u^2}du+O\left(\frac{\mu^5 b(n)}{\sqrt{\log\log n}}\right)+O\left(\frac{b(n)\log\log\log n}{g(n)}\right) \\
&\quad +O\left(b(n)\mu\sqrt{\log\log n}\, e^{-c_{15}\frac{(\log\log n)^2\log b(n)}{\log n}}\right) \\
&= \frac{b(n)}{\sqrt{2\pi}}\int_{\alpha}^{\beta} e^{-\frac{1}{2}u^2}du+O\left\{\frac{b(n)\mu^5(g(n)+\log\log\log n)}{\sqrt{\log\log n}}\right\} \\
&\quad +O\left(\frac{b(n)\log\log\log n}{g(n)}\right)+O\left(b(n)\mu\sqrt{\log\log n}\, e^{-c_{15}\frac{(\log\log n)^2\log b(n)}{\log n}}\right).
\end{aligned}$$

If we put

$$g(n) = (\log\log n)^{1/4}(\log\log\log n)^{1/2},$$

we have immediately Theorem 2.

Finally we shall prove Theorem 1. Let $\alpha < \beta$ be real numbers. If $b(n) \geq n^{1/(\log\log n)}$, then it follows from Theorem 2 that

$$\lim_{n\to\infty}\frac{1}{b(n)}A(n, b(n), \alpha, \beta) = \frac{1}{\sqrt{2\pi}}\int_{\alpha}^{\beta} e^{-\frac{1}{2}y^2}dy. \tag{15}$$

Here we use the argument in [5]. For any real $\varepsilon > 0$ let $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ be real numbers such that

$$\frac{1}{\sqrt{2\pi}}\int_{\alpha(\varepsilon)}^{\beta(\varepsilon)} e^{-\frac{1}{2}y^2}dy > 1-\varepsilon.$$

Since

$$\frac{1}{b(n)}A(n, b(n), -\infty, \alpha(\varepsilon)) \leq 1-\frac{1}{b(n)}A(n, b(n), \alpha(\varepsilon), \beta(\varepsilon)),$$

we obtain from (15)

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{b(n)}A(n, b(n), -\infty, \alpha(\varepsilon)) &\leq 1-\lim_{n\to\infty}\frac{1}{b(n)}A(n, b(n), \alpha(\varepsilon), \beta(\varepsilon)) \\ &= 1-\frac{1}{\sqrt{2\pi}}\int_{\alpha(\varepsilon)}^{\beta(\varepsilon)} e^{-\frac{1}{2}y^2}dy < \varepsilon.\end{aligned}$$

Then we have

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{b(n)}A(n, b(n), -\infty, x) &\geq \lim_{n\to\infty}\frac{1}{b(n)}A(n, b(n), \alpha(\varepsilon), x) \\ &= \frac{1}{\sqrt{2\pi}}\int_{\alpha(\varepsilon)}^{x} e^{-\frac{1}{2}y^2}dy > \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}dy-\varepsilon\end{aligned}$$

and

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{b(n)}A(n, b(n), -\infty, x) &\leq \lim_{n\to\infty}\frac{1}{b(n)}A(n, b(n), \alpha(\varepsilon), x) \\ &\quad +\limsup_{n\to\infty}\frac{1}{b(n)}A(n, b(n), -\infty, \alpha(\varepsilon)) \\ &< \frac{1}{\sqrt{2\pi}}\int_{\alpha(\varepsilon)}^{x} e^{-\frac{1}{2}y^2}dy+\varepsilon < \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}dy+\varepsilon.\end{aligned}$$

These complete the proof of Theorem 1.

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