

EXPONENTIAL SUMS OVER FINITE FIELDS

HARALD NIEDERREITER

Let \mathbb{F}_q be the finite field of order q . For $f \in \mathbb{F}_q[x_1, \dots, x_r]$ and a nontrivial additive character χ of \mathbb{F}_q define the *character sum*

$$C_1 = \sum_{a_1, \dots, a_r \in \mathbb{F}_q} \chi(f(a_1, \dots, a_r)).$$

Together with C_1 we consider *lifted character sums* corresponding to the various finite extensions \mathbb{F}_{q^s} of \mathbb{F}_q contained in a fixed algebraic closure $\overline{\mathbb{F}_q}$ of \mathbb{F}_q . First, χ is lifted via the trace to a nontrivial additive character $\chi^{(s)}$ of \mathbb{F}_{q^s} ; in detail, if Tr_s denotes the trace function from \mathbb{F}_{q^s} onto \mathbb{F}_q , then set

$$(1) \quad \chi^{(s)}(a) = \chi(\text{Tr}_s(a)) \text{ for } a \in \mathbb{F}_{q^s}.$$

Now define

$$C_s = \sum_{a_1, \dots, a_r \in \mathbb{F}_{q^s}} \chi^{(s)}(f(a_1, \dots, a_r)).$$

With these lifted character sums one sets up the *L-function*

$$L(z) = \exp\left(\sum_{s=1}^{\infty} \frac{C_s}{s} z^s\right)$$

in the complex variable z . For $r = 1$ one has the classical results of A. Weil on these *L-functions* (see [5, Ch. 5]). For general r , Grothendieck [4] proved by methods of *l*-adic cohomology that $L(z)$ is always a rational function. Bombieri [1] conjectured that $L(z)$ has the special form

$$(2) \quad L(z) = P(z)^{i-1} z^{-1}$$

with a polynomial P , provided that f satisfies some kind of nonsingularity condition. In his famous paper on the Weil conjectures, Deligne [3] proved among other results that Bombieri's conjecture is true if $\deg(f)$ is not a multiple of the characteristic of \mathbb{F}_q and the leading homogeneous part f_0 of f is nonsingular in the standard sense (i.e., there is no point over $\overline{\mathbb{F}_q}$ at which f_0 and all its first-order partial derivatives vanish simultaneously).

In a lecture given at the Oberwolfach Conference on Analytic Number Theory in 1982, S. A. Stepanov announced an elementary proof of the result of Deligne quoted above for the case where $\deg(f)$ is less than the charac-

teristic of \mathbb{F}_q (see [12]). According to the outline given by Stepanov, his method depends, first of all, on an explicit expansion of $L(z)$, where we assume for simplicity that r is odd (otherwise consider $L(z)^{-1}$):

$$(3) \quad L(z) = \exp\left(\sum_{s=1}^{\infty} \frac{C_s}{s} z^s\right) = \prod_{s=1}^{\infty} \exp\left(\frac{C_s}{s} z^s\right) = \prod_{j=1}^{\infty} \left(\sum_{i_j=0}^{\infty} \frac{1}{i_j!} \cdot \frac{C_s^{i_j}}{j^{i_j}} z^{j i_j}\right) \\ = 1 + \sum_{s=1}^{\infty} \left(\sum_{i_1+2i_2+\dots+si_s=s} \frac{C_1^{i_1} \dots C_s^{i_s}}{i_1! \dots i_s! 2^{i_2} \dots s^{i_s}}\right) z^s = : 1 + \sum_{s=1}^{\infty} \sigma_s z^s.$$

Now one has to show $\sigma_s = 0$ for all sufficiently large s . Stepanov claimed that he can do this by inserting the explicit form of the sums C_i , then fully expanding the resulting expression for σ_s and combining terms in a suitable way. In a brief note [13] summarizing the method, this point is brushed over. Since I could not get any further details from Stepanov, I tried to reconstruct his argument and I looked first for a simple test case.

It turns out that Stepanov had already used this method in his paper [11] to give an elementary proof of the Davenport-Hasse theorem for Gaussian sums over finite fields. A closer inspection of this proof reveals, however, that it breaks down at a crucial step of the argument. This raises some doubts about the validity of Stepanov's claim at the Oberwolfach conference. But, obviously, a final verdict can only be given when Stepanov publishes his proof in full detail.

In order to elaborate on the error in [11], it is necessary to first describe the Davenport-Hasse theorem. Let ψ be a multiplicative and χ an additive character of \mathbb{F}_q , not both being trivial, and use the convention $\psi(0) = 0$. The corresponding *Gaussian sum* is defined by

$$G_1 = G(\psi, \chi) = \sum_{a \in \mathbb{F}_q} \psi(a) \chi(a).$$

The character ψ is lifted by means of the formula

$$\psi^{(s)}(a) = \psi(N_s(a)) \text{ for } a \in \mathbb{F}_{q^s},$$

where N_s is the norm function from \mathbb{F}_{q^s} onto \mathbb{F}_q . With $\chi^{(s)}$ being given by (1), we consider the *lifted Gaussian sum*

$$G_s = G(\psi^{(s)}, \chi^{(s)}) = \sum_{a \in \mathbb{F}_{q^s}} \psi^{(s)}(a) \chi^{(s)}(a).$$

The Davenport-Hasse theorem expresses the following simple relation between G_s and G_1 .

Davenport-Hasse Theorem. $G_s = (-1)^{s-1} G_1^s$.

In the paper of Davenport and Hasse [2] this relation arose from the study of L -functions of an algebraic function field defined by an Artin-Schreier curve over \mathbb{F}_q . The paper contains also a proof of the formula based on the results of Stickelberger [14] concerning the factorization of Gaussian sums in cyclotomic fields. Schmid [10] has given an elementary proof of the Davenport-Hasse theorem by induction on s .

Although this is not made explicit, the method in Stepanov [11] for proving the Davenport-Hasse theorem amounts to considering an L -function corresponding to Gaussian sums and expanding it as in (3) :

$$L(z) = \exp\left(\sum_{s=1}^{\infty} \frac{G_s}{s} z^s\right) = 1 + \sum_{s=1}^{\infty} \gamma_s z^s$$

with

$$\gamma_s = \sum_{i_1+2i_2+\dots+si_s=s} \frac{G_1^{i_1} \dots G_s^{i_s}}{i_1! \dots i_s! 2^{i_2} \dots s^{i_s}}.$$

Then one tries to show $\gamma_s = 0$ for $s > 1$. In one of the key steps it is claimed in [11] that for a given solution of $i_1 + 2i_2 + \dots + si_s = s$ in non-negative integers i_1, \dots, i_s the number $N(t_1, \dots, t_s)$ of tuples

$$(a_1^{(1)}, \dots, a_{i_1}^{(1)}, \dots, a_1^{(s)}, \dots, a_{i_s}^{(s)}),$$

with the first i_1 entries being in \mathbb{F}_q , the next i_2 entries being in \mathbb{F}_{q^2}, \dots , the last i_s entries being in \mathbb{F}_{q^s} , and with the elementary symmetric polynomials in the $a_i^{(j)}$ and their conjugates over \mathbb{F}_q having prescribed values $t_1, \dots, t_s \in \mathbb{F}_q$, is independent of t_1, \dots, t_s . This statement is, however, incorrect. For instance, if $i_s = 0$ and

$$t(x) = x^s - t_1 x^{s-1} + t_2 x^{s-2} \mp \dots + (-1)^s t_s$$

is irreducible over \mathbb{F}_q , then $N(t_1, \dots, t_s) = 0$, whereas $N(0, \dots, 0) = 1$, as can be seen immediately from the factorization of $t(x)$ in its splitting field over \mathbb{F}_q . To provide another counterexample, we note that if $i_1 = s, i_2 = \dots = i_s = 0$, then $N(t_1, \dots, t_s) = 0$ whenever $t(x)$ does not split completely over \mathbb{F}_q , whereas $N(0, \dots, 0) = 1$ and $N(1, 0, \dots, 0) = s$. The proof of the Davenport-Hasse theorem in [11] is therefore fallacious. Any attempt to repair it would have to be based on a correct formula for $N(t_1, \dots, t_s)$. Such a formula will, however, be very complicated and lead to a rather involved

proof of the Davenport-Hasse theorem.

We present now a *short proof* of the Davenport-Hasse theorem using a technique in [5, Ch. 5]. Let $M = \{g \in \mathbb{F}_q[x] : g \text{ monic}\}$, $M_r = \{g \in M : \deg(g) = r\}$, $I = \{g \in M : g \text{ irreducible over } \mathbb{F}_q\}$, $I_d = \{g \in I : \deg(g) = d\}$. Define $\lambda : M \rightarrow \mathbb{C}$ by $\lambda(1) = 1$ and

$$\lambda(x^r - c_1 x^{r-1} + \cdots + (-1)^r c_r) = \psi(c_r) \chi(c_1) \text{ for } r \geq 1.$$

Then λ is multiplicative in the sense that $\lambda(gh) = \lambda(g)\lambda(h)$ for all $g, h \in M$. Splitting up G_s according to the degree of $a \in \mathbb{F}_{q^s}$ over \mathbb{F}_q , writing g_a for the minimal polynomial of a over \mathbb{F}_q , and using simple properties of Tr_s and N_s (see [5, Ch. 2]), we get for $|z| < q^{-1}$:

$$\begin{aligned} \sum_{s=1}^{\infty} \frac{G_s}{s} z^s &= \sum_{s=1}^{\infty} \frac{z^s}{s} \sum_{d|s} \sum_{\deg(a)=d} (\psi^{(d)}(a) \chi^{(d)}(a))^{s/d} \\ &= \sum_{s=1}^{\infty} \frac{z^s}{s} \sum_{d|s} \sum_{\deg(a)=d} \lambda(g_a)^{s/d} \\ &= \sum_{s=1}^{\infty} \frac{z^s}{s} \sum_{d|s} d \sum_{g \in I_d} \lambda(g)^{s/d} \\ &= \sum_{d=1}^{\infty} \sum_{g \in I_d} \sum_{s=1}^{\infty} \frac{1}{s} (\lambda(g) z^d)^s \\ &= \sum_{d=1}^{\infty} \sum_{g \in I_d} \log \frac{1}{1 - \lambda(g) z^d} \\ &= \log \prod_{g \in I} \frac{1}{1 - \lambda(g) z^{\deg(g)}}. \end{aligned}$$

In this Euler product $\lambda(g) z^{\deg(g)}$ is multiplicative as a function of g , hence

$$\begin{aligned} \sum_{s=1}^{\infty} \frac{G_s}{s} z^s &= \log \left(\sum_{g \in M} \lambda(g) z^{\deg(g)} \right) = \log \left(\sum_{r=0}^{\infty} \left(\sum_{g \in M_r} \lambda(g) \right) z^r \right) \\ &= \log(1 + G_1 z) = \sum_{s=1}^{\infty} \frac{1}{s} (-1)^{s-1} G_1^s z^s, \end{aligned}$$

and comparison of coefficients yields the Davenport-Hasse theorem.

The same method can be applied to other exponential sums. For instance, if ψ_1 and ψ_2 are two multiplicative characters of \mathbb{F}_q , not both of them trivial, and if we fix a nonzero $b \in \mathbb{F}_q$, then we can consider the *lifted Jacobi sums*

$$J_s = \sum_{a \in \mathbb{F}_q^*} \psi_1^{(s)}(a) \psi_2^{(s)}(b-a).$$

With

$$\lambda(g) = \psi_1((-1)^{\deg(g)}g(0))\psi_2(g(b)) \text{ for } g \in M$$

we get then as above :

$$\begin{aligned} \sum_{s=1}^{\infty} \frac{J_s}{s} z^s &= \log \left(\sum_{r=0}^{\infty} \left(\sum_{g \in M_r} \lambda(g) \right) z^r \right) \\ &= \log(1 + J_1 z) = \sum_{s=1}^{\infty} \frac{1}{s} (-1)^{s-1} J_1^s z^s, \end{aligned}$$

and comparison of coefficients yields $J_s = (-1)^{s-1} J_1^s$, a formula first shown by Mitchell [6].

The Davenport-Hasse theorem can be used to establish a formula of the type (2) for L -functions corresponding to a general class of multiple exponential sums. For $1 \leq i \leq r$ let F_i be a finite field, let χ_i be a nontrivial additive character of F_i , and let ψ_i be an arbitrary multiplicative character of F_i . Let H_i be a subgroup of the direct product $F_1^* \times \dots \times F_r^*$ of index m , where F^* denotes the multiplicative group of a finite field F . If $F_{i,s}$ is the extension of F_i of degree s contained in a fixed algebraic closure of F_i , let

$$\bar{N}_s : F_{1,s}^* \times \dots \times F_{r,s}^* \rightarrow F_1^* \times \dots \times F_r^*$$

be the componentwise norm function and set $H_s = \bar{N}_s^{-1}(H_1)$. For fixed $u \in F_1^* \times \dots \times F_r^*$ define

$$(4) \quad E_s = m \sum_{(a_1, \dots, a_r) \in uH_s} \chi_1^{i(s)}(a_1) \dots \chi_r^{i(s)}(a_r) \psi_1^{i(s)}(a_1) \dots \psi_r^{i(s)}(a_r).$$

Then set up the corresponding L -function

$$(5) \quad L(z) = \exp \left(\sum_{s=1}^{\infty} \frac{E_s}{s} z^s \right).$$

Theorem 1. *The L -function in (5) is of the form*

$$L(z) = P(z)^{(-1)^{r-1}}$$

with a polynomial P of degree m satisfying $P(0) = 1$.

Proof. If $u = (u_1, \dots, u_r) \in F_1^* \times \dots \times F_r^*$, we can write

$$(6) \quad E_s = m \sum_{(a_1, \dots, a_r) \in H_s} \chi_1^{i(s)}(u_1 a_1) \dots \chi_r^{i(s)}(u_r a_r) \psi_1^{i(s)}(u_1 a_1) \dots \psi_r^{i(s)}(u_r a_r).$$

For fixed s we use the Fourier expansion of the restriction of $\chi_i^{(s)}$ to $F_{i,s}^*$ with

respect to the characters λ_i of that group :

$$(7) \quad \chi_i^{(s)}(c) = \frac{1}{q_i^s - 1} \sum_{\lambda_i} G(\bar{\lambda}_i, \chi_i^{(s)}) \lambda_i(c) \text{ for all } c \in F_{i,s}^*,$$

where q_i denotes the order of F_i , the Fourier coefficients are Gaussian sums, and $\bar{\lambda}_i$ is the conjugate character of λ_i . Inserting (7) in (6) we get

$$\begin{aligned} E_s &= \frac{m}{(q_1^s - 1) \cdots (q_r^s - 1)} \sum_{(a_1, \dots, a_r) \in H_s} \phi_1^{(s)}(u_1 a_1) \cdots \phi_r^{(s)}(u_r a_r) \cdot \\ &\quad \sum_{\lambda_1, \dots, \lambda_r} G(\bar{\lambda}_1, \chi_1^{(s)}) \cdots G(\bar{\lambda}_r, \chi_r^{(s)}) \lambda_1(u_1 a_1) \cdots \lambda_r(u_r a_r) \\ &= \frac{m}{(q_1^s - 1) \cdots (q_r^s - 1)} \sum_{\lambda_1, \dots, \lambda_r} G(\bar{\lambda}_1, \chi_1^{(s)}) \cdots G(\bar{\lambda}_r, \chi_r^{(s)}) (\phi_1^{(s)} \lambda_1)(u_1) \cdots \\ &\quad (\phi_r^{(s)} \lambda_r)(u_r) \sum_{(a_1, \dots, a_r) \in H_s} (\phi_1^{(s)} \lambda_1)(a_1) \cdots (\phi_r^{(s)} \lambda_r)(a_r). \end{aligned}$$

Let A_s be the annihilator of H_s in the dual group of $F_{1,s}^* \times \cdots \times F_{r,s}^*$. Then the inner sum has the value $|H_s|$ if $(\phi_1^{(s)} \lambda_1, \dots, \phi_r^{(s)} \lambda_r) \in A_s$ and 0 otherwise. Therefore,

$$(8) \quad E_s = \frac{m |H_s|}{(q_1^s - 1) \cdots (q_r^s - 1)} \sum_{(\lambda_1, \dots, \lambda_r) \in A_s} G(\bar{\lambda}_1 \phi_1^{(s)}, \chi_1^{(s)}) \cdots G(\bar{\lambda}_r \phi_r^{(s)}, \chi_r^{(s)}) \lambda_1(u_1) \cdots \lambda_r(u_r).$$

Since \bar{N}_s is surjective, we have

$$|\ker \bar{N}_s| = \frac{(q_1^s - 1) \cdots (q_r^s - 1)}{(q_1 - 1) \cdots (q_r - 1)},$$

and from $H_1 \simeq H_s / \ker \bar{N}_s$ we get

$$(9) \quad |H_s| = |H_1| \frac{(q_1^s - 1) \cdots (q_r^s - 1)}{(q_1 - 1) \cdots (q_r - 1)}.$$

This implies

$$(10) \quad |A_s| = \frac{(q_1^s - 1) \cdots (q_r^s - 1)}{|H_s|} = \frac{(q_1 - 1) \cdots (q_r - 1)}{|H_1|} = |A_1|.$$

Since it is immediate that $(\lambda_1^{(s)}, \dots, \lambda_r^{(s)}) \in A_s$ whenever $(\lambda_1, \dots, \lambda_r) \in A_1$, it follows from (10) that A_s consists exactly of all $(\lambda_1^{(s)}, \dots, \lambda_r^{(s)})$ with $(\lambda_1, \dots, \lambda_r) \in A_1$. Using this fact as well as (9) and the definition of m , the identity (8) attains the form

$$E_s = \sum_{(\lambda_1, \dots, \lambda_r) \in A_1} G(\bar{\lambda}_1^{(s)} \psi_1^{(s)}, \chi_1^{(s)}) \cdots G(\bar{\lambda}_r^{(s)} \psi_r^{(s)}, \chi_r^{(s)}) \lambda_1^{(s)}(u_1) \cdots \lambda_r^{(s)}(u_r).$$

Now we can apply the Davenport-Hasse theorem, and taking into account that $\lambda_i^{(s)}(u_i) = (\lambda_i(u_i))^s$, we get

$$E_s = (-1)^r \sum_{(\lambda_1, \dots, \lambda_r) \in A_1} ((-1)^r G(\bar{\lambda}_1 \psi_1, \chi_1) \cdots G(\bar{\lambda}_r \psi_r, \chi_r) \lambda_1(u_1) \cdots \lambda_r(u_r))^s.$$

Since $|A_1| = m$, we can label the numbers

$$(11) \quad (-1)^r G(\bar{\lambda}_1 \psi_1, \chi_1) \cdots G(\bar{\lambda}_r \psi_r, \chi_r) \lambda_1(u_1) \cdots \lambda_r(u_r)$$

by $\omega_1, \dots, \omega_m$, so that

$$(12) \quad E_s = (-1)^r \sum_{j=1}^m \omega_j^s.$$

For the L -function in (5) we obtain then

$$\begin{aligned} L(z) &= \exp\left((-1)^r \sum_{s=1}^{\infty} \frac{z^s}{s} \sum_{j=1}^m \omega_j^s\right) = \exp\left((-1)^r \sum_{j=1}^m \sum_{s=1}^{\infty} \frac{1}{s} (\omega_j z)^s\right) \\ &= \exp\left((-1)^{r-1} \sum_{j=1}^m \log(1 - \omega_j z)\right) = P(z)^{(-1)^{r-1}} \end{aligned}$$

with

$$P(z) = (1 - \omega_1 z) \cdots (1 - \omega_m z).$$

Since the characters χ_i are nontrivial, we have $\omega_j \neq 0$ for $1 \leq j \leq m$, and the proof of Theorem 1 is complete.

The exponential sums in (4) include various classical exponential sums as special cases, such as Gaussian sums, Kummer cyclotomic periods, and products of such sums. They also include a class of character sums studied by the author in a number of papers (see [7], [8], [9]). This will be explained in the sequel.

Let (y_n) , $n = 0, 1, \dots$, be a *linear recurring sequence* in \mathbb{F}_q satisfying the linear recurrence relation

$$(13) \quad y_{n+k} = b_{k-1} y_{n+k-1} + \cdots + b_0 y_n, \quad n = 0, 1, \dots,$$

with constant coefficients $b_{k-1}, \dots, b_0 \in \mathbb{F}_q$, $b_0 \neq 0$. To exclude a trivial case, we assume $(y_0, \dots, y_{k-1}) \neq (0, \dots, 0)$. We can also assume that (13) is the linear recurrence relation of least order satisfied by (y_n) , i.e., that

$$f(x) = x^k - b_{k-1} x^{k-1} - \cdots - b_0 \in \mathbb{F}_q[x]$$

is the *minimal polynomial* of (y_n) (compare with [5, Ch. 8]). Then the least period τ of (y_n) is equal to the least positive integer e such that $f(x)$ divides $x^e - 1$. We consider now the case where f has no multiple roots. Then

$$f = f_1 \cdots f_r$$

with distinct monic irreducible polynomials f_i over $K = \mathbb{F}_q$. Let v_i be a fixed root of f_i in its splitting field F_i over K , and let $\text{Tr}_{F_i/K}$ denote the trace function from F_i onto K .

Lemma. *Under the conditions above, there exist elements $u_i \in F_i$, $1 \leq i \leq r$, such that*

$$y_n = \sum_{i=1}^r \text{Tr}_{F_i/K}(u_i v_i^n) \text{ for } n = 0, 1, \dots.$$

Proof. Let

$$(14) \quad G(x) = \sum_{n=0}^{\infty} y_n x^n$$

be the generating function of (y_n) . On account of the linear recurrence relation, it is of the form

$$G(x) = \frac{g(x)}{f^*(x)}$$

with $g \in \mathbb{F}_q[x]$, $\deg(g) < k$, and $f^*(x) = x^k f(1/x)$ being the reciprocal polynomial of f (compare with [5, Ch. 8]). By partial fraction decomposition,

$$G(x) = \sum_{i=1}^r \sum_{j=0}^{d_i-1} \frac{a_{ij}}{1 - v_i^{q^j} x},$$

where $d_i = \deg(f_i)$, and the elements $a_{ij} \in F_i$ are conjugate over K , i.e., $a_{ij} = a_{i0}^{q^j}$ for $0 \leq j \leq d_i - 1$. Expanding into formal power series, we get

$$\begin{aligned} G(x) &= \sum_{i=1}^r \sum_{j=0}^{d_i-1} a_{ij} \sum_{n=0}^{\infty} v_i^{nq^j} x^n = \sum_{n=0}^{\infty} \left(\sum_{i=1}^r \sum_{j=0}^{d_i-1} (a_{i0} v_i^n)^{q^j} \right) x^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=1}^r \text{Tr}_{F_i/K}(a_{i0} v_i^n) \right) x^n. \end{aligned}$$

and comparison of coefficients with (14) yields the result of the lemma, with $u_i = a_{i0}$.

Since $f(0) = -b_0 \neq 0$, we have $v_i \neq 0$ for $1 \leq i \leq r$, and since f is the minimal polynomial of (y_n) , we have $u_i \neq 0$ for $1 \leq i \leq r$. Now let χ be a nontrivial additive character of $K = \mathbb{F}_q$ and consider the character sum

$$(15) \quad \sum_{n=0}^{\tau-1} \chi(y_n)$$

extended over the period of (y_n) . Then writing again $d_i = \deg(f_i)$ and using the lemma,

$$\begin{aligned} \sum_{n=0}^{\tau-1} \chi(y_n) &= \sum_{n=0}^{\tau-1} \chi(\text{Tr}_{F_i/K}(u_1 v_1^n)) \cdots \chi(\text{Tr}_{F_r/K}(u_r v_r^n)) \\ &= \sum_{n=0}^{\tau-1} \chi^{(d_1)}(u_1 v_1^n) \cdots \chi^{(d_r)}(u_r v_r^n) \\ &= \sum_{(a_1, \dots, a_r) \in uH_1} \chi^{(d_1)}(a_1) \cdots \chi^{(d_r)}(a_r), \end{aligned}$$

where $u = (u_1, \dots, u_r) \in F_1^* \times \dots \times F_r^*$ and H_1 is the cyclic subgroup of $F_1^* \times \dots \times F_r^*$ generated by (v_1, \dots, v_r) . Consequently, the character sum (15) is, apart from the factor m , a sum of the form E_1 in (4), with $\chi_i = \chi^{(d_i)}$ and trivial ϕ_i for $1 \leq i \leq r$.

The identity (12), together with the form of the ω_j given by (11), immediately yields the estimate

$$|E_s| \leq m(q_1 \cdots q_r)^{s/2}$$

for the sums E_s in (4), where q_i denotes the order of F_i . If all q_i are identical, then we can establish an estimate that is in a sense best possible.

Theorem 2. *Let $F_i = \mathbb{F}_q$ for $1 \leq i \leq r$. Then there exist integers C and H with $0 < C \leq m$, $0 \leq H \leq r$, such that*

$$|E_s| \leq Cq^{sH/2} + (m - C)q^{s(H-1)/2} \text{ for all } s \geq 1.$$

Furthermore, for every $\epsilon > 0$ there exist infinitely many s with

$$|E_s| \geq (C - \epsilon)q^{sH/2}.$$

Proof. By (12) we have

$$|E_s| = \left| \sum_{j=1}^m \omega_j^s \right|,$$

where the ω_j are given by (11). For $0 \leq h \leq r$ let m_h be the number of

$(\lambda_1, \dots, \lambda_r) \in A_1$ such that $\lambda_i = \phi_i$ holds for exactly h values of i . Then

$$(16) \quad \sum_{h=0}^r m_h = m.$$

We note the fact that for a multiplicative character ψ and a nontrivial additive character χ of \mathbb{F}_q we have

$$|G(\psi, \chi)| = \begin{cases} 1 & \text{for } \psi \text{ trivial,} \\ q^{1/2} & \text{otherwise.} \end{cases}$$

Therefore,

$$(17) \quad |E_s| \leq \sum_{h=0}^r m_h q^{s(r-h)/2}.$$

Let H be the largest value of h with $m_{r-h} \neq 0$. Putting $C = m_{r-H}$, we get

$$|E_s| \leq Cq^{sH/2} + (m-C)q^{s(H-1)/2} \text{ for all } s \geq 1,$$

where we used (16).

To prove the second part of Theorem 2, let $\varepsilon > 0$ be given and let J be the set of those j , $1 \leq j \leq m$, for which $|\omega_j| = q^{H/2}$. For $j \in J$ we have

$$\omega_j = q^{H/2} e^{2\pi i \theta_j} \text{ with } \theta_j \text{ real.}$$

We note that the set J has C elements. Therefore, by Dirichlet's theorem on simultaneous diophantine approximations, there exist infinitely many s for which

$$\left| \sum_{j \in J} e^{2\pi i s \theta_j} \right| \geq C - \frac{\varepsilon}{2}.$$

Consequently,

$$\begin{aligned} |E_s| &\geq \left| \sum_{j \in J} \omega_j^s \right| - \left| \sum_{j \notin J} \omega_j^s \right| = q^{sH/2} \left| \sum_{j \in J} e^{2\pi i s \theta_j} \right| - \left| \sum_{j \notin J} \omega_j^s \right| \\ &\geq \left(C - \frac{\varepsilon}{2} \right) q^{sH/2} - (m-C)q^{s(H-1)/2} \geq (C - \varepsilon) q^{sH/2} \end{aligned}$$

for infinitely many s .

An interesting special case for applications is that of the character sums in (15), with the minimal polynomial f of (y_n) being irreducible over \mathbb{F}_q . In this case $r = 1$, $F_1 = \mathbb{F}_{q^k}$, and H_1 is the subgroup of F_1^* generated

by a root ν of f , so that $m = (q^k - 1)/\tau$. By the earlier discussion,

$$\sum_{n=0}^{\tau-1} \chi(y_n) = \frac{1}{m} E_1$$

for a sum E_1 of the form (4) with ψ_1 trivial. The lifted sum E_s , $s \geq 2$, corresponds to a subgroup H_s of $F_{1,s}^*$ of the same index m . Now H_s is cyclic of order $\tau_s = (q^{ks} - 1)/m$, so we can choose a generator $\nu^{(s)}$ of H_s . Let $f^{(s)}$ be the minimal polynomial of $\nu^{(s)}$ over \mathbb{F}_{q^s} . It is clear that $d = \deg(f^{(s)})$ divides k . Suppose d were a proper divisor of k . Then it follows that

$$\tau_s = \frac{(q^{ks} - 1)\tau}{q^k - 1} = (q^{k(s-1)} + q^{k(s-2)} + \dots + 1)\tau > q^{ks/2} > q^{sd} - 1.$$

On the other hand, $\nu^{(s)}$ is a nonzero element of the finite field of order q^{sd} , hence

$$(\nu^{(s)})^{q^{sd} - 1} = 1,$$

which implies $\tau_s \leq q^{sd} - 1$, a contradiction. Thus we have $\deg(f^{(s)}) = k$. From the earlier discussion we see that there exists a linear recurring sequence $(y_n^{(s)})$ in \mathbb{F}_{q^s} with minimal polynomial $f^{(s)}$ and least period τ_s such that

$$\sum_{n=0}^{\tau_s-1} \chi^{(s)}(y_n^{(s)}) = \frac{1}{m} E_s.$$

For $s = 1$ we write $y_n^{(1)} = y_n$, $f^{(1)} = f$, and $\tau_1 = \tau$. From (17) and the second part of Theorem 2 we obtain then the following result.

Corollary. *For all $s \geq 1$ we have*

$$(18) \quad \left| \sum_{n=0}^{\tau_s-1} \chi^{(s)}(y_n^{(s)}) \right| \leq \left(1 - \frac{\tau_s}{q^{ks} - 1} \right) q^{ks/2} + \frac{\tau_s}{q^{ks} - 1}.$$

Furthermore, for every $\varepsilon > 0$ there exist infinitely many s with

$$\left| \sum_{n=0}^{\tau_s-1} \chi^{(s)}(y_n^{(s)}) \right| \geq \left(1 - \frac{\tau_s}{q^{ks} - 1} - \varepsilon \right) q^{ks/2}.$$

In case $\tau_s = q^{ks} - 1$ (i.e., $m = 1$), the second part of the corollary provides no information. But in this case it is easy to see directly that

$$\sum_{n=0}^{\tau_S-1} \chi^{(S)}(y_n^{(S)}) = -1,$$

and so (18) is again best possible.

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MATHEMATICAL INSTITUTE
AUSTRIAN ACADEMY OF SCIENCES
Dr. IGNAZ-SEIPEL-PLATZ 2
A-1010 VIENNA, AUSTRIA

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