Math. J. Okayama Univ. 27 (1985), 127—133

HADAMARD MATRICES OF BUSH TYPE

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In [1] Bush suggested a method for constructing a Hadamard matrix of order n using a Hadamard matrix of order $\frac{1}{2}n-2$ and a skew Hadamard matrix of $\frac{1}{4}n+1$, where $n \equiv 12 \pmod{16}$. A Hadamard matrix of order n constructed by the method of Bush will be called a Hadamard matrix of Bush type of order n.

The purpose of this note is to prove two propositions on Hadamard matrices of Bush type of order n.

For basic facts on Hadamard matrices see [2].

1. Introduction. We want to construct a Hadamard matrix of order $n = 16u+12$ under certain "inductive" assumptions, where u is a non-negative integer. Obviously it suffices to construct a symmetric $2-(16u+11,\ 8u+5,\ 4u+2)$ design $D = (P, B)$, where $P = \{1,2,\dots,16u+11\}$ and B denote the sets of points and blocks of D respectively.

We make the following "inductive" assumptions: (1) There exists a Hadamard matrix L of order $8u+4$, and (2) there exists a skew Hadamard matrix R of order $4u+4$. Put $L = (\lambda(i))$, $1 \leq i \leq 8u+4$, where $\lambda(i)$ denotes the i-th row vector of L and we may assume that $\lambda(1)$ is the all one vector. Let $L(\lambda(1)) = (P(\ell), B(\ell))$ be the Hadamard 3-design associated with L at $\lambda(1)$. We put $P(\ell) = \{1,2,\dots,8u+4\}$ so that the block $\sigma(i)$ of $L(\lambda(1))$ corresponding to $\lambda(i)$ contains the point j if and only if the j-th component of $\lambda(i)$ equals 1, where $2 \leq i \leq 8u+4$ and $\sigma(i)^* = P(\ell) - \sigma(i)$ is also a block of $L(\lambda(1))$, for $2 \leq i \leq 8u+4$. Clearly we have that $\sigma(i) \cap \sigma(i)^* = \emptyset$ and $|\sigma(i) \cap \sigma(j)| = |\sigma(i) \cap \sigma(j)^*| = 2u+1$ for $i \neq j$.

We pick up any $4u+3$ distinct disjoint block pairs from the $\{\sigma(i), \sigma(i)^*\}$, $2 \leq i \leq 8u+4$. For simplicity of notation we denote them by $\{\sigma(i), \sigma(i)^*\}$, $2 \leq i \leq 4u+4$. This configuration $\mathfrak{L}$ consists of $8u+6$ blocks of size $4u+2$.

Next we may assume that R is in a skew-normalized skew form :

$$R = \begin{bmatrix} -1 & 1 & \cdot & \cdot & \cdot & 1 \\ -1 & -1 & & & & \\ \cdot & & \cdot & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \\ -1 & & & & & -1 \end{bmatrix} = (\rho(i)),\ \text{where } \rho(i) \text{ denotes}$$

the i-th row vector of R, $1 \leq i \leq 4u+4$. we label the j-th column of R by $8u+2j+2$, for $2 \leq j \leq 4u+4$, and notice that the first column is still labelled 1.

Let $D(r) = (P(r), B(r))$ be a symmetric $2-(4u+3,\ 2u+2,\ u+1)$ design which is the complement of the symmetric $2-(4u+3,\ 2u+1,\ u)$ design associated with R at $\rho(1)$ punctured at 1. We put $P(r) = \{8u+6, 8u+8, \ldots, 16u+10\}$ so that the block $\tau(i)$ of $D(r)$ corresponding to $\rho(i)$ contains the point $8u+2j+2$ if and only if the j-th component of $\rho(i)$ equals -1 $(2 \leq i, j \leq 4u+4)$. Let us define a mapping T from $B(r)$ to $P(r)$ by $\tau(i)T = 8u+2i+2$, for $2 \leq i \leq 4u+4$. Then by the skew property of R we have that $\tau(i)T \in \tau(i)$ and that $\tau(i)T \in \tau(j)$ if and only if $\tau(j)T \notin \tau(i)$ for $i \neq j$.

Now we are going to double points and blocks of $D(r)$ as follows. The block $\tau(i)$ will be developed into two blocks $\tau(i1)$ and $\tau(i2)$, $2 \leq i \leq 4u+4$. If $8u+2j+2 \in \tau(i)$ and $i \neq j$, then both $\tau(i1)$ and $\tau(i2)$ contain both $8u+2j+2$ and $8u+2j+3$. If $i = j$, then $\tau(i1)$ contains only $8u+2i+2$ and $\tau(i2)$ contains only $8u+2i+3$. Then clearly we have that $|\tau(i1) \cap \tau(i2)| = 4u+2$, for $2 \leq i \leq 4u+4$. Moreover, since $|\tau(i) \cap \tau(j)| = u+1$ and since $\tau(i)T \in \tau(j)$ if and only if $\tau(j)T \notin \tau(i)$ for $i \neq j$, we have that $|\tau(ik) \cap \tau(j\ell)| = 2u+1$ for $i \neq j$ and $1 \leq k, \ell \leq 2$. In this way we get a configuration $\mathfrak{N}$ consisting of $8u+6$ blocks of size $4u+3 = 1+2(2u+1)$.

Finally we match $\mathfrak{L}$ and $\mathfrak{N}$ together in any possible way under the condition that $\{\sigma(i), \sigma(i)^*\}$ and $\{\tau(j1), \tau(j2)\}$ should be matched if a member of $\{\sigma(i), \sigma(i)^*\}$ is matched together with a member of $\{\tau(j1), \tau(j2)\}$. For simplicity of notation we assume that $\sigma(i)$ and $\tau(i1)$, and hence $\sigma(i)^*$ and $\tau(i2)$, are matched together, $2 \leq i \leq 4u+4$.

Put $\alpha(1) = P(\mathfrak{L}) \cup \{8u+5\}$, $\alpha(2i-2) = \sigma(i) \cup \tau(i1)$ and $\alpha(2i-1) = \sigma(i)^* \cup \tau(i2)$, for $2 \leq i \leq 4u+4$. Then it is easy to see that $|\alpha(i)| = 8u+5$, $1 \leq i \leq 8u+7$ and $|\alpha(i) \cap \alpha(j)| = 4u+2$ for $i \neq j$.

So the configuration $\mathfrak{P} = (\mathrm{P}, \{\alpha(i)\}, 1 \leq i \leq 8u+7)$ is possibly a portion of a symmetric $2-(16u+11,\ 8u+5,\ 4u+2)$ design.

Now we prove the following proposition.

Proposition 1. *A necessary and sufficient condition for $\mathfrak{P}$ to be completed to a symmetric $2-(16u+11,\ 8u+5,\ 4u+2)$ design can be stated as follows.*

There exist $8u+4$ subsets $\mu(j)$ of size $4u+2$, $1 \leq j \leq 8u+4$, of $\alpha(1)$, *called blocks again, such that* $D(\ell) = (\alpha(1), \{\sigma(i), \sigma(i)^*, 1 \leq i \leq 4u+3, \mu(j), 1 \leq j \leq 8u+4\})$ *forms a* $2-(8u+5, 4u+2, 4u+1, 16u+10, 8u+4)$ *design, where the five parameters correspond to the usual notation* v, k, λ, b *and* r *respectively, with the following three conditions* :

(1) *Put* $\bar{\sigma}(i) = \sigma(i)$ or $\sigma(i)^*$, $1 \leq i \leq 4u+3$. *Then with any fixed* $\bar{\sigma}(i)$ *one half of the* $\mu(k)$ *intersects in* $2u+1$ *points and the other half of the* $\mu(k)$ *intersects in* $2u$ *points.*

(2) *With each of any fixed* $\bar{\sigma}(i)$ *and* $\bar{\sigma}(j)$ *for* $i \neq j$ *one quarter of the* $\mu(k)$ *intersects in* $2u+1$ *points and another quarter of the* $\mu(k)$ *intersects in* $2u$ *points.*

(3) *Let* a *be a point such that* $1 \leq a \leq 8u+4$. *If* a *belongs to* $\bar{\sigma}(i)$, *then exactly* $2u$ *of the* $\mu(k)$ *which intersects with* $\bar{\sigma}(i)$ *in* $2u$ *points contain* a. *If* a *does not belong to* $\bar{\sigma}(i)$, *then exactly* $2u+1$ *of the* $\mu(k)$ *which intersects with* $\bar{\sigma}(i)$ *in* $2u$ *points contain* a.

Proof. Necessity. Suppose that $\mathfrak{P}$ is completed to a symmetric $2-(16u+11, 8u+5, 4u+2)$ design D. New blocks will be denoted by $\alpha(i)$, for $8u+8 \leq i \leq 16u+11$. Put $\mu(i-8u-7) = \alpha(1) \cap \alpha(i)$ for $8u+8 \leq i \leq 16u+11$. Then $D(\ell) = (\alpha(1), \{\sigma(i), \sigma(i)^*, 1 \leq i \leq 4u+1, \mu(j), 1 \leq j \leq 8u+4\})$ is a $2-(8u+5, 4u+2, 4u+1, 16u+10, 8u+4)$ design. In fact, let a and b be any two distinct points of $\alpha(1)$. Then a belongs to $8u+5$ blocks of D including $\alpha(1)$ and $\{a, b\}$ is contained in $4u+2$ blocks of D including $\alpha(1)$. Hence a belongs to $8u+4$ blocks of $D(\ell)$ and $\{a, b\}$ is contained in $4u+1$ blocks of $D(\ell)$. So we have only to check three conditions (1), (2) and (3) on $D(\ell)$.

If $\alpha(8u+7+k)$, $1 \leq k \leq 8u+4$, contains both $8u+2i+2$ and $8u+2i+3$, where $2 \leq i \leq 4u+4$, or if it contains neither $8u+2i+2$ nor $8u+2i+3$, then $\alpha(8u+7+k) \cap \tau(i1) = \alpha(8u+7+k) \cap \tau(i2)$. Put $|\alpha(8u+7+k) \cap \tau(i1)| = x$. Then $4u+2 = |\alpha(2i-2) \cap \alpha(8u+7+k)| = |\sigma(i) \cap \mu(k)|+x = |\alpha(2i-1) \cap \alpha(8u+7+k)| = |\sigma(i)^* \cap \mu(k)|+x$. Every $\mu(k)$ contains the point $8u+5$. So $|\sigma(i) \cap \mu(k)|+|\sigma(i)^* \cap \mu(k)| = 4u+1$. Hence we have a contradiction that $4u+3 = 2x$. Thus we have that $||\alpha(8u+7+k) \cap \tau(i1)|-|\alpha(8u+7+k) \cap \tau(i2)|| = 1$, and that $|\sigma(i) \cap \mu(k)| = 2u$ or $2u+1$. Let E and F be the numbers of the $\mu(k)$ such that $|\sigma(i) \cap \mu(k)| = 2u$ and $2u+1$ respectively. Since every point of $\sigma(i)$ belongs to $4u+1$ of the $\mu(k)$, we have that $(4u+2)(4u+1) = 2u\,E+(2u+1)F$. Then E is a multiple of $2u+1$ and this fact implies that $E = F = 4u+2$ proving (1).

We notice that $|\alpha(8u+7+k)\cap\sigma(i)|=2u+1$ if and only if $8u+2i+2\notin\alpha(8u+7+k)$. Let $2\leq i\neq j\leq 4u+4$. Then, since $D(r)$ is a symmetric $2-(4u+3,\ 2u+2,\ u+1)$ design, there exist $2(u+1)-1=2u+1$ of the $\tau(\ell 1)$ and $\tau(\ell 2)$ containing the points $8u+2i+2$ and $8u+2j+2$. So $4u+2-(2u+1)=2u+1$ of the $\alpha(8u+7+k)$ contain the points $8u+2i+2$ and $8u+2j+2$, proving (2).

Let $a\in\sigma(i)$, for $1\leq a\leq 8u+4$. Now there exist exactly $2(2u+2)-1=4u+3$ of the $\tau(jk)$ containing the point $8u+2i+2$. So there exist exactly $(2u+1)+1=2u+2$ of the $\alpha(\ell)$ with $\ell\leq 8u+7$ containing both a and $8u+2i+2$. Hence there exist exactly $4u+2-(2u+2)=2u$ of the $\alpha(\ell)$ with $\ell\geq 8u+8$ containing both a and $8u+2i+2$. These are the blocks $\alpha(\ell)$ with $\ell\geq 8u+8$ intersecting with $\sigma(i)$ in $2u$ points. The rest is similar. This proves (3).

Sufficiency. Suppose that we have a $2-(8u+5,\ 4u+2,\ 4u+1,\ 16u+10,\ 8u+4)$ design $D(\ell)$ satisfying (1), (2) and (3).

Clearly $\mu(k)$ contains the point $8u+5$, for $1\leq k\leq 8u+4$. Since $\sigma(i)\cup\sigma(i)^*=\alpha(1)-\{8u+5\}$, we have that $|\sigma(i)\cap\mu(k)|=2u+1$ or $2u$ according as $|\sigma(i)^*\cap\mu(k)|=2u$ or $2u+1$ respectively, for $2\leq i\leq 4u+4$ and $1\leq k\leq 8u+4$.

We form a configuration consisting of $8u+4$ blocks $\{\nu(1),\ldots,\nu(8u+4)\}$ of size $4u+3$ based on the set of points $\{8u+6,\ 8u+7,\ldots,16u+11\}$. $\nu(k)$ contains the point $8u+2+2i$ or $8u+3+2i$ according as $|\sigma(i)\cap\mu(k)|=2u$ or $|\sigma(i)^*\cap\mu(k)|=2u$ respectively, for $2\leq i\leq 4u+4$ and $1\leq k\leq 8u+4$. Since $\nu(k)$ contains exactly one point of $\{8u+2+2i,\ 8u+3+2i\}$ for each i, such that $2\leq i\leq 4u+4$, the size of $\nu(k)$ equals $4u+3$.

We put $\alpha(8u+7+j)=\mu(j)\cup\nu(j)$, for $1\leq j\leq 8u+4$, and let $B=\{\alpha(1),\ \alpha(2),\ldots,\ \alpha(16u+11)\}$. Then we show that $D=(P,B)$ is a symmetric $2-(16u+11,\ 8u+5,\ 4u+2)$ design.

First we show that D is a 1-design. Let a be a point. If $1\leq a\leq 8u+5$, then, since $D(\ell)$ has replication number $8u+4$ and since a belongs to $\alpha(1)$, a belongs to $(8u+4)+1=8u+5$ blocks of B. So let $8u+6\leq a\leq 16u+11$. Now every point of $D(r)$ belongs to $2u+2$ blocks. One of these blocks say $\tau(i)$, contains a or $a-1$ as $\tau(i)T$. So there exists $2(2u+1)+1=4u+3$ blocks $\alpha(i)$ with $i\leq 8u+7$ containing a. Now by assumption (1) on $D(\ell)$ there exist exactly $4u+2$ of the $\mu(k)$ such that $|\sigma(i)\cap\mu(k)|=2u$ or $|\sigma(i)^*\cap\mu(k)|=2u$, according as a is even or odd respectively. So there exist $4u+2$ blocks $\alpha(i)$ with $i\leq 8u+8$ containing a.

Next we show that D is a 2-design. Let a and b be two distinct points.

If $1 \leq a, b \leq 8u+5$, then since $D(\ell)$ is a $2-(8u+5, 4u+2, 4u+1, 16u+10, 8u+4)$ design and since both a and b belong to $\alpha(1)$, a and b belong to $(4u+1)+1 = 4u+2$ blocks of B. Let $8u+6 \leq a, b \leq 16u+11$. If $\{a, b\} = \{8u+6, 8u+7\}, \{8u+8, 8u+9\}, \ldots$, or $\{16u+10, 16u+11\}$, then we may assume that a is even. Only blocks $\alpha(i)$ with $2 \leq i \leq 8u+7$ may contain $\{a, b\}$. Since the replication number of $D(r)$ is $2u+2$, and since a appears in exactly one of the $\tau(i)$ as $\tau(i)T$, $\{a, b\}$ is contained in $2(2u+2-1) = 4u+2$ blocks of B. If $\{a, b\} \neq \{8u+6+2i, 8u+7+2i\}$, $0 \leq i \leq 4u+2$, then it suffices to consider the case where a and b are even. Then $\{a, b\}$ is contained in exactly $u+1$ blocks of $D(r)$. By the skew property of T exactly one of these blocks of $D(r)$, say $\tau(j)$, contains a or b as $\tau(j)T$. So exactly $1+2(u+1-1) = 2u+1$ blocks $\alpha(i)$ with $i \leq 8u+7$ contain $\{a, b\}$. By assumption (2) on $D(\ell)$ and by the definition of $\nu(k)$, exactly $2u+1$ of the $\nu(k)$ contain $\{a, b\}$. So exactly $2u+1$ blocks $\alpha(i)$ with $i \geq 8u+8$ contain $\{a, b\}$. Finally let $1 \leq a \leq 8u+5$ and $8u+5$ and $8u+6 \leq b \leq 16u+11$. If $a = 8u+5$, then a belongs to all of the $\mu(k)$, $1 \leq k \leq 8u+4$. By assumption (3) on $D(\ell)$, b belongs to exactly $4u+2$ of the $\mu(k)$. So $\{a, b\}$ is contained in exactly $4u+2$ blocks of B. Thus we may assume that $1 \leq a \leq 8u+4$. Again we may assume that b is even. Now b belongs to exactly $2u+2$ blocks of $D(r)$ and only one of these blocks, say $\tau(k)$, contain b as $\tau(k)T$. Therefore $2u+1$ pairs of blocks $\tau(ij)$ contain b, and $\tau(k1)$, not $\tau(k2)$, contains b. So if a belongs to $\sigma(k)$, then exactly $2u+1$ blocks $\alpha(i)$ with $i \leq 8u+7$ contain $\{a, b\}$. But if a belongs to $\sigma(k)^*$, then exactly $2u$ blocks $\alpha(i)$ with $i \leq 8u+7$ contain $\{a, b\}$. Then by assumption (3) on $D(\ell)$ exactly $2u$ or $2u+1$ blocks $\alpha(i)$ with $i \geq 8u+8$ contain $\{a, b\}$ according as a belongs to $\sigma(1)$ or $\sigma(1)^*$. This completes the proof.

Definition. We call a symmetric $2-(16u+11, 8u+5, 4u+2)$ design D thus constructed a Hadamard desigh of Bush type. Furthermore we call a Hadamard matrix of order $16u+12$ associated with D a Hadamard matrix of Bush type.

Remark 1. The main point of proposition 1 is the fact that the construction of a Hadamard matrix of Bush type of order $16u+12$ is reduced to the construction of a $2-(8u+5, 4u+2, 4u+1, 16u+10, 8u+4)$ design satisfying (1), (2) and (3) for which $8u+6$ blocks are predetermined.

Remark 2. There exists some freedom to construct Hadamard matrices of Bush type of order $16u+12$: (i) The choice of a Hadamard matrix H of order $8u+4$; (ii) The choice of $4u+4$ rows from H; (iii) The choice of a skew Hadamard matrix of order $4u+4$; (iv) The choice of the mapping T; (v) The choice of $2-(8u+5,\ 4u+2,\ 4u+1,\ 16u+10,\ 8u+4)$ design and (vi) The choice of the matching between $\mathfrak{L}$ and $\mathfrak{N}$.

Remark 3. For $u=0$ it is very easy to write down a design of Bush type : $\alpha(1)=\{1,2,3,4,5\}$, $\alpha(2)=\{1,2,6,10,11\}$, $\alpha(3)=\{3,4,7,10,11\}$, $\alpha(4)=\{1,3,6,7,8\}$, $\alpha(5)=\{2,4,6,7,9\}$, $\alpha(6)=\{1,4,8,9,10\}$, $\alpha(7)=\{2,3,8,9,11\}$, $\alpha(8)=\{1,5,7,9,11\}$, $\alpha(9)=\{2,5,7,8,10\}$, $\alpha(10)=\{3,5,6,9,10\}$, and $\alpha(11)=\{4,5,6,8,11\}$. For $u=1$ there are more than ten inequivalent Hadamard matrices of Bush type.

2. The purpose of this section is to prove the following proposition.

Proposition 2. *The transpose of a Hadamard matrix of Bush type is of Bush type. More precisely, the dual of a Hadamard design of Bush type is of Bush type.*

Proof. We use the notation in the proof of Proposition 1, and consider the dual D^d of the Hadamard design of Bush type in § 1, $D=(P,B)$. It will suffice to recognize in D^d a configuration similar to $\mathfrak{P}=(P,\ \{\alpha(i)\},\ 1\leq i\leq 8u+7)$.

Let $\beta(i)$ be the set of blocks of B containing the point i of P, $1\leq i\leq 16u+11$. Let P^d and B^d denote the sets of points and blocks of D^d respectively. Then $P^d=\{\alpha(i),\ 1\leq i\leq 16u+11\}$ and $B^d=\{\beta(i),\ 1\leq i\leq 16u+11\}$.

Now the point $\alpha(1)$, the set of points $\alpha(i)$ with $8u+8\leq i\leq 16u+11$ and the block $\beta(8u+5)=\{\alpha(1),\ \alpha(i)$ with $8u+8\leq i\leq 16u+11\}$ play the roles of the point $8u+5$, $P(\mathfrak{L})$ and the block $\alpha(1)$ in D, respectively.

Furthermore, $\beta(8u+5)\cap\beta(8u+2i)$ and $\beta(8u+5)\cap\beta(8u+2i+1)$, where $3\leq i\leq 4u+5$, correspond to $\sigma(i)=\alpha(1)\cap\alpha(2i-2)$ and $\sigma(i)^*=\alpha(1)\cap\alpha(2i-1)$, where $2\leq i\leq 4u+4$, respectively. Lastly $(P(r))^d=\{\alpha(2j),\ 1\leq j\leq 4u+3\}$, $(P(r))^d\cap\beta(8u+2i)$ and T^d defined by $((P(r))^d\cap\beta(8u+2i))T^d=\alpha(2i-4)$, where $3\leq i\leq 4u+5$, correspond to $P(r)$, $\tau(i)$ and T respectively, where $2\leq i\leq 4u+4$.

The rest may be checked without difficulty.

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(*Received May 25, 1985*)