Math. J. Okayama Univ. 27 (1985), 121-125

NOTE ON SKEW POLYNOMIALS II

Dedicated to Professor Akira Hattori on his 60th birthday

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Throughout this paper, B will mean a ring with identity element 1 which has an automorphism ρ. By $B[X;\rho]$, we denote a skew polynomial ring $\sum_{i=0}^{\infty} X^iB$ whose multiplication is given $bX = X\rho(b)$ $(b \in B)$. A monic polynomial $f \in B[X;\rho]$ is called to be *separable* (resp. *Frobenius*) if $fB[X;\rho] = B[X;\rho]f$ and the factor ring $B[X;\rho]/fB[X;\rho]$ is separable (resp. Frobenius) over B.

This paper concerns with Miyashita's problem: Is any skew separable polynomial Frobenius ? (cf. [5, §3]). In [3], Ikehata proved that if the center of B is Artinian then any separable polynomial in $B[X;\rho]$ is Frobenius. Moreover, in [7], the present author proved that if the center of B is π-regular then any separable polynomial in $B[X;\rho]$ is Frobenius. In this note, we shall present some generalizations of the above results (Theorems 2 and 4).

In what follows, Z will mean the center of B. An element a of B is said to be *π-regular* if there exists an element d in B and an integer $t>0$ such that $a^tda^t = a^t$. If every element of B is π-regular then B will be called to be *π-regular*. Now, let c be an element of Z which is π-regular in B. Then $c^t = c^{2t}d$ for some $d \in B$ and an integer $t>0$. Clearly $c^td = (c^td)^2$, $c^tdB = c^tB = Bc^t = Bc^td$, and $c^t = c^{2t}c^td^2$. For any $b \in B$, we have $bc^td = c^tdbc^td = c^tdb$ and $bc^td^2 = c^tdbd = dbc^td = c^td^2b$. Hence c^td is a central idempotent of B, $c^tdZ = c^tZ = c^{2t}Z$, and c is π-regular in Z (cf. [1, Lemma 1]).

First, we shall prove the following

Lemma 1. *Let N be a nilpotent ideal of B, and let the center of the factor ring B/N be π-regular. Then Z is π-regular, and whence, for any element c in Z, there is a central idempotent $e \in B$ and an integer $t>0$ such that $eB = c^tB = c^{t+1}B$.*

Proof. Let c be an element of Z. Then $c+N$ is in the center of B/N. Hence, by our assumption, there is an integer m >0 such that $c^mB+N = c^{m+1}B+N$. This implies that $c^m+c^{m+1}b = d$ for some $b \in B$ and $d \in N$.

Since N is nilpotent, we have $(c^m+c^{m+1}b)^s=0$ for some integer $s>0$. Expanding this, we obtain $c^{ms}+c^{ms+1}b^*=0$ for some $b^*\in B$. Therefore, it follows that $c^{ms}B=c^{ms+1}B$, that is, c is π-regular in B. Thus Z is π-regular.

By Lemma 1 and [7, Th. 8], we obtain the following

Theorem 2. *Let N be a nilpotent ideal of B, and the center of B/N π-regular. Then, any separable polynomial in $B[X\,;\rho]$ is Frobenius.*

The following corollary contains the results of [3, Cor. 3], [5, Cor. to Th. 3.5] and [6, Th. 5].

Corollary 3. *Let N be a nilpotent ideal of B, and B/N satisfy the descending chain condition on two-sided ideals. Then, any separable polynomial in $B[X\,;\rho]$ is Frobenius.*

Proof. Let c be an element of the center of B/N. Then, there is an integer $t>0$ such that $c^t(B/N)=c^{t+1}(B/N)$. Hence c is π-regular in B/N, and so is in the center of B/N. This implies that the center of B/N is π-regular. Hence, our assertion follows immediately from the result of Th. 2.

Next, we shall consider separable polynomials over complete rings. First, for convenience we recall here the notion of complete rings (see e. g., [2, p. 28]). Let I be an ideal of B such that $\bigcap_{i=0}^{\infty} I^i=\{0\}$ where $I^0=B$. Then, for any real number c with $0<c<1$ define the norm $\|\ \|_I^c$ as follows

$$\|0\|_I^c=0, \text{ and } \|b\|_I^c=c^i \text{ if } b\in I^i,\ b\notin I^{i+1}.$$

Define d_I^c by

$$d_I^c(b_1,b_2)=\|b_1-b_2\|_I^c.$$

Then d_I^c is a metric on B. For $0<c,\ c'<1$, a sequence in B is a Cauchy sequence with respect to d_I^c if and only if it is a Cauchy sequence with respect to $d_I^{c'}$. From now on, a fixed c with $0<c<1$ is chosen and we write $d_I=d_I^c$. If every Cauchy sequence converges with respect to the metric d_I then B will be called to be *complete with respect to* d_I.

If B is complete with respect to d_I then, for Cauchy sequences $\{u_i\}$ and $\{v_i\}$, there holds that

(1) $\lim u_i=u$ if and only if for every integer $m>0$, there exists an integer $s>0$ such that $u_i-u\in I^m$ for $i>s$,

(2) $\lim(u_i+v_i) = \lim u_i + \lim v_i$, and $\lim u_i v_i = (\lim u_i)(\lim v_i)$.

Moreover, B will be called to be *complete* if for the Jacobson radical $J(B)$ of B, $d_{J(B)}$ is defined on B (i.e., $\bigcap_{i=0}^{\infty} J(B)^i = \{0\}$) and B is complete with respect to $d_{J(B)}$. Clearly any right Artinian ring is complete. If B is commutative and Noetherian then $d_{J(B)}$ is defined.

Now, we shall prove the following theorem which is a generalization of [3, Th. 3], [7, Th. 8] and Th. 2.

Theorem 4. *Let I be an ideal of B with $\rho(I) = I$, and the center of B/I π-regular. If d_I is defined on B and B is complete with respect to d_I then any separable polynomial in $B[X ; \rho]$ is Frobenius.*

Proof. Let $f = X^n - \sum_{i=0}^{n-1} X^i a_i$ be a separable polynomial in $B[X ; \rho]$ ($n \geq 2$). Then, by making use of the same methods as in the proof of [7, Th. 8], there are elements b_0 and b_1 in B such that $1 = a_0 b_0 + a_1 b_1$, $a_i b_i \in Z$ and $\rho(a_i b_i) = a_i b_i$ ($i = 0, 1$). Now, for an integer $m > 0$, we set $B_m = B/I^m$. Since $\rho(I^m) = I^m$, ρ induces an automorphism in B_m, which will be denoted by $\bar{\rho}$. Then by Lemma 1, there exists a central idempotent $\bar{e}_m = e_m + I^m$ in B_m such that

$$e_m B_m = (a_0 b_0)^s B_m = (a_0 b_0)^{s+1} B_m$$

for some integer $s > 0$. Noting $\rho(a_0 b_0) = a_0 b_0$, we see that $\bar{\rho}(\bar{e}_m) = \bar{e}_m$ and $\bar{\rho}(\bar{e}'_m) = \bar{e}'_m$ for $e'_m = 1 - e_m$. Moreover, since $a_0 b_0 = \rho^{-n}(b_0) a_0$ ([7, Lemma 1]), $e_m \bar{a}_0$ is inversible in $e_m B_m$. Further, expanding $(a_0 b_0 + a_1 b_1)^s$, we have $1 = (a_0 b_0)^s + a_1 b_1^*$ for some $b_1^* \in B$. Then

$$\bar{e}'_m = \bar{e}'_m((a_0 b_0)^s + a_1 b_1^*) = \bar{e}'_m a_1 b_1^*.$$

Since $a_1 b_1^* \in Z$ and $a_1 b_1^* = \rho^{-n+1}(b_1^*) a_1$ ([7, Lemma 1]), $e'_m \bar{a}_1$ is also inversible in $e'_m B_m$. We set $\bar{\alpha}_m = (e_m \bar{a}_0)^{-1} \in e_m B_m$ and $\bar{\beta}_m = (e'_m \bar{a}_1)^{-1} \in e'_m B_m$ where $\alpha_m, \beta_m \in B$. Now, let d_I be defined on B, and B complete with respect to d_I. First, we shall show that $\{e_m ; m = 1, 2, \cdots\}$ is a Cauchy sequence with respect to d_I. Let $k > m$ be arbitrary integers > 0. Then

$$e_k B_k = (a_0 b_0)^t B_k = (a_0 b_0)^{t+1} B_k$$

for some integer t > 0. Clearly

$$e_k B_m = (a_0 b_0)^t B_m = (a_0 b_0)^{t+1} B_m.$$

This gives $e_m B_m = e_k B_m$. Since $e_k + I^m$ is a central idempotent of B_m, it follows that $e_m + I^m = e_k + I^m$, and so, $e_m - e_k \in I^m$. Thus $\{e_m\}$ is a Cauchy sequence. We set $e = \lim e_m$. Since $e_m - e_m^2 \in I^m$, it follows from (1) and (2) that

$$e - e^2 = \lim(e_m - e_m^2) = 0$$

that is, e is an idempotent of B. Moreover, for any $b \in B$, we have $be_m - e_m b \in I^m$. Hence

$$be - eb = \lim(be_m - e_m b) = 0$$

and so, e is central in B. Clearly $\rho(e) - \rho(e_m) = \rho(e - e_m) \in \rho(I^m) = I^m$. Noting $\rho(e_m) - e_m \in I^m$, we obtain

$$\rho(e) - e_m = (\rho(e) - \rho(e_m)) + (\rho(e_m) - e_m) \in I^m$$

whence $\rho(e) = \lim e_m = e$. Moreover, setting $e' = 1 - e$, we have $e' = \lim(1 - e_m) = \lim e'_m$ and $\rho(e') = e'$. Now, for integer $k > m > 0$, it is easily seen that

$$e_m \alpha_m - e_k \alpha_k, \ e'_m \beta_m - e'_k \beta_k \in I^m.$$

Hence $\{e_m \alpha_m\}$ and $\{e'_m \beta_m\}$ are Cauchy sepuences. Then, it follows that

$$e - ea_0(\lim e_m \alpha_m) = \lim(e_m - e_m a_0 e_m \alpha_m) = 0,$$
$$e' - e'a_1(\lim e'_m \beta_m) = \lim(e'_m - e'_m a_1 e'_m \beta_m) = 0.$$

This shows that ea_0 and $e'a_1$ are inversible in eB and $e'B$ respectively. Since f is separable over B, ef and $e'f$ are separable in $R_0 = eB[X ; \rho | eB]$ and $R_1 = e'B[X ; \rho | e'B]$ respectively. Hence, by [3, Th. 1], ef and $e'f$ are Frobenius. Noting

$$B[X ; \rho]/fB[X ; \rho] \simeq R_0/efR_0 \oplus R_1/e'fR_1$$

it follows that f is Frobenius over B. This completes the proof.

As a direct consequence of Th. 4, we obtain the following corollary which contains the result of Cor. 3.

Corollary 5. *Let $B/J(B)$ satisfy the descending chain condition on two-sided ideals. If B is complete then any separable polynomial in $B[X ; \rho]$ is Frobenius.*

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(Received August 1, 1985)