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AN ORE EXTENSION OVER A V-HC ORDER

Dedicated to Professor Hisao Tominaga on his 60th birthday

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The polynomial ring over a hereditary noetherian prime ring (an *HNP* ring for short) is not an *HNP* ring. This leads us to the concept of *VHC* order in a simple artinian ring in [8] (in [4], Fujita defined a *v-HC* order which is a little more general concept of *VHC* order) and the following problems naturally raise; If R is a *v-HC* order in a simple artinian ring Q, then so are $R[x;\sigma]$ and $R[x;d]$, where σ is an automorphism of R and d is a derivation of R. In [9], one of the authors proved that $R[x;\sigma]$ is a *v-HC* order with enough *v*-invertible ideals if R is a *v-HC* order with enough *v*-invertible ideals.

In this paper, we define a *d-v-HC* order with enough *d-v*-invertible ideals, and we prove the following theorem; R is a *d-v-HC* order with enough *d-v*-invertible ideals if and only if $R[\mathrm{x};d]$ is a *v-HC* order with enough *v*-invertible ideals (cf. § 1 for the definitions).

The theorem will be proved in § 2 by pointing out all maximal *v*-invertible ideals of $R[x;d]$ which are derived from R and $Q[x;d]$.

In § 1, we define the concepts of *d-v*-ideals and *d-v*-invertible ideals, and give some elementary properties of them, some of which are used to prove the theorem.

In § 3, we briefly discuss on the set of all *d-v*-invertible ideals of $R[x;d]$ and the class group of $R[x;d]$ which extend Chamarie's results in the case of a Krull order. Any maximal *d-v*-ideal of a *d-v-HC* order is not necessarily a prime ideal (even a semi prime ideal) as it is seen in Eaxmple 1 of § 3, and we give some examples of *d-v-HC* orders with enough *d-v*-invertible ideals.

Concerning terminologies which are not defined in this paper, we refer to [8].

1. Throughout this paper, R will be an order with a derivation d in a simple artinian ring Q. First of all, we recall some notations and definitions in [8] and [4]. Let X and Y be subsets of Q. Then we use the following notations; $(X:Y)_l=\{s\in Q\mid sY\subseteq X\}$ and $(X:Y)_r=\{t\in Q\mid Yt\subseteq X\}$. Let I be a right R-ideal. We define $\mathrm{I}_v=(R:(R:I)_l)_r$, and if $I=I_v$, then it

is called a right v-R-ideal (a right v-ideal if there are no confusions). Similarly we define ${}_vJ=(R:(R:J)_r)_l$ for any left R-ideal J and J is called a left v-ideal if ${}_vJ=J$. An R-ideal A is called a v-ideal if ${}_vA=A=A_v$. An integral and v-ideal is simply called a v-ideal of R. A v-ideal is called a v-invertible if $(A(R:A)_r)_v=R={}_v((R:A)_l)$. Note that a v-ideal A is v-invertible if and only if $O_r(A)=R=O_l(A)$ by Lemma 1.1 of [4], where $O_r(A)=\{q\in Q\mid Aq\subseteq A\}$, a right order of A, and $O_l(A)=\{a\in Q\mid qA\subseteq A\}$, a left order of A. If A is v-invertible, then by Lemma 1.1 of [4], $(R:A)_r=(R:A)_l$ which is denoted by A^{-1}. A v-ideal A of R is called v-idempotent if ${}_v(A^2)=A=(A^2)_v$. Let $\mathscr{C}=\mathscr{C}(R)$ be a right Gabriel topology corresponding to the torsion theory cogenerated by $E(Q/R)$ ($E(M)$ denotes the injective hull of a right R-module M). Then $\mathscr{C}=\{C:$ right ideal of $R\mid(R:r^{-1}C)=R$ for any $r\in R\}$, where $r^{-1}C=\{x\in R\mid rx\in C\}$, by Porposition 5.5 of [13, p. 147]. Let I be a right R-ideal and put $cl(I)=\{q\in Q\mid qC\subseteq I$ for some $C\in\mathscr{C}\}$. If $I=cl(I)$, then I is called a right closed R-ideal. Similarly we can define the left Gabriel topology $\mathscr{C}'$ on R and a left $\mathscr{C}'$-closed ideal. Fujita considered the following conditions :

(K) : ${}_v(A(R:A)_l)=O_l(A)$ for any ideal A of R such that $A={}_vA$ and $((R:B)_rB)_v=O_r(B)$ for any ideal B of R such that $B=B_v$.

(C) : R satisfies the maximum condition on right $\mathscr{C}$-closed ideals of R as well as left $\mathscr{C}'$-closed ideals of R.

If an order R satisfies (K) and (C), then it is called a v-*HC* order. R is said to have enough v-invertible ideals if any v-ideal of R contains a v-invertible ideal of R. We note that I is closed if $I=I_v$. Hence if R satisfies (K), then R satisfies the maximum condition on one-sided v-ideals of R. An R-ideal I is called a d-stable ideal (a d-ideal) if $d(I)\subseteq I$. A v-ideal which is d-stable is called a d-v-invertible ideal. We consider the following condition:

(K') : ${}_v(A(R:A)_l)=O_l(A)$ for any d-ideal of R such that $A={}_vA$ and $((R:B)_rB)_v=O_r(B)$ for any d-ideal of R such that $B=B_v$.

If an order R satisfies (K') and (C), then it is called a d-v-*HC* order, and R is said to have enough d-v-invertible ideals if any d-v-ideal of R contains a d-v-invertible ideal of R.

We denote by $R[x;d]$ the Ore extension of R in an indeterminate x. Any element in $R[x;d]$ has the following form : $\sum_{i=0}^{n} r_ix^i$ ($r_i\in R$) with multiplication defined by $xr=rx+d(r)$ for every $r\in R$. A subset S of R is said to be a regular Ore set if any element of S is regular in R and R satisfies the left and right Ore conditions with respect to S. The quotient ring of R with respect to S is denoted by R_S, and d is extended to a derivation of R_S

in the following way : $d(ac^{-1}) = d(a)c^{-1} - ac^{-1}d(c)c^{-1}$, where $a \in R$ and $c \in S$. As it is easily seen, the following mapping : $q(x) = \sum_{i=0}^{n} q_i x^i \rightarrow \sum_{i=0}^{n} d(q_i) x^i$ is a derivation of $Q[x ; d]$ which extends the derivation d of Q. We denote it by d again. Then we note that any $R[x ; d]$-ideal I in $Q[x ; d]$ is d-stable, because $d(q(x)) = xq(x) - q(x)x \in I$ for any $q(x) \in I$. Let $\mathscr{C}$ be the set of all regular elements of R. Then $\mathscr{C}$ is a regular Ore set of $R[x ; d]$ and $R[x ; d]_{\mathscr{C}} = Q[x ; d]$. Futhermore, $Q[x ; d]$ is a prime, left and right principal ideal ring [1]. So it has a classical quotient ring $Q(Q[x ; d])$ which is simple artinian. Hence $R[x ; d]$ is an order in $Q(Q[x ; d])$.

In studying the structure of $R[x ; d]$, d-v-ideals and d-v-invertible ideals play an important rôle. So first we shall give some elementary properties of d-v-ideals and d-v-invertible ideals, and the proofs of them are all straightfoward ;

Proposition 1.1. (1) *Let I be a right (left) d-R-ideal. Then so is $(R : I)_l$ $((R : I)_r)$ and thus I_v $({}_vI)$ is also d-stable.*

(2) *Let A be a d-R-ideal. Then $AR[x ; d]$ is an $R[x ; d]$-ideal denoted by $A[x ; d]$, and $O_l(A)$ and $O_r(A)$ are also d-stable.*

(3) *Let A and B be d-R-ideals. Then so is AB and $ABR[x ; d] = A[x ; d]B[x ; d]$.*

(4) *Let I be a right R-ideal and J a left R-ideal. Then $(R[x ; d] : IR[x ; d])_l = R[x ; d](R : I)_l$ and $(R[x ; d] : R[x ; d]J)_r = (R : J)_r R[x ; d]$. In particular, $(IR[x ; d])_v = I_v R[x ; d]$ and ${}_v(R[x ; d]J) = R[x ; d]{}_vJ$.*

(5) *Let A be a d-v-ideal. Then so is $A[x ; d]$.*

(6) *Let A be a d-v-invertible ideal. Then so is $A[x ; d]$ and $(A[x ; d])^{-1} = A^{-1}[x ; d]$.*

(7) *Let A be a d-stable and v-idempotent ideal of R (a d-v-idempotent ideal). Then so is $A[x ; d]$.*

Lemma 1.2. *Let R be a d-v-HC order.*

(1) *If A is a d-ideal of R, then $A_v = {}_vA$.*

(2) *If A is a d-R-ideal and there is a d-v-invertible ideal B such that $B^{-1} \supseteq A$, then $A_v = {}_vA$.*

Proof. As in Lemma 1.5 and Corollary 1.6 of [4].

A d-v-ideal M of R is called maximal if it is maximal amongst all d-v-ideals of R. As it is easily seen from an example at the end of this paper, we can't expect that any maximal d-v-ideal is a prime ideal (even a semi-prime ideal). But we have the following nice property ;

Lemma 1.3. *Let R be an order with a derivation d and let M be a d-prime ideal of R, i. e., $AB \subseteqq M$ implies either $A \subseteqq M$ or $B \subseteqq M$, where A and B are d-ideals of R, then $M[x ; d]$ is a prime ideal.*

Proof. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of $R[x ; d]$ such that $\mathfrak{a}\mathfrak{b} \subseteqq M[x ; d]$, and assume that $\mathfrak{b} \supsetneqq M[x ; d]$. Put $C(\mathfrak{b}) = \{b_n \mid b_n x^n + \cdots + b_0 \in \mathfrak{b}\} \cup \{0\}$. Then $C(\mathfrak{b})$ is a d-ideal of R and $C(\mathfrak{b}) \supsetneqq M$. Let $a(x) = a_m x^m + \cdots + a_0 \in \mathfrak{a}$. For any $b_n \in C(\mathfrak{b})$, there is $b(x) \in \mathfrak{b}$ such that $b(x) = b_n x^n + \cdots + b_0 \in \mathfrak{b}$. Since $a(x) b(x) \in M[x ; d]$, it follows that $a_m b_n \in M$. So $a_m C(\mathfrak{b}) \subseteqq M$ and $a_m \in M$. Repeating this process, we obtain $a(x) \in M[x ; d]$, and so $\mathfrak{a} \subseteqq M[x ; d]$, proving the lemma.

As a corollary to Lemma 1. 3, we have

Corollary 1.4. *Let R be a d-v-HC order and let M be a maximal d-v-ideal of R, then M is a d-prime ideal and so $M[x ; d]$ is a prime ideal of $R[x ; d]$.*

Proof. Let A and B be d-ideals of R and $AB \subseteqq M$. Assume that $B \supsetneqq M$, then $B_v \supsetneqq M$ and B_v is d-stable. So $B_v = R$. Hence $A = Ar = AB_v \subseteqq (AB_v)_v = (AB)_v = M$ by lemma 1. 1 of [7]. It follows that M is d-prime.

Lemma 1.5. *Let R be a d-v-HC order. Then a maximal d-v-ideal of R is either v-idempotent or v-invertible.*

Proof. As in Lemma 1. 5 of [7].

A finite set of distinct maximal d-ideals $M_1, \dots, M_n$ of R which are v-idempotent is called a d-v-cycle if $O_r(M_1) = O_l(M_2), \dots, O_r(M_n) = O_l(M_1)$. A maximal d-v-ideal M which is v-invertible is also considered as a d-v-cycle, because $O_r(M) = O_l(M)$.

The following proposition will be proved by combining the methods in [7] with Corollary 1. 4, and these will be used in § 2 to study the structure of $R[x ; d]$.

Proposition 1.6. *Let R be a d-v-HC order. Then*

(1) *Let M_1 be a maximal d-v-ideal such that M_1 is v-idempotent and it*

contains a d-v-invertible ideal X of R. Then there exist maximal d-v-ideals $M_2, \ldots, M_n$ such that $M_i \supseteqq X$, M_i is v-idempotent and $M_1, \ldots, M_n$ is a d-v-cycle.

(2) *Let P be an ideal of R. Then P is a maximal d-v-invertible ideal of R (maximal amongst all d-v-invertible ideals of R) if and only if it is an intersection of a d-v-cycle.*

(3) *If R has enough d-v-invertible ideals, then the set $D_d(R)$ of all d-v-invertible ideals is a free abelian group generated by maximal d-v-invertible ideals.*

2. In this section, we shall prove the main theorem mentioned in the introduction. Let us start off the following lemma whose proof is similar to one of Theorem 3.1.8 of [3].

Lemma 2.1. *Let R be an order in Q. Then $R[x; d]$ satisfies the condition (C) if R satisfies the condition (C).*

Let R be an order in Q satisfying the maximum condition on one-sided v-ideals of R and let $\mathcal{S}$ be a set consisting of v-invertible ideals of R which is closed by "v-multiplication", i.e., if X and $Y \in \mathcal{S}$, then $(XY)_v \in \mathcal{S}$. Then $T = \cup X^{-1}$ $(X \in \mathcal{S})$ is an overring of R and we have the following :

Lemma 2.2. (1) *If $X \in \mathcal{S}$, then $X \in \mathcal{G}$, where $\mathcal{G} = \{I$: right ideal of $R \mid Hom_R(R/I, E(Q/T)) = 0\}$.*

(2) *$T = R_{\mathcal{G}}$, where $R_{\mathcal{G}}$ denotes the ring of right quotients of R with respect to $\mathcal{G}$.*

(3) *For any $I \in \mathcal{G}$, there exists $X \in \mathcal{S}$ such that $I_v \supseteqq X$.*

Proof. (1) Let X be a d-v-invertible ideal of R. If $\mathrm{Hom}(R/X, E(Q/T)) \neq 0$, then there is a non-zero f in it, in particular, $f(\bar{1}) \neq 0$, where $\bar{1} = [1+X]$ in R/X. Since Q/T is essential in $E(Q/T)$, there is a non zero element r in R such that $f(\bar{1}) \in Q/T$ and $f(\bar{1})r \neq 0$. On the other hand, $\bar{0} = f(\bar{X}) \supseteqq f(\bar{1})rX = [q+T]X$, where q is in Q but not in T. Hence $qX \subseteqq T = \cup Y^{-1}$ and $qX \subseteqq Y^{-1}$ for some $Y \in \mathcal{S}$, because X is finitely generated as a right v-ideal. It follows that $q \in (Y^{-1}X^{-1})_v = (XY)^{-1} \subseteqq T$. This is a contradiction.

(2) From (1), $R_{\mathcal{G}} \supseteqq T$. So $R_{\mathcal{G}}/T \subseteqq Q/T \subseteqq E(Q/T)$. This implies that $R_{\mathcal{G}}/T$ is $\mathcal{G}$-torsion free. On the other hand, $R_{\mathcal{G}}/R \to R_{\mathcal{G}}/T \to 0$ is exact and $R_{\mathcal{G}}/R$ is $\mathcal{G}$-torsion. Hence $R_{\mathcal{G}}/T$ is $\mathcal{G}$-torsion. It follows that $R_{\mathcal{G}}/T = 0$, i.e., $R_{\mathcal{G}} = T$.

(3) For $I \in \mathcal{G}$, we have $(R:I)_l \subseteq R_{\mathcal{G}} = T$. Since $(R:I)_l$ is finitely generated as a left v-ideal, $(R:I)_l X \subseteq R$ for some $X \in \mathcal{S}$. This implies $X \subseteq I_v$.

Remark. Put $\mathcal{G}' = \{J;$ left ideal of $R \mid \mathrm{Hom}_R(R/J, E'(Q/T)) = 0\}$ ($E'(N)$ denotes the injective hull of a left R-module N). Then, by the left version of Lemma 2.2, we have $T = {}_{\mathcal{G}'}R$ (${}_{\mathcal{G}'}R$ denotes the ring of quotients of R with respect to $\mathcal{G}'$).

Lemma 2.3. *Under the same assumption as in Lemma 2.2, let I be a right R-ideal. Then*

(1) $(IT)_v = (I_v T)_v$.

(2) *Let J be an R-ideal. Then* $(IJT)_v = (I(JT)_v)_v$.

Proof. (1) It is clear that $(IT)_v \subseteq (I_v T)_v$. To prove the converse inclusion, let c be a unit in Q such that $IT \subseteq cT$. Then $c^{-1}I \subseteq T$. Since $c^{-1}I$ is finitely generated as a right v-ideal, there exists $X \in \mathcal{S}$ such that $c^{-1}I \subseteq (c^{-1}I)_v = c^{-1}I_v \subseteq X^{-1}$. Hence $I_v \subseteq cX^{-1} \subseteq cT$ and so $I_v T \subseteq cT$. This implies that $(I_v T)_v \subseteq (IT)_v$ by Proposition 4.1 of [7] and thus $(IT)_v = (I_v T)_v$.

(2) is proved by the same way as in (1).

For a *d-v-HC* order R, we put $R_d = \cup X^{-1}$, where X runs over all *d-v*-invertible ideals of R. Then R_d is an overring of R and $d(R_d) \subseteq R_d$. We have

Lemma 2.4. *Let R be a d-v-HC order. Then*

(1) *R_d is a d-v-HC order.*

(2) *Assume that T has enough d-v-invertible ideals. Then R_d has no proper d-v-ideals of R_d and $R_d[x; d]$ is a Krull order in the sense of* [3].

Proof. (1) Because the set of all d-v-invertible ideals of R is closed by the v-multiplication, R_d satisfies (C) by Lemma 2.4 of [8] and Lemma 2.2. To prove that R_d satisfies (K'), we adopt the method used in Proposition 4.1 of [4]. Let A' be any d-ideal of R_d such that $A' = {}_vA'$. We put $A = A' \cap R$. Then by Lemma 2.2 and Lemma 2.3 of [8], ${}_{\mathcal{F}'}A = \{q \in Q \mid Jq \subseteq A$ for some $J \in \mathcal{F}'\} = A' \subseteq A_{\mathcal{F}} = \{q \in Q \mid qI \subseteq A$ for some $I \in \mathcal{F}\}$, where $\mathcal{F} = \{I$: right ideal $\mid \mathrm{Hom}_R(R/I, E(Q/R_d)) = 0\}$ and $\mathcal{F}' = \{J$: left ideal $\mid \mathrm{Hom}_R(R/J, E'(Q/R_d)) = 0\}$, ($E'(Q/R_d)$ denotes the left R-injective hull of Q/R_d). Hence ${}_v(A'(R_d : A)_l) \supseteqq {}_v({}_{\mathscr{F}'}A(R_d ; A_F)_l) \supseteqq {}_v(R_dA(R : A)_l) = {}_{\mathscr{F}'}({}_v(A(R : A)_l)) = {}_{\mathscr{F}'}O_l(A)$. Since ${}_v(A'(R_d : A)_l)$ is a right $O_l(A)$-module and $1 \in O_l(A)$, we have ${}_v(A'(R_d : A')_l) = O_l(A')$. It is proved by the similar way that $((R_d : B')_rB')_v = O_r(B')$ for any d-ideal B' of R_d such that $B' = B'_v$. Hence R_d is a *d-v-HC* order.

(2) Let A' be any d-ideal of R_d such that $A' = A'_v$. Then $A = A' = \cap R$ is a d-v-ideal of R by Lemma 2.3 of [8]. So there exists a d-v-invertible ideal X of R contained in A. Hence we have $R_d \supseteqq A' = A_{\mathscr{F}} \supseteqq X_{\mathscr{F}} = R_d$ by Lemma 2.3 of [8] and Lemma 2.2., and so $A' = R_d$. Hence R_d has no proper d-v-ideals. To prove that $R_d[x ; d]$ is a Krull order, let B be any d-ideal of R_d and let q be any element in $O_l(B)$. Then $qB \subseteqq B$ implies that $qB_v \subseteqq B_v$. But $B_v = R_d$, because B_v is a d-v-ideal, and so $q \in R_d$. Hence $O_l(B) = R_d$ and similary $O_r(B) = R_d$. So $R_d[x ; d]$ is a maximal order by Proposition 3.1.4 of [3].

Let R be an order in Q. We denote by $S(R)$, the Asano overring of R, i.e., $S(R) = \cup X^{-1}$ (X runs over all v-invertible ideals of R). Let A be an ideal of R. If $C(A) = \{c \in R \mid c \text{ is regular mod } A\}$ is a regular Ore set, then we denote $R_{C(A)}$ by R_A.

Lemma 2.5. *Let R be a d-v-HC order and let B be any maximal d-v-invertible ideal of R. Then*

(1) *$B[x ; d]$ is a semi-prime ideal of $R[x ; d]$ and intersection of a cycle in the sense of* [9].

(2) *$R[x ; d]_{B[x:d]}$ exists and is an HNP ring whose Jacobson radical $B[x ; d]R[x ; d]_{B[x:d]}$ is a unique maximal invertible ideal.*

Proof. (1) By Proposition 1.6, $B = M_1 \cap \ldots \cap M_n$, where $M_1, \ldots, M_n$ is a d-v-cycle. Hence $B[x ; d] = M_1[x ; d] \cap \ldots \cap M_n[x ; d]$ is a semi-prime ideal by Corollary 1.4. Furthermore, since $O_r(M_i[x ; d]) = O_r(M_i)[x ; d] = O_l(M_{i+1})[x ; d] = O_l(M_{i+1}[x ; d])$ and $((M_i[x ; d]^2)_v = (M_i^2[x ; d])_v = (M_i^2)_v[x ; d] = M_i[x ; d]$. Thus $M_1[x ; d], \ldots, M_n[x ; d]$ is a cycle in the sense of [9].

(2) This follows form Lemma 2.1 of [9], Lemma 2.1 and (1).

Lemma 2.6. *Let R be a d-v-HC order with enough d-v-invertible ideals. Then*

(1) *$R[x ; d] = (\cap R[x ; d]_{B[x:d]}) \cap R_d[x ; d]$, where B runs over all d-v-invertible ideals of R.*

(2) $R_d[x;d] = S(R[x;d]) \cap Q[x;d]$.

Proof. The lemma is proved by the exact same way as in Lemma 2.9 of [9].

Lemma 2.7. *Let R be a d-v-HC order with enough d-v-invertible ideals and let A' be any non-zero ideal of $Q[x;d]$. Then*

(1) *$A = A' \cap R[x;d]$ is a v-invertible ideal of $R[x;d]$.*

(2) *$A = (A_1^{n_1}\dots A_k^{n_k}) = (A_1^{n_1})_v \cap \dots \cap (A_k^{n_k})_v$ for some $n_i > 0$, where A_i is a maximal v-ideal of $R[x;d]$ which is v-invertible.*

Proof. As in Lemma 2.12 of [9].

Theorem 2.8. *Let R be an order with a derivation d in a simple artinian ring Q. Then R is a d-v-HC order with enough d-v-invertible ideals if and only if $R[x;d]$ is a v-HC order with enough v-invertible ideals.*

Proof. To prove the necessity, let R be a *d-v-HC* order with enough *d-v*-invertible ideals. Then $R[x;d]$ satisfies (C) by Lemma 2.1. Next we shall prove that $R[x;d]$ has enough v-invertible ideals. To do this let A be any ideal of $R[x;d]$ such that $A_v = A$. (i) In the case $A \cap R \neq 0$, $(A \cap R)_v$ is a v-ideal by Lemma 1.2, because A is a d-ideal. Hence there is a d-v-invertible ideal X of R such that $(A \cap R)_v \supseteqq X$. Then $X[x;d]$ is a v-invertible ideal of $R[x;d]$ which is contained in A. (ii) In the case $A \cap R = 0$. Because $Q[x;d] = R[x;d]_{\mathscr{C}}$ is hereditary, where $\mathscr{C}$ is the set of all regular elements in R, $AQ[x;d]$ is an ideal of $Q[x;d]$ by Lemma 2.3 of [4]. So $B = AQ[x;d] \cap R[x;d]$ is a v-invertibie ideal of $R[x;d]$ by Lemma 2.7. If $A = B$, then there is nothing to state any more. Assume that $B \supsetneqq A$. Then $C = \{r \in R \mid rB \subseteqq A\}$ is a non-zero and d-ideal of R, because $BQ[x;d] = AQ[x;d]$, A and B are both d-stable, and B is finitely generated as a right v-ideal. Hence $C[x;d]$ is an ideal of $R[x;d]$ such that $C[x;d]B \subseteqq A$. So $C = C_v$ by Lemma 1.1 of [8], because B is v-invertible. Since C is a v-ideal by Lemma 1.2, there is a d-v-invertible ideal D of R such that $D \subseteqq C$. Then $(D[x;d]B)_v \subseteqq A_v = A$ and $(D[x;d]B)_v$ is a d-v-invertible ideal of $R[x;d]$. Thus every ideal A of $R[x;d]$ such that $A_v = A$ contains a v-invertible ideal of $R[x;d]$. In particular, $R[x;d]$ has enough v-invertible ideals. To prove that $R[x;d]$ satisfies (K), let C be any ideal of $R[x;d]$ such that $C = C_v$. By Lemma 2.6, we have $R[x;d] = (\cap R[x;d]_{B[x;d]}) \cap Q[x;d] \cap S(R[x;d])$ and so $I_v = (\cap IR[x;d]_{B[x;d]}) \cap$

$IQ[x;d] \cap (I_v S(R[x;d]))_v$ by the same way as in Lemma 2.7 of [2], where $I = (R[x;d] : C)_r C$. Since $R[x;d]_{B[x;d]}$ and $Q[x;d]$ are hereditary, we have $IR[x;d]_{B[x;d]} = (R[x;d]_{B[x;d]} : CR[x;d]_{B[x;d]})_r CR[x;d]_{B[x;d]} = O_r(CR[x;d])$ by using Lemma 2.3 of [8] (also see Lemma 2.3 of [4]). Similarly, $IQ[x;d] = O_r(CQ[x;d])$. To prove that $(I_v S(R[x;d]))_v = S(R[x;d])$, let Y be a v-invertible ideal of $R[x;d]$ which is contained in C (the existence of Y is guarenteed by the proof above). So it follows that $S(R[x;d]) \supseteqq (CS(R[x;d]))_v \supseteqq (YS(R[x;d]))_v = S(R[x;d])$ (the last equality follows from Lemma 2.3 of [8] and Lemma 2.2), and $S(R[x;d]) \subseteqq (R[x;d] : C)_r S(R[x;d]) \subseteqq Y^{-1}S(R[x;d]) \subseteqq S(R[x;d])$. Hence $(((R[x;d] : C)_r C)_v S(R[x;d]))_v = ((R[x;d] : C)_r (CS(R[x;d]))_v)_v = S(R[x;d])$ by Lemma 2.3. Thus we have $I_v \ni 1$ and so $I_v = O_r(C)$ as desired. Similarly, we can prove $O_l(D) = {}_v(D(R[x;d] : D)_l)$ for any ideal D of R with $D = {}_vD$, finishing the proof of the necessity. The sufficiency of the theorem follows from the following Lemmas 2.9, 2.10 and 2.13.

Lemma 2.9. *Let $R[x;d]$ be a v-HC order with enough v-invertible ideals. Then R satisfies (K').*

Proof. Let A be a d-ideal of R such that ${}_vA = A$. Then $A[x;d]$ is an ideal of $R[x;d]$ which is a v-ideal. So $R[x;d]_v(A(R:A)_l) = {}_v(A[x;d]\cdot(R[x;d] : A[x;d])_l) = O_l(A[x;d]) = O_l(A)[x;d]$. Hence ${}_v(A(R:A)_l) = O_l(A)$. It is proved similarly that $((R:B)_r B)_v = O_r(B)$ for any d-ideal B of R such that $B_v = B$. Hence R satisfies (K').

Lemma 2.10. *Let $R[x;d]$ be a v-HC order with enough v-invertible ideals. Then R satisfies (C).*

Proof. Let E and E^* be the injective hulls of Q/R and $Q(R[x;d]/R[x;d])$ as a right R-module and a right $R[x;d]$-module respectively. First we show that E^* is injective as a right R-module. Let A be any right ideal of R and let f be any R-homomorphism from A to E^*. Then $AR[x;d]$ is a right ideal of $R[x;d]$ and the mapping $f' : AR[x;d] \to E^*$ given by $f'(a_m x^m + \cdots + a_0) = f(a_m)x^m + \cdots + f(a_0)$ is an $R[x;d]$-homomorphism, where $a_i \in A$. Hence there is an element t in E^* such that $f'(a(x)) = ta(x)$ for any $a(x) \in AR[x;d]$. Then $f(a) = ta$ for any $a \in A$. So E^* is injective as a right R-module. It follows that $E \subseteqq E^*$, because $Q/R \subseteqq Q(R[x;d])/R[x;d] \subseteqq E^*$. So R satisfies (C) by Proposition 2.4 of [13, p. 264].

Lemma 2.11. *Let $R[x;d]$ be a v-HC order with enough v-invertible ideals, and let M be a maximal v-ideal of $R[x;d]$. Then $M=A=A'\cap R[x;d]$ for some maximal ideal A' of $Q[x;d]$, or $M=(M\cap R)[x;d]$ and $M\cap R$ is a maximal d-v-ideal if $M\cap R\neq 0$. Furthermore, $A=A'\cap R[x;d]$ (A' is a maximal ideal of $Q[x;d]$) is a maximal v-ideal and v-invertible.*

Proof. (i) In the case $M\cap R=0$. There is a maximal ideal A' of $Q[x;d]$ such that $MQ[x;d]\subseteq A'\subsetneqq Q[x;d]$. So $M\subseteq MQ[x;d]\cap R\subseteq A\subsetneqq R[x;d]$ and A is a v-ideal. Hence $M=A$. (ii) In the case $M\cap R\neq 0$. $M\cap R$ is a d-prime ideal of R. So $(M\cap R)[x;d]$ is a prime ideal of $R[x;d]$ by Lemma 1.3. Hence $(M\cap R)[x;d]$ is a maximal v-ideal by Lemma 1.2 of [10], and so $M\cap R$ is a maximal d-v-ideal of R. Finally for any maximal ideal A' of $Q[x;d]$, $A=A'\cap R[x;d]$ is clearly a prime v-ideal. So it is a maximal v-ideal by Lemma 1.2 of [10]. It follows that either A is v-idempotent or v-invertible by Lemma 1.5 of [8]. Assume that A is v-idempotent. Then $AQ[x;d]$ is also v-idempotent. This is a contradiction. Hence A is v-invertible.

Lemma 2.12. *Let $R[x;d]$ be a v-HC order with enough v-invertible ideals and let P be a maximal v-invertible ideal of $R[x;d]$. Then either $P=A=A'\cap R[x;d]$ or $P=M_1[x;d]\cap\ldots\cap M_n[x;d]$, where A' is a maximal ideal of $Q[x;d]$, $M_1,\ldots,M_n$ is a d-v-cycle of R and $P_o=M_1\cap\ldots\cap M_n$ is a maximal d-v-invertible ideal of $R[x;d]$.*

Proof. (i) In the case $P\cap R=0$. There is a maximal ideal A' of $Q[x;d]$ such that $PQ[x;d]\subseteq A'\subsetneqq Q[x;d]$. So $P\subseteq A$. This implies that $P=A$. (ii) In the case $P\cap R\neq 0$. Since it is a d-v-ideal of R, $(P\cap R)[x;d]$ is a v-ideal of $R[x;d]$, and by Lemma 1.12 of [8], there is a cycle $N_1,\ldots,N_n$. Since N_i is a maximal v-ideal, $N_i=A$ or $N_i=M_i[x;d]$, where $M_i=N_i\cap R$ by Lemma 2.11. If $N_i=A$, then $Q[x;d]=(P\cap R)\cdot Q[x;d]\subseteq N_iQ[x;d]=A'$ by Lemma 2.3 of [8], a contradiction. So $N_i=M_i[x;d]$, and $M_1,\ldots,M_n$ is a d-cycle of R by Proposition 1.1. Hence $P=M_1[x;d]\cap\ldots\cap M_n[x;d]=P_o[x;d]$, $P_o=M_1\cap\ldots\cap M_n$ is a maximal d-v-invertible ideal by Proposition 1.6.

Lemma 2.13. *Let $R[x;d]$ be a v-HC order with enough v-invertible ideals. Then R has enough d-v-invertible ideals.*

Proof. Let V be any d-v-ideal of R. Then by Theorem 2.23 of [8] and

Lemma 2.11, we have $R[x;d]=(\cap R[x;d]_P)\cap(\cap R[x;d]_A)\cap S(R[x;d])$, where P and A are as in Lemma 2.11. Since $R[x;d]A=Q[x;d]_A$, by Proposition 1.1 of [7], it follows that $VR[x;d]_A=R[x;d]_A$. Furthermore, $(VS(R[x;d])_v=S(R[x;d])$, because $V[x;d]$ contains a v-invertible ideal of $R[x;d]$. Thus we have $V[x;d]=(\cap V[x;d]R[x;d]_P)\cap(\cap R[x;d]_A)\cap S(R[x;d])$. So there are finitely many maximal v-invertible ideals $P_1,\dots,P_k$ of $R[x;d]$ such that $R[x;d]_{P_i}\supsetneqq V[x;d]R[x;d]_{P_i}$. Since each $R[x;d]_{P_i}$ is an *HNP* ring whose Jacobson radical $P_i'=P_iR[x;d]_{P_i}$ is a unique maximal invertible ideal, we have $V[x;d]R[x;d]_{P_i}\supseteqq P_i'^{n_i}$ for some $n_i>0$. Thus it follows that $V[x;d]\supseteqq(P_1R[x;d]_{P_1})^{n_1}\cap\dots\cap(P_kR[x;d]_{P_k})^{n_k}\cap R[x;d]\supseteqq(P_1^{n_1})_v\cap\dots\cap(P_k^{n_k})_v=(P_1^{n_1}\dots P_k^{n_k})_v=(P_{10}^{n_1}\dots P_{k0}^{n_k}[x;d])_v=(P_{10}^{n_1}\dots P_{k0}^{n_k})_v[x;d]$. Hence $V\supseteqq(P_{10}^{n_1}\dots P_{k0}^{n_k})_v$, and $(P_{10}^{n_1}\dots P_{k0}^{n_k})_v$ is a d-v-invertible ideal of R.

Corollary 2.14. *Let R be a d-v-HC order with enough d-v-invertible ideals. Then $R[x;d]=(\cap R[x;d]_{B[x;d]})\cap(\cap R[x;d]_A)\cap S(R[x;d])$, where B runs over all maximal d-v-invertible ideals of R, $A=A'\cap R[x;d]$, and A' runs over all maximal ideal of $Q[x;d]$. Furthermore, $R[x;d]_{B[x;d]}$ is an HNP ring whose Jacobson radical is a unique maximal invertible ideal, $R[x;d]_A$ is a local, Dedekind prime ring and $S(R[x;d])$ is a v-simple (i. e., it has no proper v-ideal of $S(R[x;d])$) and a Krull order.*

Proof. Since $Q[x;d]$ is a Dedekind prime ring, we have $Q[x;d]=(\cap Q[x;d]_{A'})\cap S(R[x;d])$, where A' runs over all maximal ideals of $Q[x;d]$ and $Q[x;d]_{A'}$ is a local Dedekind prime ring by Theorem 3.1 of [5]. Put $A=A'\cap R[x;d]$. Then $R[x;d]_A=Q[x;d]_{A'}$. Hence the assertion follows from Theorem 2.23 of [8] and Theorem 2.8 (note that $S(R[x;d])\subseteqq S(Q[x;d])$), see the proof of Lemma 2.15 of [9]).

3. Let R be an order in Q and let A be any R-ideal such that $aR=A=Ra'$ for some $a,a'\in Q$. Then $Ra=A=aR$ by the same way as in [6, p. 37]. We denote by $D(R)$ the group consisting of all v-invertible ideals in which the multiplication is given by $X\circ Y=(XY)_v$ for any $X,Y\in D(R)$ and denote by $P(R)$ the subgroup of $D(R)$ consisting of all principal R-ideals. Put $C(R)=D(R)/P(R)$, called the class group of R. Similarly we can define $C_d(R)=D_d(R)/P_d(R)$, the d-class group of R, where $P_d(R)=D_d(R)\cap P(R)$. Now let R be a d-v-*HC* order with enough d-v-invertible ideals, then we have the following :

(1) The set $\{B[x;d], A \mid B$: maximal d-v-invertible ideal of R and $A = A' \cap R[x;d]$, where A' is a maximal ideal of $Q[x;d]\}$ is the full set of maximal v-invertible ideals of $R[x;d]$ (see the proof of Corollary 4.13 of [11]).

(2) Let I be a v-invertible ideal such that $Q[x;d] \supsetneqq I$ and $I \cap Q \neq 0$. Then $I \cap Q$ is v-invertible and $I = (I \cap Q)[x;d]$ (see the proof of Lemma 2.18 of [9]).

In case R is a Krull order, Chamarie has proved the property (2) above by using a complex lemma (see Lemma 3.3.1 of [3]) and he has obtained the following proposition in case R is a Krull order.

Proposition 3.1. ([3, Theorem 3.3.5]) *Let R be a d-v-HC order with enough d-v-invertible ideals. Then*

(1) $D(R[x;d]) \cong D_d(R) \oplus D(Q[x;d])$.

(2) *The mapping f: $D_d(R) \to D(R[x;d])$ given by $f(\mathfrak{a}) = \mathfrak{a}[x;d]$, where $\mathfrak{a} \in D_d(R)$, induces the epimorphism $\bar{f}$: $C_d(R) \to C(R[x;d])$ and if R is a domain, then $\bar{f}$ is an isomorphism.*

Proof. (1) follows from Theorem 1.13 of [8], Proposition 1.6, Theorem 2.8 and property (1) above. (2) follows from the proof of Theorem 3.3.5 of [3].

We shall end this paper with several examples. We have pointed out in the papragraph before Lemma 1.3 that a maximal d-v-ideal is not necessary to be a prime ideal as it is seen in the following example (even it is not a semi-prime ideal). Note that any maximal v-ideal is a prime ideal (see Lemma 1.4 of [8]). Let σ be any automorphism of a v-HC order R. Then we note that any maximal σ-invariant, v-ideal is also a prime ideal. Furthermore, any maximal σ-invariant, v-invertible ideal is a semi-prime ideal. But in case of a derivation type, a maximal d-v-invertible ideal is not necessary to be a semi-prime ideal. This is also seen in Example 1.

Example 1. Let k be a field of $char(k) = p \neq 0$. Put $R = K[x_1, x_2]$, the polynomial ring over k in two indeterminates x_1 and x_2. Then R is a commutative Krull ring. Let $d = x_2\partial_1 + x_1\partial_2$ be a derivation of R, where $\partial_i x_j = \delta_{ij}$. Put $A_1 = (x_1)$ and $A_2 = (x_2)$, the principal ideals generated by x_1 and x_2 respectively. Then A_1 and A_2 are maximal v-ideals (of course, these are maximal v-invertible ideals of R), but not d-stable. On the other hand, $A_1^p \cap A_2^p$ is clearly a maximal d-v-ideal of R and also a maximal d-v-invertible ideal of R, because R is a Krull order. Obviuously $A_1^p \cap A_2^p$ is not semi-prime.

Let R be a d-v-HC order with enough d-v-invertible ideals. Then Example 1 also shows that all generators of $D_d(R)$ are not necessary to be semi-prime, though $D_d(R)$ is a free abelian group.

Example 2. Let R be Krull order in the sense of [3] and let d be any derivation of R. Then R is a d-v-HC order with enough d-v-invertible ideals.

Example 3. Let R be a v-HC order with enough v-invertible ideals. Put $R = T[x]$ and let d be the usual derivation of R. Then R is a d-v-HC order with enough d-v-invertible ideals. In particular, if T is not Krull, then R is not Krull, either. Futhermore, the following result hold :

(1) $D_d(R) \cong D(T) \oplus D_d(Q[x])$, where Q is a quotient ring of T.

(2) If $Char(T) = 0$, then $D_d(Q[x]) = 0$.

(3) If $Char(T) = p \neq 0$, then $D_d(Q[x]) \neq 0$.

Proof. By Theorem 2.16 of [9], $R = T[x]$ is a v-HC order and hence it is clearly a d-v-HC order. It is proved by similar way as in the proof of Theorem 2.8 that R has enough d-v-invertible ideals. Next assume that T is not Krull. Then there is an ideal A of T such that either $O_l(A) \supsetneqq T$ or $O_r(A) \supsetneqq T$ by Proposition 3.1 of [12, p.7]. If $O_l(A) \supsetneqq T$, then $O_l(A[x]) = O_l(A)[x] \supsetneqq T[x]$. This shows that R is not maximal. (1) follows from Theorem 2.19 of [9] and Proposition 1.6. (2) Since any ideal of $Q[x]$ is principal, it is not d-stable if $Char(T) = 0$. Hence $D_d(Q[x]) = 0$. (3) The ideal (x^p), generated by x^p, is clearly d-stable. So $D_d(Q[x]) \neq 0$.

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