MAXIMAL LINEAR TOPOLOGIES
AND THE COMPLEMENT
OF LINEAR TOPOLOGIES

Dedicated to Professor Hisao Tominaga on his 60th birthday

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Introduction. The purpose of this paper is two-fold. First, we characterize the maximal left linear topology on a ring. Applying this, we again prove the equivalence of conditions on a ring, obtained by Nicholson and Sarath [6], to have a unique maximal left linear topology. Secondly, as in parallel with Meijer and Smith [7], we investigate the complement of left linear topologies on a ring $R$. Thus we consider the collection $C(R)$ of those left ideals of $R$ which do not belong to any proper left linear topology on $R$. Two extremes when $C(R)$ consists of all proper left ideals of $R$ and $C(R) = |0|$ are examined.

0. Preliminaries. Let $R$ be a ring with identity. We denote by $\mathcal{I}(R)$ the set of all left ideals of $R$, and by $R$-mod the category of all unital left $R$-modules. For $A \in \mathcal{I}(R)$ and a subset $F$ of $R$, we set $AF^{-1} = \{x \in R \mid xF \subseteq A\}$. A nonempty subset $\mathcal{I}$ of $\mathcal{I}(R)$ is called a left linear topology if the following conditions are satisfied:

T1. If $I \in \mathcal{I}$, $J \in \mathcal{I}(R)$ and $I \subseteq J$, then $J \in \mathcal{I}$.

T2. If $I$ and $J$ belong to $\mathcal{I}$, then $I \cap J \in \mathcal{I}$.

T3. If $I \in \mathcal{I}$ and $a \in R$, then $Ia^{-1} \in \mathcal{I}$.

A left linear topology $\mathcal{I}$ on $R$ is called a left Gabriel topology if $\mathcal{I}$ satisfies a further condition:

T4. If $I \in \mathcal{I}(R)$ and there exists $J \in \mathcal{I}$ such that $Ij^{-1} \in \mathcal{I}$ for every $j \in J$, then $I \in \mathcal{I}$.

A left linear topology $\mathcal{I}$ is called proper if $0 \notin \mathcal{I}$. If $\mathcal{I}_1$ and $\mathcal{I}_2$ are left linear topologies on $R$, we define $\mathcal{I}_1 \leq \mathcal{I}_2$ if every member of $\mathcal{I}_1$ is a member of $\mathcal{I}_2$. A subclass $\mathcal{P}$ of $R$-mod is called a hereditary pretorsion class if $\mathcal{P}$ is closed under isomorphisms, submodules, factor modules and direct sums. $\mathcal{P}$ is called proper if $R \notin \mathcal{P}$. A preradical $r$ for $R$-mod is called left exact if $r(N) = r(M) \cap N$ for every $M \in R$-mod and every submodule $N$ of $M$. It is called proper if $r(R) \neq R$, and is called cofaithful if $r(Q) = Q$ for every injective $Q \in R$-mod. For preradicals $r$ and $s$ for
$R$-mod, we define a preradical $r+s$ by $(r+s)(M) = r(M) + s(M)$ for all $M \in R$-mod. For a module $Q \in R$-mod, we define a preradical $t_Q$ for $R$-mod by $t_Q(M) = \sum \text{Ima. } a$ ranging over $\text{Hom}_R(Q, M)$, for each $M \in R$-mod. We remark that $t_Q$ is cofaithful if and only if $Q$ is cofaithful, i.e., $Q$ generates all injective left $R$-modules, or equivalently, $R$ can be embedded in a finite direct sum of copies of $Q$ ([1, Proposition 4.5.4]). We naturally define the ordering of hereditary pretorsion classes of $R$-mod and that of left exact preradicals for $R$-mod. It is well known that there is an order preserving bijection correspondence between left linear topologies on $R$, hereditary pretorsion classes of $R$-mod and left exact preradicals for $R$-mod (see [9, p. 145]).

1. Maximal linear topologies. It is not assured that, for a given proper left Gabriel topology on $R$, there exists a maximal left Gabriel topology containing given one. In [7, Theorem 3.4], Meijer and Smith proved that the above property on a ring $R$ holds if and only if every nonzero injective left $R$-module has a nonzero submodule whose annihilator is an $M$-ideal. If $R$ satisfies the maximum condition for ideals, then $R$ has the above property ([3, Proposition 3.2]). But we can prove the next

**Proposition 1.1.** For every proper left linear topology $\mathcal{P}$ on $R$, there exists a maximal left linear topology containing $\mathcal{P}$.

**Proof.** This is done by Zorn's lemma.

**Lemma 1.2.** For every left linear topology $\mathcal{P}$ on $R$ and left ideal $A$ of $R$, there exists a unique minimal left linear topology $\mathcal{P}^*$ containing $\mathcal{P}$ and $A$. For $J \in \mathcal{P}(R)$, $J$ belongs to $\mathcal{P}^*$ if and only if there exist $I \in \mathcal{P}$ and a finite subset $F$ of $R$ such that $J \supseteq I \cap AF^{-1}$.

**Proof.** Let $\mathcal{P}^*$ be the set of left ideals $J$ of $R$ such that there exist $I \in \mathcal{P}$ and a finite subset $F$ of $R$ satisfying $J \supseteq I \cap AF^{-1}$. It is sufficient to show that $\mathcal{P}^*$ is in fact a left linear topology. Clearly $\mathcal{P}^*$ satisfies T1. Assume $J_1$ and $J_2$ belong to $\mathcal{P}^*$. Then there exist left ideals $I_1$ and $I_2$ and finite subsets $F_1$ and $F_2$ of $R$ such that $J_i \supseteq I_i \cap AF_i^{-1}$ ($i = 1, 2$). Since $I_1 \cap I_2 \in \mathcal{P}$ and $AF_1^{-1} \cap AF_2^{-1} = A(F_1 \cup F_2)^{-1}$, we have $J_1 \cap J_2 \supseteq (I_1 \cap I_2) \cap A(F_1 \cup F_2)^{-1}$, proving $\mathcal{P}^*$ satisfies T2. Now assume $J \in \mathcal{P}^*$ and $a \in R$. Then there exist $I \in \mathcal{P}$ and a finite subset $F$ of $R$ such that $J \supseteq I \cap AF^{-1}$. Now we have $Ja^{-1} \supseteq (I \cap AF^{-1})a^{-1} = Ia^{-1} \cap (AF^{-1})a^{-1}$.
MAXIMAL LINEAR TOPOLOGIES

\[ = Ia^{-1} \cap A(aF)^{-1} \]. Since \( Ia^{-1} \in \mathcal{L} \), we obtain \( Ja^{-1} \in \mathcal{L}^* \), proving \( \mathcal{L}^* \)
satisfies T3.

Now we have a criterion of the maximality of left linear topologies.

**Theorem 1.3.** The following conditions are equivalent for a proper left linear topology \( \mathcal{L} \) on \( R \):

1. \( \mathcal{L} \) is maximal.
2. For each left ideal \( A \notin \mathcal{L} \), there exist \( I \in \mathcal{L} \) and a finite subset \( F \) of \( R \) such that \( I \cap AF^{-1} = 0 \).
3. For each left ideal \( A \notin \mathcal{L} \), there exist \( I \in \mathcal{L} \) and a natural number \( n \) such that \( R \) can be embedded in \( R/I \oplus (R/A)^m \).
4. For each left ideal \( A \notin \mathcal{L} \), there exists \( I \in \mathcal{L} \) such that \( R/I \oplus R/A \) is cofaithful.

**Proof.** \( \mathcal{L} \) is maximal if and only if, for each left ideal \( A \notin \mathcal{L} \), 0 belongs to the unique minimal left linear topology containing \( \mathcal{L} \) and \( A \). Hence by using Lemma 1.2, we have (1) \( \iff \) (2). Now assume, for each left ideal \( A \notin \mathcal{L} \), there exist \( I \in \mathcal{L} \) and a finite subset \( \{r_1, \ldots, r_n\} \) of \( R \) such that \( I \cap Ar_1^{-1} \cap \cdots \cap Ar_n^{-1} = 0 \). Then \( R \) is embedded in \( R/I \oplus R/Ar_1^{-1} \oplus \cdots \oplus R/Ar_n^{-1} \). But \( R/Ar_i^{-1} \cong (Rr_i + A)/A \leq R/A \) for each \( i = 1, \ldots, n \). Hence we have (2) \( \iff \) (3). The implication (3) \( \iff \) (4) is trivial. Finally, assume for each left ideal \( A \notin \mathcal{L} \), there exist \( I \in \mathcal{L} \) and a natural number \( n \) with a monomorphism \( f: R \to (R/I)^m \oplus (R/A)^m \). Put \( f(1) = (\tilde{s}_1, \ldots, \tilde{s}_n, \tilde{r}_1, \ldots, \tilde{r}_n) \), where \( s_i, r_i \in R \) and \( \tilde{s}_i = s_i + I \) and \( \tilde{r}_i = r_i + A \) for \( i = 1, \ldots, n \). Since \( f(x) = (x\tilde{s}_1, \ldots, x\tilde{s}_n, x\tilde{r}_1, \ldots, x\tilde{r}_n) = 0 \) implies \( x = 0 \), we have \( Is_1^{-1} \cap \cdots \cap Is_n^{-1} \cap Ar_1^{-1} \cap \cdots \cap Ar_n^{-1} = 0 \). Thus we have proved (4) \( \iff \) (2), because \( Is_1^{-1} \cap \cdots \cap Is_n^{-1} \in \mathcal{L} \).

**Corollary 1.4.** The following conditions are equivalent for a proper hereditary pretorsion class \( \mathcal{T} \) of \( R\)-mod:

1. \( \mathcal{T} \) is maximal.
2. For each (cyclic) left \( R \)-module \( M \notin \mathcal{T} \), there exist a cyclic left \( R \)-module \( C \in \mathcal{T} \) and a natural number \( n \) such that \( R \) can be embedded in \( C \oplus M^m \).
3. For each (cyclic) left \( R \)-module \( M \notin \mathcal{T} \), there exists a cyclic left \( R \)-module \( C \in \mathcal{T} \) such that \( C \oplus M \) is cofaithful.

**Corollary 1.5.** The following conditions are equivalent for a proper
left exact preradical \( r \) for \( R\text{-mod} \):
(1) \( r \) is maximal.
(2) For each (cyclic) left \( R\)-module \( M \) with \( r(M) \neq M \), there exists a cyclic left \( R \)-module \( C \) with \( r(C) = C \) such that \( t_C \) is cofaithful.

In [8] Rubin called a left ideal \( A \) of \( R \) weakly essential if \( AF^{-1} \neq 0 \) for every finite subset \( F \) of \( R \). Note that, if a left ideal \( A \) is weakly essential, then \( AX^{-1} \) is also weakly essential for every finite subset \( X \) of \( R \). We remark that every member of a proper left linear topology on \( R \) is weakly essential. Now we shall consider the case when a left linear topology is unique maximal.

**Proposition 1.6.** The following conditions are equivalent for a proper left linear topology \( \mathcal{L} \) on \( R \):
(1) \( \mathcal{L} \) is unique maximal.
(2) \( \mathcal{L} \) coincides with the set of all weakly essential left ideals of \( R \).
(3) \( \mathcal{L} \) contains all weakly essential left ideals of \( R \).

**Proof.** For a left ideal \( A \) of \( R \), we put \( \mathcal{L}_A \) a unique minimal left linear topology containing \( A \). Then \( \mathcal{L}_A \) consists of those left ideals \( B \) such that \( B \supseteq AF^{-1} \) for some finite subset \( F \) of \( R \).

(1) \( \Rightarrow \) (2). If \( A \in \mathcal{L} \), then \( AF^{-1} \in \mathcal{L} \) for every finite subset \( F \) of \( R \). Since \( \mathcal{L} \) is proper, we see that \( A \) is weakly essential. Conversely assume \( A \) is a weakly essential left ideal of \( R \). Then \( \mathcal{L}_A \) is proper and so \( \mathcal{L}_A \subseteq \mathcal{L} \). Hence we have \( A \in \mathcal{L} \).

(2) \( \Rightarrow \) (3). Clear.

(3) \( \Rightarrow \) (1). Let \( \mathcal{L}' \) be a proper left linear topology on \( R \). For each \( A \in \mathcal{L}' \), we have \( AF^{-1} \in \mathcal{L}' \) for every finite subset \( F \) of \( R \). Since \( \mathcal{L}' \) is proper, we see \( A \) is weakly essential, and so \( A \in \mathcal{L} \) by (3). Therefore we have proved \( \mathcal{L} \) is unique maximal.

The following corollary was proved by Nicholson and Sarath by using the notion of \( a \)-weak essentiality. But we can prove this directly.

**Corollary 1.7** (Nicholson and Sarath [6, Theorem 1]). The following conditions are equivalent for a ring \( R \) with the set \( \mathcal{L} \) of all weakly essential left ideals of \( R \):
(1) \( R \) has a unique maximal left linear topology.
(2) \( \mathcal{L} \) forms a left linear topology.
(3) If \( A \) and \( B \) belong to \( \mathcal{P} \), then \( A \cap B \neq 0 \).

Proof. \((1) \iff (2)\). This is clear by using Proposition 1.6.

\((2) \Rightarrow (3)\). Clear.

\((3) \Rightarrow (2)\). Clearly \( \mathcal{P} \) satisfies T1. As noted above, \( \mathcal{P} \) also satisfies T3. Now assume \( A \) and \( B \) belong to \( \mathcal{P} \). For every finite subset \( F \) of \( R \), we see that \( AF^{-1} \) and \( BF^{-1} \) belong to \( \mathcal{P} \). Hence \( (A \cap B)F^{-1} = AF^{-1} \cap BF^{-1} \neq 0 \) by (3). Thus \( A \cap B \) belongs to \( \mathcal{P} \). Therefore we showed that \( \mathcal{P} \) satisfies T2.

Example 1.8. Let \( R \) be a ring and \( \mathcal{P} \) the set of all essential left ideals of \( R \). It is well known that \( \mathcal{P} \) is a proper left linear topology on \( R \). By using Theorem 1.3, we notice that \( \mathcal{P} \) is maximal if and only if every weakly essential left ideal of \( R \) is essential. In this case, \( \mathcal{P} \) is unique maximal by Proposition 1.6. In case \( R \) is commutative, we remark that \( \mathcal{P} \) is maximal if and only if every nonzero ideal of \( R \) is essential. Thus we conclude that, if \( R \) is a commutative semiprime ring, \( \mathcal{P} \) is maximal if and only if \( R \) is prime.

2. The complement of linear topologies. In \([7]\) Meijer and Smith concerned with the collection \( N(R) \) of those left ideals of \( R \) which do not belong to any proper left Gabriel topology on \( R \). As mentioned in \([7, \text{Lemma } 2.1]\), a left ideal \( I \) belongs to \( N(R) \) if and only if \( \text{Hom}_R(R/I, E) \neq 0 \) for every nonzero injective left \( R \)-module \( E \). Now we shall consider the set

\[
\mathcal{C}(R) = \{ I \in \mathcal{L}(R) \mid I \notin \mathcal{P} \text{ for every proper left linear topology } \mathcal{P} \text{ on } R \}.
\]

Clearly \( 0 \in \mathcal{C}(R) \) and \( R \notin \mathcal{C}(R) \). If \( A \in \mathcal{P}(R) \) and \( A \leq B \) for some \( B \in \mathcal{C}(R) \), then \( A \in \mathcal{C}(R) \). Remark that \( \mathcal{C}(R) \subseteq N(R) \).

Theorem 2.1. The following statements are equivalent for a left ideal \( A \) of a ring \( R \):

1. \( A \in \mathcal{C}(R) \).
2. \( A \) is not weakly essential, i.e., \( AF^{-1} = 0 \) for some finite subset \( F \) of \( R \).
3. \( R/A \) is cofaithful.

Proof. \((1) \iff (2)\). For a left ideal \( A, A \in \mathcal{C}(R) \) if and only if \( 0 \in \mathcal{P} \) for every left linear topology \( \mathcal{P} \) containing \( A \), or equivalently
0 belongs to the unique minimal left linear topology containing \( A \). As noted in the proof of Proposition 1.6, 0 \( \in \mathcal{L}_A \) if and only if \( AF^{-1} = 0 \) for some finite subset \( F \) of \( R \).

(2) \( \iff \) (3). This is proved by the same method as is used in the proof of Theorem 1.3. (See [1, Proposition 4.5.4]).

A left linear topology \( \mathcal{L} \) is called super if \( \mathcal{L} \) contains a unique minimal member. Such a member is in fact a two-sided ideal. We denote by \( C_s(R) \) the set of those left ideals which do not belong to any proper super left linear topology on \( R \). Clearly \( C_s(R) \supseteq C(R) \). If \( R \) is left artinian, then every left linear topology on \( R \) is super, and so \( C_s(R) = C(R) \). For a ring \( R \) with Jacobson radical \( J \), it was proved in [7, Proposition 2.9] that \( N(R) \) consists of all proper left ideals of \( R \) if and only if \( J \) is right T-nilpotent and \( R/J \) is a simple artinian ring. By the definition, \( C_s(R) \) consists of all proper left ideals of \( R \) if and only if \( R \) is a two-sided simple ring.

**Theorem 2.2.** The following statements are equivalent for a ring \( R \):

1. \( C(R) \) contains all maximal left ideals of \( R \).
2. \( C(R) \) consists of all proper left ideals of \( R \).
3. \( R \) is a simple artinian ring.

**Proof.** (1) \( \implies \) (2). Clear.

(2) \( \implies \) (3). Assume (2). Then every nonzero cyclic left \( R \)-module is cofaithful by Theorem 2.1. Thus every nonzero left \( R \)-module is also cofaithful. In particular every nonzero left ideal of \( R \) is cofaithful, and so \( R \) is left strongly prime (see [5, Proposition 2.5]). Also since every faithful left \( R \)-module is cofaithful, \( \text{soc}(\_R) \neq 0 \) by [2, Proposition 1]. Hence \( R \) must be simple artinian by [5, Theorem 4.3].

(3) \( \implies \) (1). Assume \( I \) is a maximal left ideal of \( R \). Then \( \_R(R/I) \) is cofaithful and so \( I \) belongs to \( C(R) \) by Theorem 2.1.

In [7] the other extreme when \( N(R) = |0| \) was considered. It was shown in [7, Theorem 6.4] that \( N(R) = |0| \) if and only if \( R \) is a reduced ring and \( Ra + 0a^{-1} \) is essential left ideal of \( R \) for all \( a \in R \). We remark that \( C_s(R) = |0| \) if and only if every nonzero left ideal of \( R \) contains a nonzero ideal of \( R \).

**Theorem 2.3.** The following statements are equivalent for a ring \( R \):

1. \( C(R) = |0| \).
(2) Every nonzero left ideal of $R$ is weakly essential.
(3) Every nonzero cyclic left ideal of $R$ is weakly essential.
(4) For every nonzero element $a$ of $R$ and elements $r_1, \ldots, r_n$ of $R$, there exist a nonzero element $a'$ of $R$ and elements $r_i$ in $R$ such that $a'r_i = r_ia (i = 1, \ldots, n)$.

Proof. (1) $\iff$ (2). This is clear by Theorem 2.1.
(2) $\iff$ (3). Clear.
(3) $\iff$ (4). Let $A = Ra$ be a nonzero cyclic left ideal of $R$. Then $A$ is weakly essential if and only if, for every elements $r_1, \ldots, r_n$ of $R$, $Ar_1^{-1} \cap \cdots \cap Ar_n^{-1} \neq 0$ holds. This occurs if and only if, for every elements $r_1, \ldots, r_n$ of $R$, there exists a nonzero element $a'$ of $R$ such that $a'r_i \in A = Ra (i = 1, \ldots, n)$.

Corollary 2.4 (cf. [7, Corollaries 5.2 and 6.5]). If $R$ is a domain, then $C(R) = |0|$ if and only if $R$ satisfies the left Ore condition.

Proof. Assume $R$ is a left Ore domain with a classical left quotient ring $Q_0(R)$. For every nonzero element $a$ of $R$ and elements $r_1, \ldots, r_n$ of $R$, there exist nonzero elements $a'_i$ of $R$ and elements $s_i$ of $R$ such that $r_ia^{-1} = a'_i s_i (i = 1, \ldots, n)$. As is well known (see [4, p. 392]), there exist a nonzero element $a'$ of $R$ and elements $t_i$ of $R$ such that $a'_i t_i = a^{-1} t_i (i = 1, \ldots, n)$. Put $r_i = t_is_i (i = 1, \ldots, n)$. Thus we have $a'r_i = r_ia (i = 1, \ldots, n)$, and so $C(R) = |0|$. We can also show this fact by using [7, Corollary 5.2] with $C(R) \subseteq N(R)$. The reverse implication is obvious.

Remark 2.5. The property that $C(R) = |0|$ of rings $R$ is not a Morita invariant. To see this, let $K$ be a field. By Theorem 2.3, we see $C(K) = |0|$. But consider the ring $R$ of $n \times n$ matrices over $K$ for some $n > 1$. As shown in Theorem 2.2, we have $C(R) \neq |0|$. On the other hand, the property that $R$ has a unique maximal left linear topology is a Morita invariant ([6, Corollary to Theorem 2]). Hence we conclude that the above two properties on $R$ are not equivalent.

By using Theorem 2.3, we shall prove the next two propositions.

Proposition 2.6. If $R$ is a left order in a ring $Q$, then $C(Q) = |0|$ implies $C(R) = |0|$. Furthermore, if $R$ is a domain, then $C(Q) = |0|$.

Proof. Suppose there are given elements $r(\neq 0), r_1, \ldots, r_n$ in $R$.
By \( C(Q) = |0| \), there exist elements \( q(\neq 0), q_1, \cdots, q_n \) in \( Q \) such that \( qr_i = q_i r \) \((i = 1, \cdots, n)\). We can find a regular element \( r' \) in \( R \) with \( r'q(\neq 0), r'q_1, \cdots, r'q_n \in R \). Thus we have \((r'q)r_i = (r'q_i)r \) \((i = 1, \cdots, n)\), and so \( C(R) = |0| \).

Now assume \( R \) is a domain. For every elements \( q(\neq 0), q_1, \cdots, q_n \) of \( Q \), there exist a regular element \( r \) in \( R \) with \( rq(\neq 0), rq_1, \cdots, rq_n \in R \). Since \( C(R) = |0| \) by Corollary 2.4, there exist \( r'(\neq 0), r'_1, \cdots, r'_n \) in \( R \) such that \( r'(rq_i) = r'_i(rq) \) \((i = 1, \cdots, n)\). Noting that \( r'r(\neq 0) \) and \( r_ir \) \((i = 1, \cdots, n)\) belong to \( Q \), we obtain \( C(Q) = |0| \).

**Proposition 2.7.** Suppose \( R = R_1 \times \cdots \times R_n \) is a direct sum of rings \( R_i \) \((i = 1, \cdots, n)\). Then \( C(R) = |0| \) if and only if \( C(R_i) = |0| \) for all \( i = 1, \cdots, n \).

**Proof.** We may assume \( n = 2 \). Let \( S \) and \( T \) be rings. Assume \( C(S) = |0| \) and \( C(T) = |0| \). Let \((s, t)(\neq 0), (s_1, t_1), \cdots, (s_n, t_n) \) be elements of \( S \times T \). We may assume that \( s \neq 0 \). By \( C(S) = |0| \), there exist \( s'(\neq 0), s_1', \cdots, s_n' \) in \( S \) such that \( s's_i = s_is \) \((i = 1, \cdots, n)\). If \( t = 0 \), then we have \((s', t)(s_1, t_1) = (s_1, t_1)(s, t) \) \((i = 1, \cdots, n)\). If \( t \neq 0 \), by \( C(T) = |0| \), there exist \( t'(\neq 0), t'_1, \cdots, t'_n \) in \( T \) such that \( t't_i = t_i t \) \((i = 1, \cdots, n)\), and so we have \((s', t')(s_1, t_1) = (s_1, t_1)(s, t) \) \((i = 1, \cdots, n)\). Therefore we have \( C(S \times T) = |0| \).

Conversely assume \( C(S \times T) = |0| \). To show \( C(S) = |0| \), let \( s(\neq 0), s_1, \cdots, s_n \) be elements of \( S \). For the elements \((s, 0), (s_1, 1), \cdots, (s_n, 1) \) in \( S \times T \), there exist elements \((s', t')(\neq 0), (s_1', t_1'), \cdots, (s_n', t_n') \) in \( S \times T \) such that \((s', t')(s_1, 1) = (s_1, t_1)(s, 0) \) \((i = 1, \cdots, n)\). Then we have \( s's_i = s_is \) \((i = 1, \cdots, n)\) and \( s' \neq 0 \) because \( t' = 0 \). Therefore we showed \( C(S) = |0| \).

**Example 2.8.** There may be many rings \( R \) such that \( C(R) \) are not extreme. To give such an example, we shall calculate \( C(R) \) where \( R \) is the \( 2 \times 2 \) upper triangular matrix ring over a field \( K \). There are three types of minimal left ideals of \( R \), namely \( A = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix} \) and \( B = \left[ \begin{array}{cc} xa & xb \\ 0 & 0 \end{array} \right] \) for some fixed nonzero elements \( a \) and \( b \) of \( K \). Let \( e_{11}, e_{12} \) and \( e_{22} \) be matrix units in \( R \). Since \( Ae_{12}^{-1} \cap Ae_{22}^{-1} = 0 \), \( A \) belongs to \( C(R) \). Also since \( Be_{11}^{-1} \cap Be_{22}^{-1} = 0 \), \( B \) belongs to \( C(R) \). But since \( C \) is an ideal of \( R \), it is weakly essential and so \( C \) does not belong to \( C(R) \). Now
let $I$ be a left ideal of $R$ which contains $A$ or some $B$ strictly. Then $I$ also contains $C$ and so $I$ does not belong to $C(R)$. Thus we conclude that $C(R)$ consists precisely of $A$ and those left ideals $B$.

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