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ON LEFT EXACT PRERADICALS

Dedicated to Professor Hisao Tominaga on his 60th birthday

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In [7], we have given the conditions for a left exact preradical to be a radical and to be stable by means of the notions of weakly divisibility and divisibility.

In the first part of this note, we study the divisible hulls of modules. Especially, for a left exact preradical r, we give a necessary condition for every module to have the r-divisible hull (Theorem 1.8). Next we characterize left exact preradicals which satisfy the condition that every divisible module is injective (Theorem 1.11). Finally, we investigate left exact preradicals for which every weakly divisible module is injective (Theorem 2.6).

Throughout this note, R means a ring with identity and modules mean unitary left R-modules, unless otherwise stated. The category of all modules is denoted by R-mod and the injective hull of a module A by $E(A)$. For the terminologies and basic properties of preradicals and torsion theories, we refer to [8]. For each preradical r, we denote the r-torsion class (resp. r-torsionfree class) by $T(r)$ (resp. $F(r)$). Also the left linear topology corresponding to a left exact preradical r is denoted by $L(r)$. Now, for two preradicals r and s, we shall say that r *is larger than* s if $r(A) \supseteq s(A)$ for all modules A. For a preradical r, $\bar{r}$ means the smallest radical larger than r. As is well-known, $F(r) = F(\bar{r})$.

1. The divisible hulls of modules. Let r be a preradical. We call a module A *r-weakly divisible* (resp. *r-divisible*) if the functor $\mathrm{Hom}_R(-, A)$ preserves the exactness of all sequences of modules $0 \to L \to M \to N \to 0$ with M in $T(r)$ (resp. N in $T(r)$).

The following lemma is well-known.

Lemma 1.1. *Let r be an idempotent preradical and A a module. Then the following are equivalent :*

1) *A is divisible.*
2) $r(E(A)/A) = 0$.
3) *A is $\bar{r}$-divisible.*

Let r be a preradical and let A be a module. We call a module D an *r-divisible hull of A* if D is an r-divisible module containing A, A is essential in D and D/A is in $T(r)$.

As is easily seen, an r-divisible hull of A is the smallest r-divisible module containing A.

Lemma 1.2. *For a preradical r and a module A, an r-divisible hull of A (if it exists) is unique up to isomorphism.*

Proof. Let both D and D' be r-divisible hulls of A. Consider the diagram

$$\begin{array}{ccccc} O & \longrightarrow & A & \xrightarrow{\;i\;} & D \\ & & {\scriptstyle j}\big\downarrow & & \\ & & D' & & \end{array}$$

where i and j are inclusions. Since D/A is in $T(r)$ and A is essential in D, there exists a monomorphism $g: D \to D'$ such that $g \circ i = j$. Since $A \subseteq g(D) \subseteq D'$ and $D'/g(D)$ is in $T(r)$, $g(D)$ is a direct summand of D'. Since $g(D)$ is essential in D', $g(D) = D'$. Hence $D \simeq D'$.

We may use $D(A)$ to denote the r-divisible hull of A, if it exists.

Proposition 1.3. *If r is an idempotent preradical and a module A has the r-divisible hull, then $D(A)/A = r(E(A)/A)$.*

Proof. We put $r(E(A)/A) = B/A$. Since $A \subseteq D(A) \subseteq E(A)$ and $D(A)/A$ is in $T(r)$, $D(A) \subseteq B$ and $B/D(A)$ is in $T(r)$. Also since $D(A)$ is a direct summand of B and is essential in B, $B = D(A)$. Hence $D(A)/A = r(E(A)/A)$.

Corollary 1.4. *Let r be an idempotent preradical and A a module. A has the r-divisible hull if and only if $r(E(A)/A) = \bar{r}(E(A)/A)$.*

Corollary 1.5. *If r is an idempotent radical, then every module has the r-divisible hull.*

We can provide an idempotent preradical for which there exists a module having no divisible hull (see Example 1.9).

Lemma 1.6. *For a left exact preradical r, the following conditions are equivalent:*

1) *r is a radical.*
2) *$r(H/r(H)) = O$ for all r-weakly divisible modules H.*
3) *$r(D/r(D)) = O$ for all r-divisible modules D.*
4) *$r(E/r(E)) = O$ for all injective modules E.*

Proof. 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) are clear. 4) $\Rightarrow$ 1). Let A be a module. Then $A/r(A) = A/(r(E(A)) \cap A) \simeq (A+r(E(A)))/r(E(A)) \subseteq E(A)/r(E(A)) \in F(r)$. Thus by assumption r is a radical.

Now let r be a left exact preradical. We define a left exact preradical r_2 larger than r by $r_2(A)/r(A) = r(A/r(A))$ for each module A.

Proposition 1.7. *For a left exact preradical r, the following conditions are equivalent:*

1) *r_2 is a radical.*
2) *$r(A/r(A)) = \bar{r}(A/r(A))$ for all modules A.*
3) *$r(H/r(H)) = \bar{r}(H/r(H))$ for all r-weakly divisible modules H.*
4) *$r(D/r(D)) = \bar{r}(D/r(D))$ for all r-divisibie modules D.*
5) *$r(E/r(E)) = \bar{r}(E/r(E))$ for all injective modules E.*
6) *$r_2(E)$ is r-divisible for all injective modules E.*

Proof. 1) $\Rightarrow$ 2). Let A be a module. Since $r_2(A/r_2(A)) = O$ and $F(r) = F(\bar{r})$, $\bar{r}(A/r_2(A)) = O$. Thus $\bar{r}(A/r(A)/r_2(A)/r(A)) = O$, namely, $r_2(A)/r(A) \supseteq \bar{r}(A/r(A))$. Therefore $r(A/r(A)) = \bar{r}(A/r(A))$. 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 5) $\Rightarrow$ 6) are clear. 6) $\Rightarrow$ 1). Let E be an injective module. We put $r(E/r_2(E)) = B/r_2(E)$. Since $r_2(E)$ is r-divisible, there exists a submodule Y of B such that $B = r_2(E) \oplus Y$. Also since $r(B) = r(r_2(E))$, Y is in $F(r)$. Clearly since Y is in $T(r)$, $B = r_2(E)$. Hence $E/r_2(E)$ is in $F(r)$. However $F(r) = F(r_2) = F(\bar{r})$ and thus r_2 is a radical by Lemma 1.6.

Theorem 1.8. *Let r be a left exact preradical. If every module has the r-divisible hull, then r_2 is a radical.*

Proof. Let E be an injective module. We show that $r_2(E)$ is r-divisible. By Proposition 1.3, $D(r(E))/r(E) = r(E(r(E))/r(E))$, and so it is contained in $r(E/r(E)) = r_2(E)/r(E)$. Thus $D(r(E)) \subseteq r_2(E)$. Since $D(r(E))$ is r-divisible and $r_2(E)/D(r(E))$ is in $T(r)$, $D(r(E))$ is a direct summand of $r_2(E)$. Hence there exists a submodule X of $r_2(E)$ such that $r_2(E) = D(r(E)) \oplus X$. Since $r(E) \subseteq r_2(E) \subseteq E$ and $r(E) \subseteq D(r(E)) \subseteq E$, $r(r_2(E)) = r(D(r(E)))$. And so X is in $F(r)$. Also since $X \simeq r_2(E)/D(r(E))$, X is in $T(r)$. Therefore $r_2(E)$ is r-divisible.

The converse of this theorem is not necessarily true (see Example 1.13).

Now we give an example of a left exact preradical r for which there exists a module having no r-divisible hull.

Example 1.9. Let K be a field and R the ring of all 3×3 upper triangular matrices over K. If we put $T = \begin{pmatrix} O & K & K \\ O & O & K \\ O & O & O \end{pmatrix}$, then T is an ideal of R and $T \supsetneqq T^2 \supsetneqq T^3 = O$. Let r be a left exact preradical corresponding to the left linear topology which has the smallest element T. By [4, Theorem 6], r_2 is not a radical. Thus by Theorem 1.8 there exists a module which does not have the r-divisible hull.

Now let $Z(A)$ be the singular submodule of a module A. As is well-known, Z is a left exact preradical and $L(Z)$ is the set of essential left ideals of R. Furthermore if B is an essential submodule of a module A, then $Z(A/B) = A/B$. Also the smallest radical larger than Z is called the *Goldie torsion radical* and is denoted by G.

Lemma 1.10. *For a left exact preradical r, the following conditions are equivalent:*

1) *r is larger than Z.*
2) *If B is an essential submodule of a module A, then $r(A/B) = A/B$.*
3) *$r(E(A)/A) = E(A)/A$ for all modules A.*
4) *For each module A, the injective hull $E(A)$ of A is the r-divisible hull of A.*

Proof. The implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) are clear. 4) $\Rightarrow$ 1). Let I be an element of $L(Z)$. Since I is essential in R, $E(I) = E(R)$. Also since $E(I)/I = E(R)/I$ is in $T(r)$ by assumption and r is left exact, R/I is in $T(r)$. Thus I is in $L(r)$. Hence r is larger than Z.

Now we consider the following condition (*) :

(*) *Every r-divisible module is injective.*

Each one of them in above lemma implies $(*)$. For, let A be an r-divisible module. Then $r(E(A)/A) = O$ by Lemma 1.1. By assumption $r(E(A)/A) = E(A)/A$. Thus $E(A) = A$.

But the converse is not true in general (see Example 2.4). As concerns the equivalent conditions of $(*)$, we state it in the following theorem.

Theorem 1.11. *Let r be a left exact preradical. Then the following conditions are equivalent:*

1) *Every r-divisible module is injective.*
2) *$\bar{r}$ is larger than $\dot{G}$.*
3) *If a module A has the r-divisible hull, then it is the injective hull of A.*

Proof. 1) $\Rightarrow$ 2). Let I be an essential left ideal of R. Then $\bar{r}$ is a left exact radical and I has the $\bar{r}$-divisible hull $\bar{D}(I)$. Since $\bar{D}(I)$ is r-divisible and is essential extension of I, $\bar{D}(I) = E(I) = E(R)$. Hence $R/I \subseteqq \bar{D}(I)/I$ and R/I is in $T(\bar{r})$. I is in $L(\bar{r})$ and so $\bar{r}$ is larger than Z. Thus $\bar{r}$ is larger than $\bar{Z} = G$. 2) $\Rightarrow$ 3). Assume that a module A has the r-divisible hull $D(A)$. Since $D(A)$ is also the $\bar{r}$-divisible hull of A and $\bar{r}$ is larger than Z, $D(A)$ is injective by Lemma 1.10. Thus $D(A) = E(A)$. 3) $\Rightarrow$ 1) is clear.

Corollary 1.12. *Let r be a left exact preradical for which each module has the r-divisible hull. Then every r-divisible module is injective if and only if r is larger than Z.*

Example 1.13. Let R be the 3×3 upper triangular matrix ring over a field K. If we put $T = \begin{pmatrix} K & K & K \\ O & O & K \\ O & O & O \end{pmatrix}$, then T is an ideal of R and $T^2 = \begin{pmatrix} K & K & K \\ O & O & O \\ O & O & O \end{pmatrix} = soc({}_RR)$. Let r be a left exact preradical corresponding to the left linear topology which has the smallest element T. Since $T \neq T^2$, r is not a radical. On the other hand, since R is artinian and $T^2 = T^3$, $r_2 = G$ by [4, Theorem 6]. If R has the r-divisible hull $D(R)$, then $D(R)$ is the r_2-divisible hull of R. By Lemma 1.10, $D(R) = E(R)$. But $r(E(R)/R) \neq E(R)/R$. Thus R does not have the r-divisible hull.

2. Weakly divisible and divisible modules. We call a preradical r

stable if $T(r)$ is closed under injective hulls.

Proposition 2.1. *For a left exact radical r, the follwing conditions are equivalent*:

1) *Every r-torsion r-divisible module is injective.*
2) *Every r-torsion r-weakly divisible module is injective.*
3) *r is stable.*

Proof. Refer to [7, Proposition 2.6] for the equivalence of 2) and 3). 1) $\Rightarrow$ 3). Let A be an r-torsion module. Since $r(E(A))$ is in $T(r)$ and is r-divisible, it is injective. Also $r(E(A)) \supseteq r(A) = A$ implies that $r(E(A))$ is essential in $E(A)$. Hence $r(E(A)) = E(A)$ and thus r is stable. 2) $\Rightarrow$ 1) is clear.

Now combining Theorem 1.8, Lemma 1.10 and Corollary 1.12 with Proposition 2.1, we readily obtain.

Corollary 2.2. [5, Proposition 2.1] *Let r be a left exact preradical. If r is larger than Z, then r_2 is a radical and stable.*

Proposition 2.3. *For a left exact preradical r for which every module in $F(r)$ is injective, the following conditions are equivalent*:

1) *Every r-divisible module is injective.*
2) *Every r-torsion r-divisible module is injective.*
3) *Every $\bar{r}$-torsion r-divisible module is injective.*

Proof. 1) $\Rightarrow$ 2) $\Rightarrow$ 3) are clear. 3) $\Rightarrow$ 1). Let A be an r-divisible module. Then $\bar{r}(E(A)/A) = O$ and $A \supseteq \bar{r}(E(A))$. Since $\bar{r}(E(A)) = \bar{r}(A)$ is r-divisible and $\bar{r}$-torsion, $\bar{r}(A)$ is injective. There exists a submodule B of A such that $A = B \oplus \bar{r}(A)$. Since r is left exact, we have $r(\bar{r}(A)) = r(A)$. And so $r(A) = r(B) \oplus r(\bar{r}(A)) = r(B) \oplus r(A)$. Thus B is in $F(r)$. Hence A is injective.

We give an example to show that the assumption that every module in $F(r)$ is injective cannot be removed in the preceding proposition.

Example 2.4. Let R be the 3×3 upper triangular matrix ring over a field K. If we put $I = \begin{pmatrix} K & K & K \\ O & K & K \\ O & O & O \end{pmatrix}$, then I is an idempotent two-sided ideal and R/I is flat as a right R-module. Let r be a left exact radical corresponding to the left Gabriel topology which has the smallest element I. Since R/I is a flat right R-module, r is stable. Thus r satisfies 2) by Proposition 2.1. Also since $I \supsetneqq soc\ ({}_RR)$, r is properly smaller than G. Hence r does not satisfy 1).

Let r be a preradical. We call r *splitting* if $r(A)$ is a direct summand of A for each module A. Also r is called *pseudo-cohereditary* if every homomorphic image of $A/(A \cap r(E(A)))$ is in $F(r)$ for each module A.

We quote the following.

Lemma 2.5. [6, Theorem 3.3 and Corollary 3.5]

(1) *For an idempotent preradical r, it is pseudo-cohereditary if and only if every r-weakly divisible module is r-divisible.*

(2) *For a left exact preradical r, it is pseudo-cohereditary if and only if r is an exact radical.*

For a class $\mathfrak{C}$ of modules, we put
$\mathfrak{C}^l = \{A \in R\text{-mod} \mid \mathrm{Hom}_R(A,X) = O \text{ for all } X \text{ in } \mathfrak{C}\}$ and
$\mathfrak{C}^r = \{A \in R\text{-mod} \mid \mathrm{Hom}_R(X,A) = O \text{ for all } X \text{ in } \mathfrak{C}\}$.

Theorem 2.6. *For a left exact preradical r, the following conditions are equivalent:*

1) *Every r-weakly divisible module is injective.*

2) *r is splitting (stable) and every module in $F(r)$ is injective.*

3) *r is splitting (stable) and $R/r(R)$ is a semisimple artinian ring.*

4) *r is pseudo-cohereditary and every r-divisible module is injective.*

5) *$(T(r)^l, T(r), F(r))$ is a 3-fold torsion theory with length 2 and r is larger than Z.*

6) *$(T(r), F(r), F(r)^r)$ is a 3-fold torsion theory with length 2 and r is larger than Z.*

Proof. 1) $\Rightarrow$ 2). Since r is a left exact preradical, $F(r)$ is closed under injective hulls and so every module in $F(r)$ is r-weakly divisible. By assumption it is injective. Now let A be a module. Since r is stable by [7, Proposition 2.6], if $r(A)$ is essential in A, then $r(A) = A$. Thus we assume that $r(A)$ is not essential in A. Then there exists a submodule B of A such that $r(A) \oplus B$ is essential in A. Since $r(A) \cap B = O$, B is in $F(r)$. Thus B is injective. Hence there exists a submodule C of A such that

$A = B \oplus C$. Since $r(A) = r(C)$, $r(A) \subseteq C$. Also since $r(A)$ is essential in C, $C = r(C)$. Hence $C = r(A)$, namely, $A = r(A) \oplus B$. Therefore r is splitting. 2) $\Rightarrow$ 3) follows from [1, Theorem 3.2]. 3) $\Rightarrow$ 1). Let A be an r-weakly divisible module. Since r is splitting, there exists a submodule B of A such that $A = r(A) \oplus B$. B is in $F(r)$ and so is injective by [2, Theorem 2.2]. Since $r(E(A)) = r(A)$, $r(A)$ is injective. Hence A is injective. 1) $\leftrightarrow$ 4) follows from Lemma 2.5. 1) $\Rightarrow$ 5). $T(r)^l$ is a hereditary torsion class by [7, Theorem 2.7]. Also r is a cotorsion radical by Lemma 2.5. Thus $T(r)^l = F(r)$ by [3, Theorem 2.7]. 5) $\Rightarrow$ 6). Since $T(r)^l = F(r)$, $T(r) = T(r)^{lr} = F(r)^r$. 6) $\Rightarrow$ 4) follows from Corollary 1.12 and Lemma 2.5.

Corollary 2.7. *For a left exact radical r, the following conditions are equivalent*:

1) *Every r-weakly divisible module is injective.*

2) *Every r-divisible module is injective and every module in $F(r)$ is injective.*

3) *Every r-torsion r-divisible module is injective and every module in $F(r)$ is injective.*

Proof. The implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3) are clear. 3) $\Rightarrow$ 1). By Proposition 2.1, r is stable. Therefore r is splitting by [1, Theorem 3.2]. Thus the implication follows from Theorem 2.6.

Corollary 2.8. *For a left exact radical larger than G, the following conditions are equivalent*:

1) *Every r-weakly divisible module is injective.*
2) *r is pseudo-cohereditary.*
3) *Every module in $F(r)$ is injective.*
4) *$R/r(R)$ is a semisimple artinian ring.*
5) *$(T(r)^l, T(r), F(r))$ is a 3-fold torsion theory with length 2.*
6) *$(T(r), F(r), F(r)^r)$ is a 3-fold torsion theory with length 2.*

Now we fix a module M. Let T be the left annihilator of M in R and let r be a left exact preradical corresponding to the left linear topology which has the smallest element T.

We call a module A *strongly M-injective* if every homomorphism of any submodule of M^J into A can be extended to a homomorphism of M^J into A for any index set J.

A strongly M-injective module is nothing but an r-weakly divisible module [7, Theorem 1.3]. By Theorem 2.6 we readily obtain.

Corollary 2.9. *The following conditions are equivalent*:

1) *Every strongly M-injective module is injective.*

2) $T = Re$ *for some central idempotent* e *in* R *and* T *is a direct summand of* $soc({}_RR)$.

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REFERENCES

[1] E. P. ARMENDARIZ : Quasi-injective modules and stable torsion classes, Pacific J. Math. 31 (1969), 277−280.

[2] P. E. BLAND : Divisible and codivisible modules, Math. Scand. 34 (1974), 153−161.

[3] Y. KURATA : On an n-fold torsion theory in the category ${}_R$M , J. Algebla 22 (1972), 559−572.

[4] Y. KURATA and T. SHUDŌ : On linear topologies having the smallest elements, Hokkaido Math. J. 10, Special Issue (1981), 424−434.

[5] J. N. MANOCHA : On rings with essential socle, Comm. Algebla, 4 (11)(1976), 1077−1086.

[6] S. MORIMOTO : Weakly divisible and divisible modules, Tsukuba J. Math. 6 (1982), 195−200.

[7] S. MORIMOTO : Note on strongly M-injective modules, Math. J. Okayama Univ. 25 (1983), 165−171.

[8] Bo STENSTRÖM : Rings of Quotients, Grundl. Math. Wiss. 217, Springer-Verlag Berlin, 1975.

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