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RINGS OF QUOTIENTS FOR RINGS WITH SETS OF LOCAL UNITS

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Introduction. In this paper we introduce suitable notion of rings of quotients for rings with sets of local units and prove that the rings of quotients of two Morita equivalent rings with sets of local units are again rings with sets of local units and they are Morita equivalent. Since it has been observed in [6] that two rings with sets of local units are Morita equivalent if and only if they are the right and left operator rings of a gamma ring with right and left local units, we construct the gamma ring of quotients for gamma rings with right and left local units. Using our construction of rings of quotients for rings with sets of local units which coincide with the rings of quotients for rings with unities ([8]) we prove that right and left operator rings of a gamma ring of quotients are nothing but the rings of quotients for the right and left operator rings of the original gamma ring. Hence we extend the result of Turnidge [9, Th. 2.4] to rings with sets of local units.

1. Preliminaries. Throughout this paper all rings considered are associative without assuming the existence of unity elements. We recall the following definitions and theorems which will be needed later.

Definition 1.1. Let R be any ring. A set E of commuting idempotents in the ring R is called a *set of local units* for R (abreviated slu) if for each $r \in R$ there exists $e \in E$ with $er = re = r$.

Let E be a set of local units for R. Then for $e_1, e_2 \in E$, $e_1 \leq e_2 \Leftrightarrow e_1e_2 = e_1$ defines a binary relation on E. This binary relation '$\leq$' is a partial order relation on E and $(E, \leq)$ is an upward directed set. For $e_1 \leq e_2$ in E

$$\dot{e}_{2e_1} : e_1Re_1 \to e_2Re_2$$

is a canonical injection which preserves addition and multiplication and

$$R = \varinjlim_E eRe \qquad ([1])$$

Definition 1.2. Let R be a ring and let I be an upward directed set. Suppose that for each $i \in I$ there exists a left R-module X_i, and for each pair $i \leq j$ in I there exist

$$A_{ji} : X_i \to X_j \text{ and } B_{ij} : X_j \to X_i.$$

We call the collection $\{X_i, A_{ji}, B_{ij} \mid i \in I\}$ *compatible* in the category of left R-modules in case

(1.2.1) $A_{ii} = B_{ii} = 1_{X_i}$ for each $i \in I$

(1.2.2) $A_{kj}A_{ji} = A_{ki}$ and $B_{ij}B_{jk} = B_{ik}$ for each $i \leq j \leq k$ in I

(1.2.3) $B_{ij}A_{ji} = 1_{X_i}$ for each $i \leq j$ in I

(1.2.4) $A_{kj}B_{jk}A_{ki}B_{ik} = A_{ki}B_{ik}A_{kj}B_{jk}$ for each $k \geq i, j$ in I.

Definition 1.3. Let $\{X_i, A_{ji}, B_{ij} \mid i \in I\}$ be compatible in R-Mod. By

$$\varinjlim_I \mathrm{End}_R(X_i)$$

we mean the limit as constructed in [1, Prop. 2.3].

Definition 1.4. Let R be a ring with slu. By a *progenerator* for R we mean a compatible set $\{X_i, A_{ji}, B_{ij} \mid i \in I\}$ in R-Mod such that

(a) for each $i \in I$, X_i is a finitely generated projective left R-module

(b) $X = \varinjlim_I (X_i, A_{ji})$ is a generator for R-Mod.

We need the following

Theorem 1.1 ([1, Th. 4.2]). *Let R and S be two rings with slu. Then R and S are Morita equivalent if and only if there exists a progenerator $\{X_i, A_{ji}, B_{ij} \mid i \in I\}$ for R such that*

$$S = \varinjlim_I \mathrm{End}_R(X_i).$$

Definition 1.5. Let A and Γ be two additive abelian groups. If there exists a mapping

$$A \times \Gamma \times A \to A$$

($(a, \alpha, b) \to a\alpha b$ for all $a, b \in A$ and $\alpha \in \Gamma$) which is additive in each variable and associative, then we call A a *Γ-ring*.

Definition 1.6 ([4]). Let A be a Γ-ring. Consider the mapping

$$[\alpha, a] : A \to A$$

defined by $b[\alpha, a] = b\alpha a$ for fixed $\alpha \in \Gamma$ and $a \in A$. Then $[\alpha, a] \in \mathrm{End}(A)$. Again the mapping

$$\Gamma \times A \to \mathrm{End}(A)$$

defined by $(\alpha, a) \to [\alpha, a]$ is $\mathbf{Z}$-bilinear. Hence there exists a linear mapping

$$\Gamma \otimes_Z A \to \mathrm{End}(A)$$

which takes $\sum_i (\alpha_i \otimes a_i) \to \sum_i [\alpha_i, a_i]$ for $\alpha_i \in \Gamma$ and $a_i \in A$. Due to the associativity of A, the image of $\Gamma \otimes_Z A$ in $\mathrm{End}(A)$ is an associative ring with the following rule of multiplication

$$\textstyle\sum_i [\alpha_i, a_i] \sum_j [\beta_j, b_j] = \sum_{i,j} [\alpha_i, a_i\beta_j b_j].$$

we denote this ring by $\mathrm{R}(A, \Gamma)$ or simply by R and call it the *right operator ring* of A. Similarly, we can construct the *left operator ring* of A, denoted $\mathrm{L}(A, \Gamma)$ or simply by L, by considering the mapping

$$[a, \alpha] : A \to A$$

defined by $[a, \alpha]b = a\alpha b$ for fixed $\alpha \in \Gamma$ and $a \in A$, and with the following rule of multiplication

$$\textstyle\sum_i [a_i, \alpha_i] \sum_j [b_j, \beta_j] = \sum_{i,j} [a_i\alpha_i b_j, \beta_j].$$

Then A is a faithful L-R bimodule.

Let A be a Γ-ring with the left and right operator rings L and R. Then

Definition 1.7. Let A be a Γ-ring. Then A is said to be a Γ-ring with *right local units* (resp. *left local units*) if the following conditions are satisfied:

1) For every $a \in A$ and $\beta \in \Gamma$,

(a) there exist $y_i (1 \leq i \leq n)$ in A and $\alpha_i (1 \leq i \leq n)$ in Γ (resp. there exist $y_p (1 \leq p \leq m)$ in A and $\alpha_p (1 \leq p \leq m)$ in Γ) such that

$$a \textstyle\sum_i [\alpha_i, y_i] = a \, (\text{resp. } \sum_p [y_p, \alpha_p] a = a);$$

(b) $y_j \sum_i [\alpha_i, y_i] = y_j$, $j \in \{1, 2, \ldots, n\}$ (resp. $\sum_p [y_p, \alpha_p] y_q = y_q$ for $q \in \{1, 2, \ldots, m\}$);

(c) $(x \sum_i [\alpha_i, y_i])\beta a = x\beta a$ (resp. $a\beta(\sum_p [y_p, \alpha_p] x) = a\beta x$) for all x in A.

2) Given a_1, a_2 in A (resp. b_1, b_2 in A) and β_1, β_2 in Γ (resp. θ_1, θ_2 in Γ), there exist $\sum_i [\alpha_i, y_i]$, $\sum_j [\delta_j, w_j]$ in R (resp. $\sum_p [y_p, \alpha_p]$, $\sum_q [w_q, \beta_q]$ in L) respectively satisfying (1) and for every x in A

$$x(\textstyle\sum_i [\alpha_i, y_i] \sum_j [\delta_j, w_j]) = x(\sum_j [\delta_j, w_j] \sum_i [\alpha_i, y_i])$$

(resp. $(\sum_p [y_p, \alpha_p] \sum_q [w_q, \beta_q])x = (\sum_q [w_q, \beta_q] \sum_p [y_p, \alpha_p]x)$).

We record the following theorems, without proof, for reference.

Theorem 1.2 ([7, Lemmas 2.2 and 2.4]). *Let A be a Γ-ring with right and left local units. Then the right and left operator rings of A are rings with slu.*

Theorem 1.3 ([7, Th. 2.1]). *Let A be a Γ-ring with right and left local units. Then the right and left operator rings R and L of A are Morita equivalent rings.*

Definition 1.8. A Γ-ring A is said to be a *left* (resp. *right*) *Γ-weakly semiprime* Γ-ring if $[x, \Gamma] = 0$ (resp. $[\Gamma, x] = 0$) for $x \in A$ implies $x = 0$. A is said to be *Γ-weakly semiprime* if it is both left and right Γ-weakly semiprime.

Definition 1.9. A Γ-ring A is said to be *weakly semiprime* if it is Γ-weakly semiprime and further satisfies the condition $A\Gamma A = A$.

Definition 1.10 ([6, p. 5]). Assume that a ring S satisfies the conditions

(1.10.1) $S^2 = S$

(1.10.2) $Sa = 0$ or $aS = 0$ implies $a = 0$, $a \in S$.

Then two rings S and T satisfying the above conditions are said to be Γ-context equivalent, $S \underset{\Gamma}{\sim} T$, if there exists a weakly semiprime Γ-ring A such that its left and right operator rings are isomorphic to S and T respectively.

2. Since it has been observed in [7, Th. 2.2] that given two rings R and S with slu which are Morita equivalent, then there exists a Γ-ring with right and left local units (for a suitably chosen Γ) whose left and right operator rings are isomorphic to R and S respectively, we begin with the study of localization for gamma rings.

Throughout this chapter by a Γ-ring A we mean A is a Γ-ring with left and right local units. If L and R designate the left and right operator rings of A, then L and R are rings with slu (Th. 1.2), say F and E respectively.

The following results are easily obtained as in the case of Γ-rings with left and right unities [4, pp. 191—193].

Lemma 2.1. (a) *If A is a Γ-ring with left and right local units, then $A\Gamma A = A$ and A is a weakly semiprime Γ-ring.*

(b) *The lattices of all right ideals of A and its left operator ring L are isomorphic via the mapping $I \to I^{+'}$, where I is a right ideal of A and*

$$I^{+'} = \{\textstyle\sum_i [a_i, \alpha_i] \in L \mid (\sum_i [a_i, \alpha_i])A \subseteq I\}.$$

(c) *The lattices of all left ideals of A and its right operator ring R are isomorphic via the mapping $J \to J^{*'}$, where J is a left ideal of A and*

$$J^{*'} = \{\textstyle\sum_j [\beta_j, b_j] \in R \mid A(\sum_j [\beta_j, b_j]) \subseteq J\}.$$

(d) *The lattices of all two sided ideals of A and L are isomorphic via the mapping $I \to I^{+'}$, where I is a two sided ideal of A. By symmetry, we have*

(e) *The lattices of all two sided ideals of A and R are isomorphic via the mapping $J \to J^{*'}$, where J in a two sided ideal of A.*

We recall Gabriel topology for a Γ-ring.

Definition 2.1. A nonempty family $\mathcal{F}(A)$ of right ideals of A is said to be a *topology* on A, if

(1) $I \in \mathcal{F}(A)$ implies $(I : x)_\alpha \in \mathcal{F}(A)$ for all $x \in A$ and $\alpha \in \Gamma$,

(2) $I \in \mathcal{F}(A)$, $I \subset J$ implies $J \in \mathcal{F}(A)$ for all right ideals J of A,

(3) $I, J \in \mathcal{F}(A)$ implies $I \cap J \in \mathcal{F}(A)$,

(4) $(I : x)_\alpha \in \mathcal{F}(A)$ for all $\alpha \in \Gamma$ and $x \in J$, $J \in \mathcal{F}(A)$ implies $I \in \mathcal{F}(A)$ where

$$(I : x)_\alpha = \{y \in A \mid x\alpha y \in I\}.$$

Then we obtain the following results as in the case of Γ-rings with left and right unities [7, pp. 34—44].

Proposition 2.1. (a) *Let $\mathcal{F}(A)$ be a topology on A. Then*

$$\mathcal{F}(R) = \{\textit{right ideals } P \textit{ of } R \mid P^{(\alpha)} \in \mathcal{F}(A) \textit{ for all } \alpha \in \Gamma\}$$

is a topology on R where,

$$P^{(\alpha)} = \{x \in A \mid [\alpha, x] \in P\}.$$

(b) *Let $\mathcal{F}(R)$ be a topology on R. Then*

$$\mathcal{F}(A) = \{\textit{right ideals } I \textit{ of } A \mid [I : x]_R \in \mathcal{F}(R) \textit{ for all } x \in A\}$$

is a topology on A where,

$$[I : x]_R = \{r \in R \mid xr \in I\}.$$

Moreover, we have

(c) *Let $\mathcal{F}(A)$ be a topology on A. Then*

$$\mathcal{F}(L) = \{\textit{all right ideals } I^{+'} \mid I \in \mathcal{F}(A)\}$$

is a topology on L where,

$$I^{+'} = \{\textstyle\sum_i [a_i, \alpha_i] \in L \mid (\sum_i [a_i, \alpha_i])A \subseteq I\}.$$

(d) *Let $\mathcal{F}(L)$ be a topology on L. Then*

$$\mathcal{F}(A) = \{Q^+ \mid Q \textit{ is a right ideal in } L \textit{ and } Q \in \mathcal{F}(L)\}$$

is a topology on A where,

$$Q^+ = \{x \in A \mid [x, \Gamma] \subset Q\}.$$

Combining all the above propositions, we obtain the following

Theorem 2.1. *There is a one to one order preserving correspondence between the topologies on A; L; R.*

Proof. Follows from (a), (b), (c) and (d) of Prop. 2. 1.

Let $\mathcal{F}(A)$ be a topology on A. Then there exist corresponding topologies $\mathcal{F}(R)$ and $\mathcal{F}(L)$ on R and L respectively.

For any $M \in \mathrm{ob}^{\text{Mod-}R}$, we define

$$\mathrm{t}(M) = \{m \in M \mid \mathrm{Ann}_R(m) \in \mathcal{F}(R)\}.$$

If $\mathrm{Ann}_R^{(\alpha)}(m) = \{a \in A \mid m\alpha a = m[\alpha, a] = 0\}$, then it is easily seen that

$$\mathrm{t}(M) = \{m \in M \mid \mathrm{Ann}_R^{(\alpha)}(m) \in \mathcal{F}(A) \text{ for all } \alpha \in \Gamma\}.$$

Likewise

$$\mathrm{t}'(M) = \{m \in M \mid \mathrm{Ann}_L^{(\alpha)}(m) \in \mathcal{F}(A) \text{ for all } \alpha \in \Gamma\},$$

where $\mathrm{Ann}_L^{(\alpha)}(m) = \{a \in \mathrm{A} \mid a\alpha m = [a, \alpha]m = 0\}$ for $M \in \mathrm{ob}^{\text{Mod-}L}$. Since A is an R-module, as in [7, Lemma 3. 1. 25], we have

Proposition 2.2. *Let $\mathcal{F}(A)$ be a topology on A and $\mathcal{F}(R)$ and $\mathcal{F}(L)$ be the corresponding topologies on R and L respectively. If* t *and* t′ *are the functors, corresponding to the respective topologies, on* Mod-*R and* Mod-*L, then*

(2.2.1) $\mathrm{t}(A)$ *is a two sided ideal of* A.
(2.2.2) $\mathrm{t}(R) = (\mathrm{t}(A))^{*'} = [\Gamma, \mathrm{t}(A)]$.
(2.2.3) $\mathrm{t}'(L) = (\mathrm{t}(A))^{+'} = [\mathrm{t}(A), \Gamma]$.

We call an R-module M torsion free if $\mathrm{t}(M) = 0$. Then we have the following

Corollary 2.1. *The following statements are equivalent.*
(C1) *A is* t*-torsion free.*
(C2) *R is* t*-torsion free.*
(C3) *L is* t'*-torsion free.*

We observe that if S' is a two sided proper ideal in a ring S with slu U, then $U \not\subset S'$. Then $\bar{S} = S/S'$ becomes a ring with slu $\bar{U}$. If A is not t-torsion free, then $\bar{A} = A/\mathrm{t}(A)$ represents the quotient Γ-ring of A where the multiplication is defined by $\bar{x}a\bar{y} = \overline{xay} = xay + \mathrm{t}(A)$ ([2]) for all $\bar{x}$, $\bar{y} \in \bar{A}$ and $\alpha \in \Gamma$. Let $\mathrm{R}(\bar{A}, \Gamma)$ and $\mathrm{L}(\bar{A}, \Gamma)$ designate the respective right and left operator rings of the Γ-ring $\bar{A}$, then $\mathrm{R}(\bar{A}, \Gamma) = [\Gamma, \bar{A}]$ and $\mathrm{L}(\bar{A}, \Gamma) = [\bar{A}, \Gamma]$ respectively.

On the other hand if $\bar{R} = R/(\mathrm{t}(A))^{*'}$ and $\bar{L} = L/(\mathrm{t}(A))^{+'}$, then both $\bar{R}$ and $\bar{L}$ are rings with slu $\bar{E}$ and $\bar{F}$ respectively and we have the following

Lemma 2.2. *The mappings*

$$\pi_1 : \mathrm{R}(\bar{A}, \Gamma) \to \bar{R}$$

defined via $\pi_1(\sum_i [\alpha_i, \bar{x}_i]) = \sum_i \overline{[\alpha_i, x_i]}$ *and*

$$\pi_2 : \mathrm{L}(\bar{A}, \Gamma) \to \bar{L}$$

defined via $\pi_2(\sum_j [\bar{y}_j, \beta_j]) = \sum_j \overline{[y_j, \beta_j]}$ *are ring isomorphisms.*

3. Having established the necessary one to one correspondence between topologies on A, L and R, we now proceed to the construction of right Γ-ring of quotients for Γ-rings with sets of local units.

Let $\mathcal{F}(A)$, $\mathcal{F}(R)$ and $\mathcal{F}(L)$ be (fixed) corresponding topologies on A, R and L and let t, t' be the corresponding functors defined in Mod-R and Mod-L. We recall the following

Definition 3.1. For any subset $X \subset \bar{R}$ and $\alpha \in \Gamma$, we define

$$X^{(\alpha)} = \{\bar{x} \in \bar{A} \mid \overline{[\alpha, x]} \in X\}.$$

The following lemma follows as in [7, Lemma 3. 2. 1].

Lemma 3.1. (a) *If P is a right ideal of R, then*
$(P^{(\alpha)}+\mathrm{t}(A))/\mathrm{t}(A)$ *is contained in* $((P+\mathrm{t}(R))/\mathrm{t}(R))^{\langle\alpha\rangle}$ *for all* $\alpha \in \Gamma$.

(b) *If I, $J \in \mathcal{I}(A)$ and $g: J \to \bar{A}$ is an R-homomorphism, then*

$$g^{-1}((I+\mathrm{t}(A))/\mathrm{t}(A)) \in \mathcal{I}(A).$$

(c) *If P, $Q \in \mathcal{I}(R)$ and $g: Q \to \bar{A}$ is an R-homomorphism, then*

$$g^{-1}((P^{(\alpha)}+\mathrm{t}(A))/\mathrm{t}(A)) \in \mathcal{I}(R) \text{ for all } \alpha \in \Gamma.$$

(d) *If P, $Q \in \mathcal{I}(R)$ and $g: Q \to \bar{A}$ is an R-homomorphism, then*

$$g^{-1}(((P+\mathrm{t}(R))/\mathrm{t}(R))^{\langle\alpha\rangle}) \in \mathcal{I}(R).$$

Let us recall that $\mathcal{I}(R)$ becomes a downwards directed set with the partial order relation '$\leq$' defined as $P \leq Q \Leftrightarrow Q \subseteq P$, P, $Q \in \mathcal{I}(R)$. For each $P \leq Q$, if we define

$$\Omega_{QP}: \mathrm{Hom}_R(P, \bar{R}) \to \mathrm{Hom}_R(Q, \bar{R})$$

by $\Omega_{QP}(g) = g \mid Q$, $g \in \mathrm{Hom}_R(P, \bar{R})$, then $\{\mathrm{Hom}_R(P, \bar{R}), \Omega_{QP}\}_{P \in \mathcal{I}(R)}$ becomes a directed system of abelian groups. Let

$$R' = \varinjlim_{P \in \mathcal{I}(R)} \mathrm{Hom}_R(P, \bar{R})$$

represent the direct limit of this directed system. Then R' is an abelian group.

We introduce the rule of multiplication in R' as in [8, IX(Lemma 1. 6)] by the following rule.

Let $\langle g \rangle$, $\langle h \rangle \in R'$ have the representations $g \in \mathrm{Hom}_R(P, \bar{R})$ and $h \in \mathrm{Hom}_R(Q, \bar{R})$ respectively. Then

$$\langle g \rangle \langle h \rangle = \langle \bar{g}h \rangle$$

where $\langle \bar{g}h \rangle$ is represented by the composite homomorphism

$$h^{-1}((P+\mathrm{t}(R))/\mathrm{t}(R)) \to (P+\mathrm{t}(R))/\mathrm{t}(R) \to R/\mathrm{t}(R).$$

With this rule of multiplication R' becomes an associative ring.

Let

$$R\mathrm{Hom}_R(P, \bar{R}) = \{\textstyle\sum_i r_i * g_i \mid r_i \in R,\ g_i \in \mathrm{Hom}_R(P, \bar{R})\}$$

where $(\sum_i r_i * g_i)(r) = \sum_i r_i g_i(r)$ for $r \in P$. Then $\{R\mathrm{Hom}_R(P, \bar{R}), \Omega_{QP}\}_{P \in \mathcal{F}(R)}$ is again a directed system and let

$$R'' = \varinjlim_{P \in \mathcal{F}(R)} R\mathrm{Hom}_R(P, \bar{R}).$$

Moreover, we have

Lemma 3.2. *R'' is a subring of R'.*

Proof. It is quite a routine work and hence omited.

For any element $r \in R$, we have a right R-homomorphism

$$h_r : R \to \bar{R}$$

defined by $h_r(r_1) = \overline{rr_1}$, $r_1 \in R$. Since R is a ring with slu E, for every $r \in R$, there exists an $e \in E$ such that $er = re = r$. So $(e * h_r)(r_1) = e(\overline{rr_1}) = (\overline{er})r_1 = \overline{rr_1} = h_r(r_1)$ implies $e * h_r = h_r \in R\mathrm{Hom}_R(P, \bar{R})$. In particular, $e * h_e = h_e \in R\mathrm{Hom}_R(P, \bar{R})$ for all $e \in E$. Hence we have a well defined homomorphism

$$\theta : R \to R'' = \varinjlim_{P \in \mathcal{F}(R)} R\mathrm{Hom}_R(P, \bar{R})$$

via $\theta(r) = \langle h_r \rangle$, which gives the following

Lemma 3.3. *θ is a ring homomorphism of R into R''. In particular, if R is torsion-free as a right R-module, then θ is a monomorphism.*

From now onwards we identify $\langle h_e \rangle$ with e in R and note that $\langle h_e \rangle$ in R'' is an idempotent and we have a natural inclusion

$$t_{e'e} : eR''e \to e'R''e'.$$

The set $\{eR''e, t_{e'e}\}_{e \in E}$ is a directed system of subrings of R''. If

$$R_{\mathcal{F}} = \varinjlim_{e \in E} eR''e$$

then $R_{\mathcal{F}} \subset R''$.

Proposition 3.1. *$R_{\mathcal{F}}$ is a ring with slu E.*

Proof. It follows by the construction of $R_{\mathcal{F}}$ and E itself will be the set of commuting idempotents.

By the one to one correspondence between the topologies of right ideals on R and L, there is a topology $\mathcal{F}(L)$ on L corresponding to the topology

$\mathcal{F}(R)$ on R. Then as in the previous case, if

$$L' = \varinjlim_{P' \in \mathcal{F}(L)} \mathrm{Hom}_L(P', \bar{L})$$

and

$$L'' = \varinjlim_{P' \in \mathcal{F}(L)} L\mathrm{Hom}_L(P', \bar{L}),$$

then L' is an associative ring and L'' is a subring of L'. Moreover, there is a ring homomorphism from L into L'' which is a ring monomorphism when L is torsion free as right L-module. If F is the slu for L, then

$$L_{\mathcal{F}} = \varinjlim_{f \in F} fL''f$$

and $L_{\mathcal{F}} \subset L''$. As in Proposition 3.1, we have

Proposition 3.2. *L is a ring with slu F.*

Let

$$A' = \varinjlim_{P \in \mathcal{F}(R)} \mathrm{Hom}_R(P, \bar{A})$$

which is an additive abelian group. By Lemma 3.1(d), if P, $Q \in \mathcal{F}(R)$ and $g: Q \to \bar{A}$ is an R-homomorphism, then

$$g^{-1}(((P+\mathrm{t}(R))/\mathrm{t}(R))^{\langle\alpha\rangle}) \in \mathcal{F}(R).$$

with the help of this, we define a composition

$$A' \times \Gamma \times A' \to A'$$

as follows. Let $\langle g \rangle$, $\langle h \rangle \in A'$ and $\alpha \in \Gamma$. If $\langle g \rangle$ and $\langle h \rangle$ have representations

$$g: P \to \bar{A} \text{ and } h: Q \to \bar{A}$$

respectively, we define, as in [7, Th. 3.2.4],

$$\langle g \rangle \alpha \langle h \rangle = \langle \bar{g} \phi_\alpha h \rangle$$

where the representation of $\langle \bar{g}\phi_\alpha h \rangle$ is given by the following composite homomorphism

$$h^{-1}(((P+\mathrm{t}(R))/\mathrm{t}(R))^{\langle\alpha\rangle}) \to ((P+\mathrm{t}(R))/\mathrm{t}(R))^{\langle\alpha\rangle} \to$$
$$(P+\mathrm{t}(R))/\mathrm{t}(R) \to A/\mathrm{t}(A)$$

where $\bar{g}: (P+\mathrm{t}(R))/\mathrm{t}(R) \to A/\mathrm{t}(A)$ is the homomorphism induced by $g: P \to \bar{A}$, ϕ_α is the restriction of $\phi_\alpha: \bar{A} \to \bar{R}$ ($\phi_\alpha(\bar{a}) = \overline{[\alpha, a]}$).

Since $h^{-1}(((P+\mathrm{t}(R))/\mathrm{t}(R))^{\langle\alpha\rangle}) \in \mathcal{J}(R)$, $\langle\bar{g}\phi_\alpha h\rangle \in A'$. This multiplication is well defined and the rest of the proof that A' is a Γ-ring follows exactly as in [7, Th. 3. 2. 4]. If

$$L\mathrm{Hom}_R(P, \bar{A}) = \{\textstyle\sum_j 1_j * \mu_j \mid 1_j \in L \text{ and } \mu_j \in \mathrm{Hom}_R(P, \bar{A})\}$$

then the set $L\mathrm{Hom}_R(P, \bar{A})$ is the set of all R-homomorphisms of the form $1 * \mu$ where $(1 * \mu)(r) = 1\mu(r) \in \bar{A}$. As $\mathcal{J}(R)$ is a downwards directed set, the collection $\{L\mathrm{Hom}_R(P, \bar{A})\}_{P\in\mathcal{J}(R)}$ forms a directed system of abelian groups and let

$$A'' = \varinjlim_{P\in\mathcal{J}(R)} L\mathrm{Hom}_R(P, \bar{A}).$$

Proposition 3.3. *A'' is a Γ-subring of A'.*

Proof. It is routine and hence omitted.

With each $a \in A$, we associate a map

$$h_a : R \to \bar{A}$$

defined by $h_a(r) = \bar{a}r$ for all $r \in R$. h_a is clearly an R-homomorphism. Now, as there exists an $f \in F \subset L$ such that $fa = af = a$, $f * h_a(r) = f(\bar{a}r) = (f\bar{a})r = \bar{a}r = h_a(r)$ implying that $f * h_a = h_a \in L\mathrm{Hom}_R(P, \bar{A})$. If

$$\pi : A \to A''$$

via $\pi(a) = h_a$, then π is a well defined mapping and

Lemma 3.4. *π is a Γ-ring homomorphism from A into A''. In particular if A is* t*-torsion free, then π is a ring monomorphism with* $\ker \pi = \mathrm{t}(A) = 0$.

Proof. π is a group homomorphism is clear. To prove it is a Γ-ring homomorphism, let $a, b \in A$ and $\alpha \in \Gamma$. Then $\langle h_a\rangle \alpha \langle h_b\rangle = \langle \bar{h}_a\phi_\alpha h_b\rangle$ is represented by the composite mapping

$$(h_b)^{-1}((R/\mathrm{t}(R))^{\langle\alpha\rangle}) \to (R/\mathrm{t}(R))^{\langle\alpha\rangle} \to R/\mathrm{t}(R) \to A/\mathrm{t}(A).$$

So $\langle h_{a\alpha b}\rangle$ and $\langle \bar{h}_a\phi_\alpha h_b\rangle$ coincide on $h_b^{-1}((R/\mathrm{t}(R))^{\langle\alpha\rangle})$, since if

$$r \in (h_b)^{-1}((R/\mathrm{t}(R))^{\langle\alpha\rangle}),$$

then

$$(\bar{h}_a\phi_\alpha h_b)(r) = h_a(\overline{[\alpha, br]}) = h_a([\alpha, br]) = \\ \bar{a}([\alpha, br]) = (\overline{a\alpha b})(r) = h_{a\alpha b}(r).$$

Hence $\langle h_{a\alpha b}\rangle = \langle h_a\rangle \alpha \langle h_b\rangle$ proving that π is a Γ-ring homomorphism. If A is t-torsion free, then π becomes a ring monomorphism, for if $\langle h_a\rangle = 0$, then $aP = 0$ for some $P \in \mathscr{F}(R)$. So $a \in \mathrm{t}(A)$. But when A is t-torsion free right R-module, $\mathrm{t}(A) = 0$, hence $a = 0$. Thus the Lemma follows.

If we designate the right and left operator rings of A'', as a Γ-ring, by $\mathrm{R}(A'', \Gamma)$ and $\mathrm{L}(A'', \Gamma)$ respectively then we have the following

Lemma 3.5. *R is embedded into* $\mathrm{R}(A'', \Gamma)$.

Proof. Let

$$\Phi : R \to \mathrm{R}(A'', \Gamma)$$

be defined as follows. If $r \in R$ and $r = \sum_i [\alpha_i, x_i]$, then $\Phi(r) = \sum_i [\alpha_i, \langle h_{x_i}\rangle]$. Then it is easily verified that Φ is a ring monomorphism.

Likewise, we can prove the following

Lemma 3.6. *L is embedded into* $\mathrm{L}(A'', \Gamma)$.

By Lemma 3.5, every element $e = \sum_i [\alpha_i, x_i] \in E$ goes to $\Phi(e) = \sum_i [\alpha_i, \langle h_{x_i}\rangle] \in \mathrm{R}(A'', \Gamma)$. Let

$$A''_e = \{ \langle \lambda \rangle \in A'' \mid \langle \lambda \rangle e = \langle \lambda \rangle \}.$$

A''_e is not empty, because $\langle h_{x_i}\rangle \in A''_e$ for $1 \leq i \leq n$, and A''_e is a left L-submodule of A'', since A'' is a left L-module. Further, for any $\langle \lambda \rangle \in A''_e$,

$$\begin{aligned}\langle \lambda \rangle \textstyle\sum_i [\alpha_i, \langle h_{x_i}\rangle] = \langle \lambda \rangle &\Rightarrow \textstyle\sum_i [\lambda, \alpha_i] \langle h_{x_i}\rangle = \langle \lambda \rangle \\ &\Rightarrow \textstyle\sum_i \Psi_{\alpha_i}(\langle \lambda \rangle) \langle h_{x_i}\rangle = \langle \lambda \rangle\end{aligned}$$

where $\Psi_{\alpha_i} \in \mathrm{Hom}_L(A'', L)$. Therefore A''_e is a finitely generated left L-submodule of A''.

For $e \leq e'$ in $E \subset R$, $ee' = e \Rightarrow \Phi(e) \leq \Phi(e')$ and $\Phi(e)\Phi(e') = \Phi(e)$ in $\mathrm{R}(A'', \Gamma)$. So we have an inclusion mapping

$$t_{e'e} : A''_e \to A''_{e'}.$$

Then $\{A''_e, t_{e'e}\}$ is a directed system of submodules of A''. If

$$A_{\mathscr{F}} = \varinjlim_{e \in E} A''_e,$$

then $A_{\mathscr{F}}$ is a Γ-subring of A''.

Let $\mathrm{R}(A_{\mathscr{F}}, \Gamma)$ and $\mathrm{L}(A_{\mathscr{F}}, \Gamma)$ designate the right and left operator rings of $A_{\mathscr{F}}$ as a Γ-ring. With each $[\lambda] \in A_{\mathscr{F}}$, $\lambda = \sum_i 1_i * \lambda_i : P \to \bar{A}$, and $\delta \in \Gamma$, we associate a map ϕ_δ such that

$$\phi_\delta \lambda : P \to \bar{R}$$

defined by $(\phi_\delta \lambda)(t) = \overline{[\delta, \lambda(t)]}$. Then

Lemma 3.7. (a) *Let* $\lambda = \sum_i 1_i * \lambda_i : P \to \bar{A}$ *and* $\delta \in \Gamma$, *then* $\phi_\delta \lambda \in R\mathrm{Hom}_R(P, \bar{R})$.

(b) *If* $\langle \lambda \rangle \in A_e''$ *and* $\delta \in \Gamma$, *then* $[\phi_\delta \lambda] \in R_\mathfrak{I}$.

(c) *If* $g \in R\mathrm{Hom}_R(P, \bar{R})$, *then for any* $x \in A$, $h_x g \in L\mathrm{Hom}_R(P, \bar{A})$.

Proof. (a) If $t \in P$, then $(\phi_\delta \lambda)(t) = \sum_i \overline{[\delta, 1_i \lambda_i(t)]}$. Since each 1_i is of the form $\sum_j [w_j^{(i)}, \beta_j^{(i)}]$, we have

$$(\phi_\delta \lambda)(t) = \sum_{i,j} \overline{[\delta, w_j^{(i)} \beta_j^{(i)} \lambda_i(t)]} = \sum_{i,j} \overline{[\delta, w_j^{(i)}][\beta_j^{(i)}, \lambda_i(t)]}$$
$$= (\sum_{i,j} r_{ij} \phi_{\beta_j^{(i)}} \lambda_i)(t)$$

implies $\phi_\delta \lambda = \sum_{i,j} r_{ij} * \phi_{\beta_j^{(i)}} \lambda_i \in R\mathrm{Hom}_R(P, \bar{R})$. (b) and (c) follow easily.

Hence we have the following

Proposition 3.4. *The mapping*

$$\Psi : \mathrm{R}(A_\mathfrak{I}, \Gamma) \to R_\mathfrak{I}$$

via $\Psi(\sum_m [\delta_m, [\lambda_m]]) = \sum_m [\phi_{\delta_m} \lambda_m]$ *is a ring isomorphism.*

The proof follows as in [7, Prop. 3. 2. 6]. Similarly, we can prove the following

Proposition 3.5. $\mathrm{L}(A_\mathfrak{I}, \Gamma) \cong L_\mathfrak{I}$.

Since both $R_\mathfrak{I}$ and $L_\mathfrak{I}$ are rings with slu, they satisfy the conditions in Def. 1. 10. Our aim is to prove $L_\mathfrak{I}$ and $R_\mathfrak{I}$ are Morita equivalent. First we prove the following

Proposition 3.6. $A_\mathfrak{I}$ *is a weakly semiprime* Γ*-ring.*

Proof. Follows from the fact that for every $x \in A_\mathfrak{I}$ there exists an $e \in E$ such that $xe = x$.

Now we prove the main result.

Theorem 3.1. $L_\mathfrak{I}$ *and* $R_\mathfrak{I}$ *are Morita equivalent rings.*

Proof. $R_\mathfrak{I}$ and $L_\mathfrak{I}$ are Γ-context equivalent since there exists a weakly semiprime Γ-ring $A_{\bar{\gamma}}$ such that

$$\mathrm{R}(A_{\bar{\gamma}}, \Gamma) \cong R_{\bar{\gamma}} \text{ and } \mathrm{L}(A_{\bar{\gamma}}, \Gamma) \cong L_{\bar{\gamma}},$$

Therefore, $\mathrm{R}(A_{\bar{\gamma}}, \Gamma)$ and $\mathrm{L}(A_{\bar{\gamma}}, \Gamma)$ are both rings with slu. Let

$$H_e = \{[\lambda] \in A_{\bar{\gamma}} \mid [\lambda]e = [\lambda]\}.$$

Then H_e can be easily seen to be a left $\mathrm{L}(A_{\bar{\gamma}}, \Gamma)$-submodule of $A_{\bar{\gamma}}$ which is finitely generated projective ([6, Th. 2. 1]). Moreover we have

$$_{\mathrm{L}(A_{\bar{\gamma}}, \Gamma)}A_{\bar{\gamma}} = \varinjlim_E (H_e, A_{e'e}) \text{ where}$$
$$A_{e'e} : H_e \to H_{e'}$$

is the natural inclusion. Then set $\{H_e, A_{e'e}, B_{ee'} \mid e \in E\}$, where

$$B_{ee'} : H_{e'} \to H_e$$

is the right multiplication by e, is compatible in $\mathrm{L}(A_{\bar{\gamma}}, \Gamma)$-Mod and $A_{\bar{\gamma}}$ is a generator for $\mathrm{L}(A_{\bar{\gamma}}, \Gamma)$-Mod. Therefore the set $\{H_e, A_{e'e}, B_{ee'} \mid e \in E\}$ is a progenerator for $\mathrm{L}(A_{\bar{\gamma}}, \Gamma)$-Mod (Def. 1. 4) and

$$e\mathrm{R}(A_{\bar{\gamma}}, \Gamma)\mathrm{e} \cong \mathrm{End}_{\mathrm{L}(A_{\bar{\gamma}}, \Gamma)}(H_e) \qquad ([6,\ \mathrm{Th.}\ 2.\ 1]).$$

Since $\mathrm{R}(A_{\bar{\gamma}}, \Gamma) \cong \varinjlim_E (eR''e)$ and $eR''e \cong \mathrm{End}_{\mathrm{L}(A\ ,\Gamma)}(H_e)$, we obtain

$$R \cong \varinjlim_E (\mathrm{End}_{\mathrm{L}(A_{\bar{\gamma}}, \Gamma)}(H_e)).$$

Hence $\mathrm{R}(A_{\bar{\gamma}}, \Gamma)$ and $\mathrm{L}(A_{\bar{\gamma}}, \Gamma)$ are Morita equivalent by Theorem 1. 1.

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