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SOME STUDIES ON GENERALIZED P.P. RINGS AND HEREDITARY RINGS

Dedicated to Professor Hisao Tominaga on his 60th birthday

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Introduction. In [3], G. M. Bergman gave characterizations of semihereditary and hereditary commutative rings in terms of their Pierce stalks. In preparation for these, he also gave characterizations of commutative p.p. rings by means of their Pierce stalks and classical quotient rings. The purpose of this paper is to present some generalizations of these results.

The first section is preliminaries on Boolean spectra and Pierce sheaves derived mainly from [10] and [12]. In section 2 we give a theorem on normal, generalized p.p. rings with classical quotient rings, from which the result of G. M. Bergman on commutative p.p. rings are deduced. In section 3 we give characterizations of semihereditary and hereditary normal rings with classical quotient rings.

Throughout this paper the word "ring" will mean "non-zero associative ring with identity element". For any ring R, (right or left) R-modules are unital and subrings of R contain the same identity element as R. $E(R)$ denotes the set of all idempotents of R. A ring R is called *normal* if every idempotent of R is central. For any non-empty subset A of a right (resp. left) R-module M, we set $r_R(A) = \{r \in R \mid Ar = 0\}$ (resp. $l_R(A) = \{r \in R \mid rA = 0\}$). If $r_R(a) = l_R(a)$ for an element $a \in R$, then we write it $\mathrm{ann}_R(a)$. A *right* (resp. *left*) *p.p. ring* is a ring in which every principal right (resp. left) ideal is projective. A ring R is called a *generalized right* (resp. *left*) *p.p. ring* if for any element $a \in R$, there is a positive Integer n such that a^nR (resp. Ra^n) is projective. A ring which is both (generalized) right and left p.p. is said to be a (*generalized*) *p.p. ring*.

1. Preliminaries. In this section we will summarize fundamental facts about Boolean spectra and Pierce sheaves which are needed in later sections. Let R be a ring and let $B(R)$ be the set of all central idempotents of R. $B(R)$ forms a Boolean ring under the operations

$$e \dot{+} f = e + f - 2ef,$$
$$e \cdot f = ef \text{ (product in } R\text{)}.$$

The *Boolean spectrum* of R is the set $\operatorname{Spec} B(R)$ consisting of all prime (equivalently, maximal) ideals of $B(R)$ endowed with the Zarisky topology, which is a totally disconnected, compact Hausdorff space. For simplicity we write $X(R)$ for Spec $B(R)$. For any central idempotent e of R we set $V_e = \{x \in X(R) \mid e \in x\}$. As is well known, the set $\{V_e \mid e \in B(R)\}$ coincides with the open and closed subsets of $X(R)$ and forms a basis for the topology of $X(R)$. The space $X(R)$ has the following partition property ([10, p. 12], see also [8, Chap. I]):

If $\{U_i\}_{i\in I}$ is an open covering of $X(R)$, then there is a finite set $\{V_1,\cdots, V_r\}$ of open and closed subsets of $X(R)$ such that

(1) for each $1 \leq j \leq r$, there is an $i \in I$ such that $V_j \subseteq U_i$;

(2) $V_j \cap V_k = \phi$ for $j \neq k$;

(3) $\bigcup_{j=1}^{r} V_j = X(R)$.

For e, $f \in B(R)$, we define $e \leq f$ if $ef = e$. Then $B(R)$ becomes an ordered set, in fact a lattice, where lattice operations are given by

$$e \vee f = e+f-ef,$$
$$e \wedge f = ef.$$

For any e, $f \in B(R)$, the following hold :

(a) $V_e \cup V_f = V_{ef}$.

(b) $V_e \cap V_f = V_{e\vee f}$.

(c) $V_e \subseteq V_f$ if and only if $f \leq e$, that is, $1-e \leq 1-f$.

(d) $X(R) = V_e \cup V_{1-e}$ and $V_e \cap V_{1-e} = \phi$.

We begin with the following elementary

Lemma 1. *Let R be a ring and let $\mathfrak{a}$ be a subset of $B(R)$. Then $\mathfrak{a}$ is an ideal of the Boolean ring $B(R)$ if and only if $\mathfrak{a}$ is an ideal of the lattice $B(R)$.*

Proof. The proof is immediate from identities $e \vee f = e(1-f)\dot{+}f$, $e\dot{+}f = e(1-f) \vee f(1-e)$ in $B(R)$.

Let R be a ring and let $\mathfrak{a}$, $\mathfrak{b}$ be ideals of the Boolean ring $B(R)$. As is easily seen, the sum $\mathfrak{a}\dot{+}\mathfrak{b} = \{e\dot{+}f \mid e \in \mathfrak{a},\ f \in \mathfrak{b}\}$ in $B(R)$ is contained in the sum $\mathfrak{a}+\mathfrak{b}$ in R and, if $\mathfrak{a}\mathfrak{b} = 0$, then they are equal. Although they do not coincide in general, for simplicity we write $\mathfrak{a}+\mathfrak{b}$ instead of $\mathfrak{a}\dot{+}\mathfrak{b}$. Similarly we do for any family of ideals of $B(R)$.

Let $\{e_1,\cdots,e_n\}$ be a finite set of orthogonal idempotents of a ring R.

The set $\{e_1,\cdots,e_n\}$ is said to be *complete* if $e_1+\cdots+e_n = 1$.

The next lemma is well known.

Lemma 2. *Let $e_1,\cdots,e_n \in B(R)$ and let $f_i = 1-e_i$ for $i = 1,\cdots,n$. Then $X(R) = \bigcup_{i=1}^{n} V_{e_i}$ and $V_{e_i} \cap V_{e_j} = \phi$ for $i \neq j$ if and only if $\{f_1,\cdots,f_n\}$ is a complete set of orthogonal idempotents.*

Let R be a ring and let M be a right R-module. The Pierce sheaf of R (resp. M) has $X(R)$ as the base space and has $R_x = R/Rx$ (resp. $M_x = M/Mx$) as the stalk at x for each $x \in X(R)$. The ring R is then isomorphic to the ring of global sections of this sheaf. The module M has an analogous sheaf representation [10, Theorems 4.4, 4.5, pp. 17−19]. For $m \in M$ and $x \in X(R)$, m_x denotes the image of m in the stalk M_x. The support of an element $m \in M$ is the set $\operatorname{supp} m = \{x \in X(R) \mid m_x \neq 0_x\}$, which is always closed (see Lemma 4 below). For any non-empty subset A of M, the support of A is defined to be $\operatorname{supp} A = \bigcup_{a \in A} \operatorname{supp} a$. For any $e \in B(R)$, it is easily verified that $\operatorname{supp} e = V_{1-e}$ and $\{a \in R \mid \operatorname{supp} a \subseteq V_e\} = R(1-e)$ (cf. Lemma 7 (2)).

Lemma 3 ([12, (2.8)]). *Let A be a non-empty finite subset of a right R-module M and let $x \in X(R)$. If $a_x = 0_x$ for all $a \in A$, then there is an $e \in x$ such that $a(1-e) = 0$ for all $a \in A$.*

Lemma 4 ([10, Lemma 4.3, p. 16] or [12, (2.9)]). *Let M be a right R-module and let $a, b \in M$, $x \in X(R)$. If $a_x = b_x$, then there is an $e \in B(R)$ with $x \in V_e$ such that $a_y = b_y$ for all $y \in V_e$.*

Lemma 5 ([12, (2.10)]). *Let A be a subset of $B(R)$. Then the following are equivalent:*

1) $X(R) = \bigcup_{e \in A} V_e$.

2) $B(R) = \sum_{e \in A} B(R)(1-e)$.

3) $R = \sum_{e \in A} R(1-e)$.

Lemma 6. *Let A be a subset of $B(R)$ and let $f \in B(R)$. Then the following are equivalent:*

1) $V_f = \bigcup_{e \in A} V_e$.

2) $B(R)(1-f)=\sum_{e\in A} B(R)(1-e)$.

3) $R(1-f)=\sum_{e\in A} R(1-e)$.

Proof. Easy consequence of Lemma 5.

Lemma 7 ([12, (2.9)]). *Let M be a right R-module and let $a, b \in M$, $e \in B(R)$.*

(1) *If $a_x=b_x$ for all $x \in V_e$, then $a(1-e)=b(1-e)$.*

(2) *If $a_x=b_x$ for all $x \in X(R)$, then $a=b$.*

Let $f: M \to M'$ be an R-module homomorphism. Then for every $x \in X(R)$, f induces an R_x-module homomorphism $f_x: M_x \to M'_x$. If f is a monomorphism, then so is f_x. Hence for any submodule N of M, we regard N_x as a submodule of M_x for every $x \in X(R)$.

Lemma 8 ([12, (2.11)]). *Let N be a submodule of a right R-module M. If $N_x=M_x$ for all $x \in X(R)$, then $N=M$.*

2. Generalized p.p. rings. Let M be a right module over a ring R and let $x \in X(R)$. Then the stalk M_x is a right R_x-module in the natural way. As for annihilators of elements of M_x in R_x, the following holds.

Lemma 9. *Let $a_1, \cdots, a_n$ be finitely many elements of a right R-module M. Then for every $x \in X(R)$, we have*

$$r_{R_x}(\{(a_1)_x, \cdots, (a_n)_x\}) = (r_R(\{a_1, \cdots, a_n\}))_x.$$

Proof. Since the inclusion "$\supseteq$" is obvious, we prove the inverse inclusion. Let r_x be an arbitrary element of $r_{R_x}(\{(a_1)_x, \cdots, (a_n)_x\})$. Then $(a_i r)_x = 0_x$ $(i=1, \cdots, n)$, whence there is an $e \in x$ such that $a_i r(1-e)=0$ for every $i=1, \cdots, n$ (Lemma 3). Consequently $r(1-e) \in r_R(\{a_1, \cdots, a_n\})$, so that $r_x \in (r_R(\{a_1, \cdots, a_n\}))_x$, as desired.

Lemma 10. *Let R be a ring with classical right quotient ring Q. Suppose that for any element $a \in R$ there is a positive integer n such that for every integer $m \geq n$, $\operatorname{supp} a^m = \operatorname{supp} a^n$ is open and closed, and that for any $x \in X(R)$ every zero-divisor of the stalk R_x is nilpotent. Then for any $x \in X(R)$, the stalk Q_x is a classical right quotient ring of R_x.*

Proof. Let $x \in X(R)$ and let a_x be an arbitrary non-zero-divisor of R_x.

By hypothesis there is a positive integer n such that for every integer $m \geq n$, $\operatorname{supp} a^m = \operatorname{supp} a^n = V_e$ with some $e \in B(R)$. Put $a' = a^n + e$. Then for every $y \in V_e$, $a'_y = a^n_y$ is non-nilpotent by the choice of n, hence it is a non-zero-divisor of R_y by hypothesis. If $y \notin V_e$, then $a'_y = 1_y$. Thus a'_y is a non-zero-divisor of R_y for every $y \in X(R)$. Consequently, by Lemma 7 (2), a' is a non-zero-divisor of R. Since $x \in V_e$ we have $a'_x = a^n_x$, whence it follows that a_x is invertible in Q_x. The rest of the proof is obvious.

A ring R is called *strongly π-regular* if it satisfies one of the following equivalent conditions (see, [5, Théorème 1]):

1) For every element $a \in R$, there is a positive integer n and an element $b \in R$ such that $a^{n+1}b = a^n$.

2) For every element $a \in R$, there is a positive integer n and an element $b \in R$ such that $ba^{n+1} = a^n$.

Proposition 1. *Let R be an algebra over a commutative ring C. Then the following are equivalent:*

1) *R is a strongly π-regular ring.*

2) *For any $x \in X(C)$, the stalk R_x is either zero or a strongly π-regular ring.*

Proof. 1) $\Rightarrow$ 2). Trivial.

2) $\Rightarrow$ 1). Assume 2) and let a be an arbitrary element of R. By hypothesis for every $x \in X(C)$ there is a positive integer $n(x)$ and an element $b^{(x)} \in R$ such that $a_x^{n(x)} = a_x^{n(x)+1} b_x^{(x)}$. By Lemma 4 there is an idempotent $e(x) \in C$ with $x \in V_{e(x)}$ such that $a_y^{n(x)} = a_y^{n(x)+1} b_y^{(x)}$ for all $y \in V_{e(x)}$. Since $X(C) = \bigcup_{x \in X(C)} V_{e(x)}$, by the partition property there are finitely many idempotents $e_1, \cdots, e_r \in C$ such that $V_{e_i} \cap V_{e_j} = \phi$ for $i \neq j$, $X(C) = \bigcup_{i=1}^{r} V_{e_i}$ and for every $i = 1, \cdots, r$, there is an $x_i \in X(C)$ with $V_{e_i} \subseteq V_{e(x_i)}$. Clearly we have $a_y^{n(x_i)} = a_y^{n(x_i)+1} b_y^{(x_i)}$ for all $y \in V_{e_i}$ $(i = 1, \cdots, r)$. Let $n = \max\{n(x_1), \cdots, n(x_r)\}$. Then it is easy to see that $a_y^n = a_y^{n+1} b_y^{(x_i)}$ for all $y \in V_{e_i}$ $(i = 1, \cdots, r)$. Put $b = \sum_{i=1}^{r} f_i b^{(x_i)}$, where $f_i = 1 - e_i$ $(i = 1, \cdots, r)$. As is easily verified, $b_y = (f_i)_y b_y^{(x_i)} = b_y^{(x_i)}$ for all $y \in V_{e_i}$ $(i = 1, \cdots, r)$. Hence we have $a_y^n = a_y^{n+1} b_y$ for all $y \in V_{e_i}$ $(i = 1, \cdots, r)$, that is, $a_x^n = a_x^{n+1} b_x$ for all $x \in X(C)$. By Lemma 7 (2) we conclude that $a^n = a^{n+1}b$, showing that R is a strongly π-regular ring.

Remark. Let R be a ring and let C be a subring of the center of R. Then, as is easily seen, the stalk of R at any point of $X(C)$ is non-zero. Furthermore for any prime ideal P of R, $P \cap B(C) = x \in X(C)$ and so R/P is a homomorphic image of R_x. Hence in this case Proposition 1 follows from [6, Theorem 2.1].

Let R be a ring. R is called *local* if $R/J(R)$ is a division ring, where $J(R)$ is the Jacobson radical of R. R is said to be a *π-regular ring* if for every element $a \in R$, there is a positive integer n and an element $b \in R$ such that $a^n b a^n = a^n$. As was shown by G. Azumaya [2, Corollary], every strongly π-regular ring is π-regular.

Theorem 1. *Let R be a normal ring with classical right quotient ring Q. Then the following are equivalent:*

1) *R is a generalized p.p. ring and for any $x \in X(R)$, the set of nilpotent elements of the stalk Q_x is invariant under right multiplication by elements of Q_x.*

2) (i) *For any $a \in R$, there is a positive integer n such that for every integer $m \geq n$, $\operatorname{supp} a^m = \operatorname{supp} a^n$ is open and closed.*

(ii) *For any $x \in X(R)$, every zero-divisor of the stalk R_x is nilpotent.*

(iii) *For any $x \in X(R)$, the set of nilpotent elements of the stalk Q_x is invariant under right multiplication by elements of Q_x.*

3) *For any $x \in X(R)$, the stalk Q_x is a local ring with Jacobson radical nil.*

4) *Q is a π-regular ring and $E(Q) = E(R)$.*

Proof. 1) $\Rightarrow$ 2). Assume 1) and let a be an arbitrary element of R. By [9, Corollary 4 and Lemma 3], there is a positive integer n and an idempotent $e \in R$ such that for every integer $m \geq n$, $\operatorname{ann}_R(a^m) = \operatorname{ann}_R(a^n) = Re$. Let $m \geq n$ be an arbitrary integer. Then by Lemma 9 (and its left-right symmetry) we have $\operatorname{ann}_{R_x}(a_x^m) = R_x e_x$ for every $x \in X(R)$. Evidently $a_x^m \neq 0_x$ if and only if $e_x \neq 1_x$. Hence we have $\operatorname{supp} a^m = \operatorname{supp}(1-e) = V_e$, which is open and closed. Since e is a central idempotent, we have $e_x = 0_x$ or 1_x, leading to $\operatorname{ann}_{R_x}(a_x^n) = 0_x$ or R_x. This implies a_x is either a non-zero-divisor or nilpotent.

2) $\Rightarrow$ 3). Assume 2) and take any $x \in X(R)$. Let $q_x = (ab^{-1})_x = a_x(b_x)^{-1}$ $(a, b \in R)$ be an arbitrary element of Q_x. If a_x is a zero-divisor,

then it is nilpotent by (ii), hence q_x is nilpotent by (iii). If a_x is a non-zero-divisor, then by Lemma 10 a_x is invertible in Q_x, and so q_x is invertible. Thus Q_x is a local ring with Jacobson radical nil.

3) ⇒ 4). Assuming 3), we first show that Q is a π-regular ring. For any $x \in X(R)$, we easily see that Q_x is a strongly π-regular ring, hence by Proposition 1 Q is a strongly π-regular ring. Therefore Q is a π-regular ring. Next we show that $E(Q) = E(R)$. Let $e \in E(Q)$ and let $x \in X(R)$. Since Q_x is a local ring, it has no idempotents other than 0_x and 1_x. Hence we have $e_x = \varepsilon_x^{(x)}$, where $\varepsilon^{(x)} = 0$ or 1. By Lemma 4 there is an idempotent $e(x) \in R$ with $x \in V_{e(x)}$ such that $e_y = \varepsilon_y^{(x)}$ for all $y \in V_{e(x)}$. Since $X(R) = \bigcup_{x \in X(R)} V_{e(x)}$, by the partition property there are finitely many idempotents $e_1, \cdots, e_r \in R$ such that $V_{e_i} \cap V_{e_j} = \phi$ for $i \neq j$, $X(R) = \bigcup_{i=1}^{r} V_{e_i}$ and $e_y = \varepsilon_y^{(i)}$ for all $y \in V_{e_i}$ $(i = 1, \cdots, r)$, where each $\varepsilon^{(i)} = 0$ or 1. By Lemma 7 (1) we have $(e - \varepsilon^{(i)})f_i = 0$, where $f_i = 1 - e_i$ $(i = 1, \cdots, r)$. Since $\sum_{i=1}^{r} f_i = 1$ (Lemma 2), we then have $0 = \sum_{i=1}^{r} (e - \varepsilon^{(i)})f_i = e - \sum_{i=1}^{r} \varepsilon^{(i)} f_i$. Hence $e = \sum_{i=1}^{r} \varepsilon^{(i)} f_i \in R$, as desired.

4) ⇒ 1). Assume 4) and let q be an arbitrary element of Q. Since Q is a normal π-regular ring, as in the proof of [9, Theorem 1] we have $q^n Q = Qq^n = Qe$ for some positive integer n and $e \in E(Q) = E(R)$. For any $x \in X(R)$ we then have $q_x^n Q_x = Q_x q_x^n = Q_x e_x$, and $e_x = 0_x$ or 1_x. If $e_x = 0_x$, then q_x is nilpotent. If $e_x = 1_x$, then q_x is invertible. Thus Q_x is a local ring with Jacobson radical nil. It is immediate from [9, Theorem 2] that R is a generalized p.p. ring.

From the preceding theorem we obtain the following three corollaries, the first of which contains [3, Lemma 3.1].

Corollary 1. *Let R be a normal ring with classical right quotient ring Q. Then the following are equivalent:*

1) *R is a p.p. ring.*

2) (i) *For any $a \in R$,* supp$a$ *is open and closed.*

(ii) *For any $x \in X(R)$, the stalk R_x is a (not necessarily commutative) integral domain.*

3) *For any $x \in X(R)$, the stalk Q_x is a division ring.*

4) *Q is a von Neumann regular ring and $E(Q) = E(R)$.*

Proof. This follows from Theorem 1, Lemma 10 and [7, Theorem 1].

Corollary 2. *Let R be a normal ring. Then the following are equivalent:*

1) *R is a generalized p.p. ring in which every non-zero-divisor is invertible.*

2) (i) *For any $a \in R$, there is a positive integer n such that for every integer $m \geq n$, $\operatorname{supp} a^m = \operatorname{supp} a^n$ is open and closed.*

(ii) *For any $x \in X(R)$, every zero-divisor of the stalk R_x is nilpotent.*

(iii) *Every non-zero-divisor of R is invertible.*

3) *For any $x \in X(R)$, the stalk R_x is a local ring with Jacobson radical nil.*

4) *R is a π-regular ring.*

The next contains [3, Corollary 3.2].

Corollary 3. *Let R be a normal ring. Then the following are equivalent:*

1) *R is a p.p. ring in which every non-zero-divisor is invertible.*
2) *For any $x \in X(R)$, the stalk R_x is a division ring.*
3) *R is a von Neumann regular ring.*

Remark. The equivalence of 1) and 3) in Corollary 3 has been obtained in [7, Corollary 2]. Incidentally, in [7, Remark], "torsion free" should be read as "torsion free divisible".

3. Semihereditary and hereditary rings. Let R be a subring of a ring S and let A be a submodule of S considered as a right R-module. As in the commutative case, we define A to be *invertible in* S if there are finitely many elements $a_1,\cdots,a_n \in A$, $s_1,\cdots,s_n \in S$ such that $s_iA \subseteq R$ for $i = 1,\cdots,n$ and $\sum_{i=1}^{n} a_is_i = 1$. It follows from [4, Proposition 3.1, p. 132] that if A is invertible in S, then A is finitely generated and projective as a right R-module.

Lemma 11. *Let R be a subring of a ring S and let $A = a_1R+\cdots+a_nR$ be a finitely generated submodule of the right R-module S with $a_i \in A$ ($i = 1,\cdots,n$). Then A is invertible in S if and only if there are elements $s_1,\cdots,$*

$s_n \in S$ *such that* $s_i A \subseteq R$ for $i = 1, \cdots, n$ *and* $\sum_{i=1}^{n} a_i s_i = 1$.

Proof. Since the if part is obvious, we prove the only if part. Suppose that A is invertible in S and let $a_1', \cdots, a_m' \in A$, $s_1', \cdots, s_m' \in S$ be elements such that $s_j' A \subseteq R$ for $j = 1, \cdots, m$ and $\sum_{j=1}^{m} a_j' s_j' = 1$. Write $a_j' = \sum_{i=1}^{n} a_i c_{ij}$ with $c_{ij} \in R$ $(i = 1, \cdots, n;\ j = 1, \cdots, m)$. Then it is easy to see that $\sum_{i=1}^{n} a_i \left(\sum_{j=1}^{m} c_{ij} s_j'\right) = 1$ and $\left(\sum_{j=1}^{m} c_{ij} s_j'\right) A \subseteq R$ $(i = 1, \cdots, n)$, completing the proof.

Proposition 2. *Let R be a subring of a ring S and let A be a submodule of the right R-module S. Then the following are equivalent:*

1) *A is invertible in S.*

2) *A is finitely generated over R and for any $x \in X(R)$, A_x* is invertible in the stalk S_x.

Proof. 1) $\Rightarrow$ 2). Trivial.

2) $\Rightarrow$ 1). Let $A = a_1 R + \cdots + a_n R$ with $a_i \in A$ $(i = 1, \cdots, n)$ and suppose that A_x is invertible in S_x for any $x \in X(R)$. Let $x \in X(R)$. By Lemma 11 there are elements $s_1^{(x)}, \cdots, s_n^{(x)} \in S$ such that $(s_i^{(x)})_x A_x \subseteq R_x$ for $i = 1, \cdots, n$ and $\sum_{i=1}^{n} (a_i)_x (s_i^{(x)})_x = 1_x$. By Lemma 4 there is an $e(x) \in B(R)$ with $x \in V_{e(x)}$ such that $(s_i^{(x)})_y A_y \subseteq R_y$ $(i = 1, \cdots, n)$ and $\sum_{i=1}^{n} (a_i)_y (s_i^{(x)})_y = 1_y$ for all $y \in V_{e(x)}$. (Note that A is finitely generated.) Since $X(R) = \bigcup_{x \in X(R)} V_{e(x)}$, by the partition property there are finitely many central idempotents $e_1, \cdots, e_r \in R$ such that $V_{e_j} \cap V_{e_k} = \phi$ for $j \neq k$, $X(R) = \bigcup_{j=1}^{r} V_{e_j}$ and for every $j = 1, \cdots, r$, there is an $x_j \in X(R)$ such that $V_{e_j} \subseteq V_{e(x_j)}$. Clearly we have $(s_i^{(x_j)})_y A_y \subseteq R_y$ $(i = 1, \cdots, n)$ and $\sum_{i=1}^{n} (a_i)_y (s_i^{(x_j)})_y = 1_y$ for all $y \in V_{e_j}$ $(j = 1, \cdots, r)$. Put $f_j = 1 - e_j$ for $j = 1, \cdots, r$. By Lemma 7 (1) we have $s_i^{(x_j)} f_j A \subseteq R$ $(i = 1, \cdots, n;\ j = 1, \cdots, r)$ and $(\sum_{i=1}^{n} a_i s_i^{(x_j)}) f_j = f_j$ $(j = 1, \cdots, r)$. Setting $s_i = \sum_{j=1}^{n} s_i^{(x_j)} f_j$ for $i = 1, \cdots, n$, we have $s_i A \subseteq R$ $(i = 1, \cdots, n)$ and $\sum_{i=1}^{n} a_i s_i = \sum_{j=1}^{r} f_j = 1$ (Lemma 2). Thus A is invertible in S.

Let R be a subring of a ring Q. R is called a *right order* in Q if every element of Q has the form ab^{-1} with some $a, b \in R$. From the proof of [11, Lemma 1.2] we obtain the following

Lemma 12. *Let R be a right order in a ring Q and let A be a submodule of the right R-module Q which contains an invertible element of Q. If A is R-projective, then A is invertible in Q.*

Let α be a cardinal. A ring is called *right α-hereditary* if every right ideal generated by at most α elements is projective as a right module. Thus a right semihereditary ring is a right α-hereditary ring for every finite cardinal α and a right hereditary ring is a right α-hereditary ring for every cardinal α.

Lemma 13. *Let α be an arbitrary cardinal. If R is a right α-hereditary ring, then for any $x \in X(R)$ the stalk R_x is also a right α-hereditary ring.*

Proof. Let A_x be a right ideal of R_x generated by a family of elements $\{(a_i)_x\}_{i\in I}$ of cardinal at most α, where $a_i \in A$ for each $i \in I$ and A is a right ideal of R containing Rx. Setting $B = \sum_{i\in I} a_i R$, we have $A = B + Rx$ and hence isomorphisms $B \otimes_R R_x \simeq B_x \simeq A_x$. By hypothesis B is R-projective and so A_x is R_x-projective. Therefore R_x is a right α-hereditary ring.

A ring which has a classical right quotient ring is called a *right Ore ring*. We can now prove the following theorem which contains [3, Theorem 4.1].

Theorem 2. *Let R be a normal, right Ore ring. Then R is right semihereditary if and only if*

(i) *R is p.p., and*

(ii) *for any $x \in X(R)$, the stalk R_x is right semihereditary.*

Proof. The only if part being immediate from Lemma 13, we prove the if part. Suppose that R satisfies conditions (i) and (ii). Let Q be a classical right quotient ring of R and let A be a non-zero finitely generated right ideal of R. By Corollary 1 supp A is open and closed and hence supp $A = V_e$ for some idempotent $e \in R$. Then we have $A \subseteq R(1-e)$ and, as is easily verified, $R(1-e)$ is a right Ore, normal p.p. ring. We now claim that for any $y \in X(R(1-e))$, $(R(1-e))_y$ is a right semihereditary ring. To see this, first note that $B(R) = B(Re) \oplus B(R(1-e))$. Hence for any $y \in X(R(1-e))$, $x = B(Re) \oplus y \in X(R)$ and $R_x = R/Rx = (Re \oplus R(1-e))/$

$(Re \oplus R(1-e)y) \simeq R(1-e)/R(1-e)y = (R(1-e))_y$. Thus $(R(1-e))_y$ is a right semihereditary ring. Viewing A as a right ideal of $R(1-e)$ as well as a right ideal of R, for any $y \in X(R(1-e))$ we have $x = B(Re) \oplus y \in V_e$ and $A_y \simeq A_x$, which is non-zero. Furthermore if A is $R(1-e)$-projective, then A is R-projective. Hence we may suppose that supp $A = X(R)$. For any $x \in X(R)$, R_x is a right semihereditary integral domain and Q_x is a classical right quotient ring of R_x (Corollary 1). Therefore A_x is a non-zero projective right ideal of R_x and so it is invertible in Q_x for all $x \in X(R)$ (Lemma 12). By Proposition 2 A is invertible in Q, hence is R-projective. This completes the proof.

Lemma 14. *Let M be a right R-module and let N, N' be submodules of M. Then for any $x \in X(R)$,*

$$N_x \cap N'_x = (N \cap N')_x.$$

Proof. Since the inclusion "$\supseteq$" is obvious, we prove the inverse inclusion. In order to do this, let $a_x = a'_x$ ($a \in N$, $a' \in N'$) be an arbitrary element of $N_x \cap N'_x$. By Lemma 3 there is an $e \in x$ such that $(a-a')(1-e) = 0$, that is, $a(1-e) = a'(1-e)$, which belongs to $N \cap N'$. Hence $a_x = (a(1-e))_x \in (N \cap N')_x$, as desired.

Lemma 15. *Let A, B be right ideals of a ring R. If* supp $A \cap$ supp $B = \phi$, *then $A \cap B = 0$. If R is a right Ore, normal p.p. ring, then the converse holds.*

Proof. Suppose first that $A \cap B \neq 0$. Then by Lemma 7 (2) there is an $x \in X(R)$ such that $(A \cap B)_x \neq 0_x$, whence $x \in$ supp $A \cap$ supp B. Therefore supp $A \cap$ supp $B \neq \phi$, as was to be shown.

Next suppose that R is a right Ore, normal p.p. ring and that supp $A \cap$ supp $B \neq \phi$. Take any $x \in$ supp $A \cap$ supp B. Then A_x and B_x are non-zero right ideals of R_x. By Corollary 1 R_x is a right Ore domain. Consequently by the preceding lemma we have $(A \cap B)_x = A_x \cap B_x \neq 0_x$, whence $A \cap B \neq 0$, completing the proof.

Let X be a set and let $\{X_i\}_{i \in I}$ be a family of subsets of X. Recall that $\{X_i\}_{i \in I}$ is a *partition* of X provided (i) $X_i \neq \phi$ for every $i \in I$, (ii) $X_i \cap X_j = \phi$ for $i \neq j$ and (iii) $X = \bigcup_{i \in I} X_i$.

Proposition 3. *Let R be a right Ore, normal p.p. ring and let A be a* *non-zero right ideal of R. Then there is a bijection from the set of all decompositions of A into direct sums of non-zero right ideals to the set of all partitions of* supp A *consisting of open sets. The map is given by*

$$A = \bigoplus_{i \in I} A_i \mapsto \{\mathrm{supp}\, A_i\}_{i \in I},$$

where each A_i *is a non-zero right ideal. The converse of this map is given by*

$$\{U_i\}_{i \in I} \mapsto A = \bigoplus_{i \in I} A_i,$$

where $\{U_i\}_{i \in I}$ *is a partition of* supp A *consisting of open sets and* $A_i = \{a \in A \mid \mathrm{supp}\, a \subseteq U_i\}$ for every $i \in I$.

Proof. The proof consists of three parts.

Part 1. Let $A = \bigoplus_{i \in I} A_i$, where each A_i is a non-zero right ideal of R. Then supp $A = \bigcup_{i \in I}$ supp A_i and the supp A_i are non-empty disjoint open sets (Lemma 7 (2), Corollary 1, Lemma 15). For every $i \in I$ set $B_i = \{a \in A \mid \mathrm{supp}\, a \subseteq \mathrm{supp}\, A_i\}$. As is easily seen, each B_i is a right ideal of R containing A_i. Hence by the modular law we have $B_i = A_i \oplus ((\bigoplus_{j \neq i} A_j) \cap B_i)$. On the other hand supp $B_i \subseteq$ supp A_i, hence in view of Lemma 15 and Lemma 7 (2) we get $B_i = A_i$.

Part 2. Let U be an open set of $X(R)$ and let $U = \bigcup_{i \in I} U_i$, where the U_i are disjoint open sets. Put $\mathfrak{A} = \{a \in R \mid \mathrm{supp}\, a \subseteq U\}$ and $\mathfrak{A}_i = \{a \in R \mid \mathrm{supp}\, a \subseteq U_i\}$ for every $i \in I$. Evidently $\mathfrak{A}$ and the $\mathfrak{A}_i$ are ideals of R. We claim that $\mathfrak{A} = \bigoplus_{i \in I} \mathfrak{A}_i$. Let $a \in \mathfrak{A}$. Since supp a is open and closed (Corollary 1), there is an idempotent $e \in R$ such that supp $a = V_e$. We then have supp $a = V_e = \bigcup_{i \in I} (V_e \cap U_i)$ and $X(R) = V_e \cup V_{1-e} = (\bigcup_{i \in I} (V_e \cap U_i)) \cup V_{1-e}$. Note that the $V_e \cap U_i$ and V_{1-e} are disjoint open sets. Hence every $V_e \cap U_i$, being the complement of an open set, is open and closed. Consequently $V_e \cap U_i = V_{e_i}$ for some idempotent $e_i \in R$. It follows from supp $a = V_e$ that $a \in R(1-e)$, which is equal to $\sum_{i \in I} R(1-e_i)$ by Lemma 6. Now $1-e_i \in \mathfrak{A}_i$ for every $i \in I$, since supp $(1-e_i) = V_{e_i} \subseteq U_i$. Hence $a \in \sum_{i \in I} \mathfrak{A}_i$, showing that $\mathfrak{A} = \sum_{i \in I} \mathfrak{A}_i$. That the sum is direct follows immediately from Lemma 15.

Part 3. Setting supp $A = U$, we see that U is an open set (Corollary 1). Let $U = \bigcup_{i \in I} U_i$, where the U_i are non-empty disjoint open sets. Put $\mathfrak{A} =$

$\{a \in R \mid \operatorname{supp} a \subseteq U\}$, $\mathfrak{A}_i = \{a \in R \mid \operatorname{supp} a \subseteq U_i\}$ and $A_i = A \cap \mathfrak{A}_i$ for every $i \in I$. We claim that $A = \bigoplus_{i \in I} A_i$. Since $A \supseteq \sum_{i \in I} A_i = \bigoplus_{i \in I} A_i$ is obvious, we have only to show that $A \subseteq \bigoplus_{i \in I} A_i$. For an arbitrary element $a \in A$, we know that supp $a = V_e$ for some idempotent $e \in R$ (Corollary 1). Noting that $\operatorname{supp}(1-e) = V_e \subseteq U$, we see that $1-e \in \mathfrak{A} = \bigoplus_{i \in I} \mathfrak{A}_i$. Hence we have $a \in A \cap R(1-e) = A(1-e) \subseteq \bigoplus_{i \in I} A_i$, whence $A \subseteq \bigoplus_{i \in I} A_i$, as was to be shown. We then have $U = \bigcup_{i \in I} \operatorname{supp} A_i$. On the other hand, $U = \bigcup_{i \in I} U_i$ is a disjoint union and supp $A_i \subseteq U_i$ for every $i \in I$, whence it follows that supp $A_i = U_i$ for all $i \in I$.

Lemma 16. *Let M be a right R-module and let $M = \bigoplus_{i \in I} M_i$, where $\{M_i\}_{i \in I}$ is a family of submodules of M. Then*

(1) $M_x = \bigoplus_{i \in I} (M_i)_x$ *for all* $x \in X(R)$.

(2) *For any element $m = \sum_{i \in I} m_i$ of M, where $m_i \in M_i$ $(i \in I)$ and $m_i = 0$ for all but a finite number of i, we have* supp $m = \bigcup_{i \in I}$ supp m_i.

Proof. (1) First note that $Mx = \bigoplus_{i \in I} M_i x$ for any $x \in X(R)$. The assertion follows from the isomorphism $M_x = M/Mx \simeq \bigoplus_{i \in I} (M_i)_x$ and the identification of each $(M_i)_x$ with its image in M_x.

(2) Immediate from (1).

The following theorem, which contains [3, Proposition 4.2], is essential in the proof of Theorem 4.

Theorem 3. *Let R be a right Ore, normal p.p. ring and let α be an infinite cardinal. Then the following are equivalent:*

1) *Every right ideal of R generated by at most α elements is a direct sum of finitely generated right ideals.*

2) (i) *For any $x \in X(R)$, the stalk R_x is right Noetherian.*

(ii) *For any non-zero-divisor $a \in R$, a_x is invertible in the stalk R_x for all but a finite number of $x \in X(R)$.*

(iii) *$B(R)$ is α-hereditary.*

Proof. 1) $\Rightarrow$ 2). Assuming 1), we first show (i). In order to do this, it suffices to prove that every countably generated right ideal of R_x ($x \in$

$X(R))$ is finitely generated. Let $x \in X(R)$ and let A_x be an arbitrary non-zero, countably generated right ideal of R_x, where A is a countably generated right ideal of R. By hypothesis we then have $A = \bigoplus_{i \in I} A_i$, where each A_i is a finitely generated right ideal of R. Since $x \in \operatorname{supp} A = \bigcup_{i \in I} \operatorname{supp} A_i$ and the supp A_i are disjoint (Lemma 15), there is a unique $i \in I$ such that $x \in \operatorname{supp} A_i$. Then $A_x = (A_i)_x$, which is finitely generated, as was to be shown.

Secondly, we show (ii). Let $a \in R$ be a non-zero-divisor and let A be any countably generated right ideal of R containing a. Noting that supp $A = X(R)$ (Lemma 3), which is compact, we see by Proposition 3 that A cannot be an infinite direct sum of non-zero right ideals. Hence by hypothesis it follows that A is finitely generated.

Now suppose that $\Gamma = \{x \in X(R) \mid a_x \text{ is not invertible in } R_x\}$ is an infinite set and let $\{x_i\}_{i=1}^{\infty}$ be a countably infinite subset of Γ. Take any $e_1 \in x_2 \backslash x_1$. Then either e_1 or $1-e_1$ is contained in infinitely many x_i. Let e_1 be contained in infinitely many x_i and let $\{x_i^{(1)}\}_{i=1}^{\infty}$ be the infinite subset of $\{x_i\}_{i=1}^{\infty}$ consisting of those x_i such that $e_1 \in x_i$. Then as in the above argument, we can pick an idempotent e_2 such that e_2 is contained in infinitely many $x_i^{(1)}$ but not contained in all of them. Continuing this process, we obtain an infinite ascending chain

$$aR \subsetneq aR + e_1R \subsetneq aR + e_1R + e_2R \subsetneq \cdots$$

of right ideals of R. (To show that the inclusion is strict everywhere, we use the fact that each R_x is right Noetherian.) Then $aR + \sum_{i=1}^{\infty} e_iR$ is a countably generated right ideal containing a, but it is not finitely generated. This contradiction completes the proof of (ii).

Finally we show (iii). Let $\mathfrak{a}$ be a non-zero ideal of $B(R)$ generated by at most α elements. Then $\mathfrak{a}R$ is an ideal of R having the same set of generators as $\mathfrak{a}$. Hence by hypothesis $\mathfrak{a}R = \bigoplus_{i \in I} \mathfrak{A}_i$, where each $\mathfrak{A}_i$ is a finitely generated right ideal of R. Then $\operatorname{supp} \mathfrak{a} = \operatorname{supp} \mathfrak{a}R = \bigcup_{i \in I} \operatorname{supp} \mathfrak{A}_i$, and the supp $\mathfrak{A}_i$ are disjoint open sets (Corollary 1, Lemma 15). Setting $\mathfrak{a}_i = \{a \in \mathfrak{a} \mid \operatorname{supp} a \subseteq \operatorname{supp} \mathfrak{A}_i\}$ for each $i \in I$, we see that every $\mathfrak{a}_i$ is an ideal of $B(R)$ and $\mathfrak{a} = \bigoplus_{i \in I} \mathfrak{a}_i$ (Proposition 3). For any $i \in I$ we have $\mathfrak{a}_iR \subseteq \mathfrak{a}R$ and supp $\mathfrak{a}_iR = \operatorname{supp} \mathfrak{a}_i \subseteq \operatorname{supp} \mathfrak{A}_i$, hence by Proposition 3 we get $\mathfrak{a}_iR \subseteq \mathfrak{A}_i$. On the other hand, $\mathfrak{a}R = \sum_{i \in I} \mathfrak{a}_iR = \bigoplus_{i \in I} \mathfrak{A}_i$, whence it follows that $\mathfrak{a}_iR = \mathfrak{A}_i$ for every

$i \in I$. We now claim that every $\mathfrak{a}_i$ is a principal ideal. Indeed, there is a finitely generated ideal $\mathfrak{b}_i$ of $B(R)$ with $\mathfrak{b}_i \subseteq \mathfrak{a}_i$ such that $\mathfrak{A}_i = \mathfrak{b}_i R$. Then $\mathfrak{b}_i$ is generated by an idempotent and hence is a direct summand of $\mathfrak{a}_i$. Let $\mathfrak{a}_i = \mathfrak{b}_i \oplus \mathfrak{c}_i$, where $\mathfrak{c}_i$ is an ideal of $B(R)$. We observe that supp $\mathfrak{b}_i =$ supp $\mathfrak{A}_i =$ supp $\mathfrak{a}_i =$ supp $\mathfrak{b}_i \bigcup$ supp $\mathfrak{c}_i$ and supp $\mathfrak{b}_i \bigcap$ supp $\mathfrak{c}_i = \phi$ (Lemma 15), whence supp $\mathfrak{c}_i = \phi$, so that $\mathfrak{c}_i = 0$ (Lemma 7 (2)). Thus $\mathfrak{a}_i = \mathfrak{b}_i$ is principal, as claimed. Therefore the ideal $\mathfrak{a}$, being the direct sum of principal (hence projective) ideals, is projective. This establishes (iii).

2) ⇒ 1). Assume 2). Let A be a right ideal of R generated by a family of elements $\{a_i\}_{i \in I}$ of cardinal at most α. We know by Corollary 1 that for every $i \in I$, supp $a_i = V_{e_i}$ for some idempotent $e_i \in R$, whence it follows that supp $A = \bigcup_{i \in I} V_{e_i}$. Set $\mathfrak{a} = \sum_{V_e \subseteq \text{supp} A} B(R)(1-e)$, which is equal to $\sum_{i \in I} B(R)(1-e_i)$ (see, Lemma 6). By (iii) $\mathfrak{a}$ is projective, hence has an orthogonal family of generators $\{f_j\}_{j \in J}$ [3, Lemma 1.1]. Then supp $A =$ supp $\mathfrak{a} = \bigcup_{j \in J} V_{1-f_j}$ (disjoint), hence by Proposition 3 $A = \bigoplus_{j \in J} A_j$, where each A_j is a right ideal of R with supp $A_j = V_{1-f_j}$. To complete the proof, it suffices to show that each A_j is finitely generated. This follows from the following lemma.

Lemma 17. *Let R be a right Ore, normal p.p. ring which satisfies conditions* (i) *and* (ii) *of Theorem* 3. *If A is a right ideal of R with open and closed support, then A is finitely generated.*

Proof. By hypothesis there is an idempotent $e \in R$ such that supp $A = V_e$. By virtue of Lemma 15 we see that $A \bigcap eR = 0$. Setting $A' = A \oplus eR$, we have supp $A' = X(R)$. It suffices to show that A' is finitely generated. By the partition property there are finitely many idempotents $e_1, \cdots, e_n \in R$ such that $V_{e_i} \bigcap V_{e_j} = \phi$ for $i \neq j$, $X(R) = \bigcup_{i=1}^{n} V_{e_i}$ and for every $i = 1, \cdots, n$, there is an $a_i \in A'$ such that $V_{e_i} \subseteq$ supp a_i. Let $f_i = 1 - e_i$ for $i = 1, \cdots, n$. We know by Lemma 2 that $\{f_1, \cdots, f_n\}$ is a complete set of orthogonal idempotents. Put $a = \sum_{i=1}^{n} a_i f_i \in A'$. Noting that supp $(a_i f_i) =$ supp $f_i = V_{e_i}$ for every $i = 1, \cdots, n$, we see by Lemma 16 (2) that supp $a = \bigcup_{i=1}^{n}$ supp $(a_i f_i) = \bigcup_{i=1}^{n} V_{e_i} = X(R)$. Since R_x is an integral domain for all $x \in X(R)$ (Corollary 1), it then follows from Lemma 7(2) that a is a non-zero-divisor. Let $\{x \in$

$X(R) \mid a_x$ is not invertible in $R_x\} = \{x_1,\cdots,x_m\}$, which is finite by (ii). By (i) A'_{x_j} is a finitely generated right ideal of R_{x_j} for $j = 1,\cdots,m$, so we write $A'_{x_j} = \sum_{k=1}^{n(j)} (a_k^{(j)})_{x_j} R_{x_j}$ with $a_k^{(j)} \in A'$ ($j = 1,\cdots,m$; $k = 1,\cdots,n(j)$). Setting $A_0 = aR + \sum_{j=1}^{m}\sum_{k=1}^{n(j)} a_k^{(j)}R$, which is a right ideal contained in A', we have $(A_0)_x = A'_x$ for all $x \in X(R)$. It follows from Lemma 8 that $A_0 = A'$, showing that A' is a finitely generated right ideal. This completes the proof.

Corollary 4 (cf. [3, Corollary 4.3]). *Let R be a right Ore, normal p.p. ring. Then the following are equivalent:*

1) *Every countably generated right ideal of R is a direct sum of finitely generated right ideals.*

2) *R satisfies conditions* (i) *and* (ii) *of Theorem* 3.

3) *Every right ideal of R containing a non-zero-divisor is finitely generated.*

4) *Every right ideal of R with open and closed support is finitely generated.*

Proof. Noting that every Boolean ring is $\aleph_0$-hereditary [3, Lemma 1.1], the corollary follows from Theorem 3, Lemma 17 and their proofs.

We state here the following result of F. Albrecht [1, Theorem], which will be needed in the proof of Theorem 4: Let R be a right semihereditary ring. Then every projective right R-module is a direct sum of submodules, each of which is isomorphic to a finitely generated right ideal of R.

Theorem 4 (cf. [3, Theorem 4.4]). *Let R be a normal, right Ore ring. Then R is right hereditary if and only if*

(i) *R is p.p.,*

(ii) *for any $x \in X(R)$, the stalk R_x is right hereditary,*

(iii) *for any non-zero-divisor $a \in R$, a_x is invertible in the stalk R_x for all but a finite number of $x \in X(R)$, and*

(iv) *$B(R)$ is hereditary.*

More generally, for any infinite cardinal α, R is right α-hereditary if and only if R satisfies (i), (ii), (iii) *and*

(iv') *$B(R)$ is α-hereditary.*

Proof. Let α be an infinite cardinal. If R is right α-hereditary, then by the above result of F. Albrecht, every right ideal of R generated by at most α

elements is a direct sum of finitely generated right ideals. Hence the only if part follows from Theorem 3 and Lemma 13.

Conversely, suppose that R satisfies conditions (i), (ii), (iii) and (iv'). It follows from (i) and (ii) that R is right semihereditary (Theorem 2). For any $x \in X(R)$, R_x is a right Ore domain (Corollary 1). This together with (ii) implies that R_x is right Noetherian (Lemma 12). Therefore, every right ideal of R generated by at most α elements, being a direct sum of finitely generated (hence projective) right ideals (Theorem 3), is projective. This completes the proof.

Corollary 5 (cf. [3, Corollary 4.5]). *Let R be a normal, right Ore ring. Then the following are equivalent:*

1) *R is right $\aleph_0$-hereditary.*

2) *R satisfies conditions* (i), (ii) *and* (iii) *of Theorem* 4.

3) *R is p.p. and every right ideal of R containing a non-zero-divisor is (finitely generated and) projective.*

4) *R is p.p. and every right ideal of R with open and closed support is finitely generated and projective.*

Proof. The equivalence of 1) and 2) is a direct consequence of Theorem 4. 1) ⇒ 3) and 4) ⇒ 1) follow from Corollary 4. 3) ⇒ 4) follows from Lemma 12 and the proof of Lemma 17.

Corollary 6 (cf. [3, Corollary 4.6]). *Let α be any cardinal.*

(1) *Let R be a strongly regular ring. Then the following are equivalent:*

(a) *R is right (α-)hereditary.*

(b) *R is left (α-)hereditary.*

(c) *$B(R)$ is (α-)hereditary.*

(2) *Let S be a normal, right Ore, right (α-)hereditary ring and let R be a von Neumann regular subring of S. Then R is right (and left) (α-) hereditary.*

Proof. (1) (a) ⇒ (c) is immediate from Theorem 4. (c) ⇒ (a) follows from Corollary 4 and Theorem 3.

(2) By Theorem 4 $B(S)$ is (α-)hereditary, hence by [3, Proposition 1.2] $B(R)$ is (α-)hereditary. Since R is a strongly regular ring, the conclusion follows from (1).

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