ON STRONGLY $\pi$-REGULAR RINGS
AND PERIODIC RINGS

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Throughout, $R$ will represent a ring with Jacobson radical $J$. Let $N = N(R)$ be the set of nilpotent elements in $R$. Given an integer $q > 1$, we put $E_q = \{ u \in R \mid u^q = u \}$. In particular $E = \bigcup_{q > 1} E_q$. We denote by $E_\infty$ the set of potent elements in $R$: $E_\infty = \bigcup_{q > 1} E_q$. Now, let $x$ be an element of $R$.

First, $x$ is said to be regular if $xyx = x$ for some $y \in R$, while $x$ is said to be right (resp. left) regular if $x^3y = x$ (resp. $yx^3 = x$) for some $y \in R$. If $x$ is strongly regular if it is both right and left regular. We denote by $S$ the set of strongly regular elements in $R$. Next, $x$ is called a right p. p. element if there exists an $e \in E$ such that $xe = x$ and $r(x) = r(e)$, where $r(*)$ denotes the right annihilator of $*$ in $R$, and we denote by $P$ the set of right p. p. elements in $R$. If every element in $E$ is central, $R$ is said to be normal. A ring $R$ is called periodic if for every $x \in R$ there exist distinct positive integers $h$, $k$ such that $x^h = x^k$, or equivalently, $x^n \in E_\infty$ for some positive integer $n$. $R$ is called $\pi$-regular (resp. strongly $\pi$-regular) if there exists a positive integer $n$ such that $x^n$ is regular (resp. strongly regular). Finally, $R$ is called a generalized right p. p. ring if for every $x \in R$ there exists a positive integer $n$ such that $x^n \in P$. Obviously, every periodic ring is strongly $\pi$-regular, and every $\pi$-regular ring is a generalized right p. p. ring (see [5]).

The present objective is to prove the following theorems.

**Theorem 1.** If $R$ is normal, then the following are equivalent:

1) $R$ is a generalized right p. p. ring.
2) Every $x \in R$ can be written in the form $x = u + a$, where $u \in P$, $a \in N$ and $ua = au$.

**Theorem 2.** (1) The following are equivalent:

1) $R$ is strongly $\pi$-regular.
2) Every $x \in R$ can be written in the form $x = u + a$, where $u \in S$, $a \in N$ and $ua = au$.

(2) Let $R$ be a normal (strongly) $\pi$-regular ring. Then $N$ forms an ideal, $R/N$ is strongly regular, and $R$ is a subdirect sum of nil rings and/or local rings with radical nil.
Theorem 3. (1) The following are equivalent:

1) \( R \) is periodic.

2) For every \( x \in R \) there exists a positive integer \( n \) and an \( x' \in \langle x \rangle \) such that \( x^n = x^{n+1}x' \).

3) Every \( x \in R \) can be written in the form \( x = u + a \), where \( u \in E_a \), \( a \in N \) and \( ua = au \).

(2) Let \( R \) be a normal periodic ring. Then \( N \) forms an ideal, \( R/N \) is a subdirect sum of periodic fields, and \( R \) is a subdirect sum of nil rings and/or local rings \( R_i \), such that \( R_i/N(R_i) \) is a periodic field.

Proof of Theorem 1. 1) \( \Rightarrow \) 2). Given \( x \in R \), we can find a positive integer \( m \) and an \( e \in E \) such that \( x^m e = x^m \) and \( r(x^m) = r(e) \). Clearly, \( (xe) e = xe \) and \( r(xe) = r(e) \), and so \( xe \in P \). Since \( e \) is central, we have \( (x - xe)^m = 0 \). Therefore \( x \) is the sum of \( xe \in P \) and \( x - xe \in N \).

2) \( \Rightarrow \) 1). Let \( x \) be an arbitrary element of \( R \). Then \( x = u + a \), where \( ue = u \) and \( r(u) = r(e) \) for some (central) idempotent \( e \), \( a^n = 0 \) for some positive integer \( m \) and \( ua = au \). Since \( (x - u)^m = 0 \), we can easily see that \( x^n e = x^n \). Furthermore, \( r(x) \subseteq r(u^n) \subseteq r(e) \). Hence \( x^n y = 0 \) implies \( x^{n-1} ey = ex^{n-1}y = 0 \), and therefore \( x^{n-2}e^2y = 0 \); eventually, \( ey = 0 \). This proves that \( r(x^n) = r(e) \) and \( x^n \in P \).

In advance of proving Theorem 2, we state the following easy lemma.

Lemma 1. Let \( R \) be a normal (strongly) \( \pi \)-regular ring. If \( R \) is indecomposable, then \( R \) is either a nil ring or a local ring with radical \( N \).

Proof. Suppose \( R \neq N \). Then \( R \) has a unity and every element of \( R \) is either nilpotent or invertible. Hence \( R \) is a local ring with radical \( N \).

Proof of Theorem 2. (1) 1) \( \Rightarrow \) 2). Let \( x \) be an arbitrary element of \( R \). Then, by [1, Lemma 1], there exists a positive integer \( n \) and \( y \in R \) such that \( x^n y = x^n \) and \( xy = yx \). It is easy to see that \( (x^{n+1}y)^2 x^{n-1}y = x^{n+1}y \) and \( (x - x^{n+1}y)^n = 0 \). Therefore \( x \) is the sum of \( x^{n+1}y \in S \) and \( x - x^{n+1}y \in N \).

2) \( \Rightarrow \) 1). Let \( x = u + a \), where \( u \in S \), \( a \in N \) and \( ua = au \). Again by [1, Lemma 1], there exists a \( t \in R \) such that \( u^2t = u \), \( ut = tu \) and \( at = ta \). Since \( x - x^2t = (u + a) - (u + a)^2t = a - a(2u + a)t \in N \) and \( xt = tx \), we conclude that \( x \) is strongly \( \pi \)-regular.

(2) As was claimed in [4, Remark], every homomorphic image of \( R \) is normal and \( \pi \)-regular. Since every primitive local ring is a division ring, Lemma 1 shows that \( R/J \) is a subdirect sum of division rings and reduced.
Hence $J$ coincides with $N$ and $R/J$ is strongly regular by (1). As is well known, $R$ is a subdirect sum of subdirectly irreducible rings, and so the latter assertion is clear by Lemma 1.

Proof of Theorem 3. (1) By a theorem of Chacron (see, e.g., [2, Theorem 1]). 2) implies 1).

1) $\Rightarrow$ 3). Let $x$ be an arbitrary element of $R$. Then there exists a positive integer $k$ such that $x^k = x^k$. It is easy to see that $x = x^{k+1} + (x-x^{k+1})$, $x^{k+1} = x^{k+1}$ and $(x-x^{k+1})^k = 0$.

3) $\Rightarrow$ 2). Let $x = u + a$, where $u \in E_q$ for some $q > 1$, $a \in N$ and $ua = au$. Since $x-x^q = (u+a) - (u+a)^q = a + ay$ with some $y \in \langle u, a \rangle$, we see that $x-x^q \in N$, proving 2).

(2) This is an easy consequence of Theorem 2 (2) and Jacobson's commutativity theorem.

Corollary 1. Let $R$ be a normal ring. If every $x \in R$ can be written in the form $x = u + a$, where $u \in E_q$, $a \in N$ and $ua = au$, then $N$ forms an ideal, $R/N$ is a subdirect sum of finite fields of order at most $q$, and $R$ is a subdirect sum of nil rings and/or local rings $R_i$ such that $R_i/N(R_i)$ is a finite field of order at most $q$.

As is well known, if every $x \in R$ may be written in at most one way in the form $x = e + a$, where $e \in E$ and $a \in N$, then $R$ is normal. Hence we have

Corollary 2 (A. Yaquob). Suppose every element $x$ of $R$ can be written uniquely in the form $x = e + a$, where $e \in E$ and $a \in N$. Then $N$ forms an ideal, $R/N$ is Boolean, and $R$ is a subdirect sum of nil rings and/or local rings $R_i$ such that $R_i/N(R_i)$ is GF(2).

Remark 1. Suppose every element $x$ in a periodic ring $R$ may be written in at most one way in the form $x = u + a$, where $u \in E_\infty$ and $a \in N$, then $R = E_\infty \oplus N$ by [3, Theorem 3]. Now, let $R = \mathbb{Z}/4\mathbb{Z}$. Then $R$ satisfies the hypothesis in Corollary 2, but $E$ does not form an ideal.

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REFERENCES


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