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ON STRONGLY π-REGULAR RINGS AND PERIODIC RINGS

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Throughout, R will represent a ring with Jacobson radical J. Let $N=N(R)$ be the set of nilpotent elements in R. Given an integer $q>1$, we put $E_q=\{u\in R \mid u^q=u\}$; in particular $E=E_2$. We denote by E_∞ the set of potent elements in R : $E_\infty=\bigcup_{q>1} E_q$. Now, let x be an element of R. First, x is said to be *regular* if $xyx=x$ for some $y\in R$, while x is said to be *right* (resp. *left*) *regular* if $x^2y=x$ (resp. $yx^2=x$) for some $y\in R$: x is *strongly regular* if it is both right and left regular. We denote by S the set of strongly regular elements in R. Next, x is called a *right p. p. element* if there exists an $e\in E$ such that $xe=x$ and $r(x)=r(e)$, where $r(*)$ denotes the right annihilator of $*$ in R, and we denote by P the set of right p. p. elements in R. If every element in E is central, R is said to be *normal*. A ring R is called *periodic* if for every $x\in R$ there exist distinct positive integers h, k such that $x^h=x^k$, or equivalently, $x^n\in E_\infty$ for some positive integer n ; R is called *π-regular* (resp. *strongly π-regular*) if there exists a positive integer n such that x^n is regular (resp. strongly regular). Finally, R is called a *generalized right p. p. ring* if for every $x\in R$ there exists a positive integer n such that $x^n\in P$. Obviously, every periodic ring is strongly π-regular, and every π-regular ring is a generalized right p. p. ring (see [5]).

The present objective is to prove the following theorems.

Theorem 1. *If R is normal, then the following are equivalent :*

1) *R is a generalized right p. p. ring.*

2) *Every $x\in R$ can be written in the form $x=u+a$, where $u\in P$, $a\in N$ and $ua=au$.*

Theorem 2. (1) *The following are equivalent :*

1) *R is strongly π-regular.*

2) *Every $x\in R$ can be written in the form $x=u+a$, where $u\in S$, $a\in N$ and $ua=au$.*

(2) *Let R be a normal (strongly) π-regular ring. Then N forms an ideal, R/N is strongly regular, and R is a subdirect sum of nil rings and/or local rings with radical nil.*

Theorem 3. (1) *The following are equivalent :*

1) *R is periodic.*

2) *For every $x \in R$ there exists a positive integer n and an $x' \in \langle x \rangle$ such that $x^n = x^{n+1}x'$.*

3) *Every $x \in R$ can be written in the form $x = u + a$, where $u \in E_\infty$, $a \in N$ and $ua = au$.*

(2) *Let R be a normal periodic ring. Then N forms an ideal, R/N is a subdirect sum of periodic fields, and R is a subdirect sum of nil rings and/or local rings R_i such that $R_i/N(R_i)$ is a periodic field.*

Proof of Theorem 1. 1) ⇨ 2). Given $x \in R$, we can find a positive integer m and an $e \in E$ such that $x^m e = x^m$ and $r(x^m) = r(e)$. Clearly, $(xe)e = xe$ and $r(xe) = r(e)$, and so $xe \in P$. Since e is central, we have $(x - xe)^m = 0$. Therefore x is the sum of $xe \in P$ and $x - xe \in N$.

2) ⇨ 1). Let x be an arbitrary element of R. Then $x = u + a$, where $ue = u$ and $r(u) = r(e)$ for some (central) idempotent e, $a^m = 0$ for some positive integer m and $ua = au$. Since $(x - u)^m = 0$, we can easily see that $x^m e = x^m$. Furthermore, $r(x) \subseteq r(u^m) \subseteq r(e)$. Hence $x^m y = 0$ implies $x^{m-1}ey = ex^{m-1}y = 0$, and therefore $x^{m-2}e^2 y = 0$; eventually, $ey = 0$. This proves that $r(x^m) = r(e)$ and $x^m \in P$.

In advance of proving Theorem 2, we state the following easy lemma.

Lemma 1. *Let R be a normal (strongly) π-regular ring. If R is indecomposable, then R is either a nil ring or a local ring with radical N.*

Proof. Suppose $R \neq N$. Then R has a unity and every element of R is either nilpotent or invertible. Hence R is a local ring with radical N.

Proof of Theorem 2. (1) 1) ⇨ 2). Let x be an arbitrary element of R. Then, by [1, Lemma 1], there exists a positive integer n and $y \in R$ such that $x^{2n}y = x^n$ and $xy = yx$. It is easy to see that $(x^{n+1}y)^2 x^{n-1}y = x^{n+1}y$ and $(x - x^{n+1}y)^n = 0$. Therefore x is the sum of $x^{n+1}y \in S$ and $x - x^{n+1}y \in N$.

2) ⇨ 1). Let $x = u + a$, where $u \in S$, $a \in N$ and $ua = au$. Again by [1, Lemma 1], there exists a $t \in R$ such that $u^2 t = u$, $ut = tu$ and $at = ta$. Since $x - x^2 t = (u + a) - (u + a)^2 t = a - a(2u + a)t \in N$ and $xt = tx$, we conclude that x is strongly π-regular.

(2) As was claimed in [4, Remark], every homomorphic image of R is normal and π-regular. Since every primitive local ring is a division ring, Lemma 1 shows that R/J is a subdirect sum of division rings and reduced.

Hence J coincides with N and R/J is strongly regular by (1). As is well known, R is a subdirect sum of subdirectly irreducible rings, and so the latter assertion is clear by Lemma 1.

Proof of Theorem 3. (1) By a theorem of Chacron (see, e.g., [2, Theorem 1]), 2) implies 1).

1) $\Rightarrow$ 3). Let x be an arbitrary element of R. Then there exists a positive integer k such that $x^{2k} = x^k$. It is easy to see that $x = x^{k+1} + (x - x^{k+1})$, $(x^{k+1})^{k+1} = x^{k+1}$ and $(x - x^{k+1})^k = 0$.

3) $\Rightarrow$ 2). Let $x = u + a$, where $u \in E_q$ for some $q > 1$, $a \in N$ and $ua =$ au. Since $x - x^q = (u+a) - (u+a)^q = a + ay$ with some $y \in \langle u, a \rangle$, we see that $x - x^q \in N$, proving 2).

(2) This is an easy consequence of Theorem 2 (2) and Jacobson's commutativity theorem.

Corollary 1. *Let R be a normal ring. If every $x \in R$ can be written in the form $x = u + a$, where $u \in E_q$, $a \in N$ and $ua = au$, then N forms an ideal, R/N is a subdirect sum of finite fields of order at most q, and R is a subdirect sum of nil rings and/or local rings R_i such that $R_i/N(R_i)$ is a finite field of order at most q.*

As is well known, if every $x \in R$ may be written in at most one way in the form $x = e + a$, where $e \in E$ and $a \in N$, then R is normal. Hence we have

Corollary 2 (A. Yaqub). *Suppose every element x of R can be written uniquely in the form $x = e + a$, where $e \in E$ and $a \in N$. Then N forms an ideal, R/N is Boolean, and R is a subdirect sum of nil rings and/or local rings R_i such that $R_i/N(R_i)$ is* GF(2).

Remark 1. Suppose every element x in a periodic ring R may be written in at most one way in the form $x = u + a$, where $u \in E_\infty$ and $a \in N$, then $R = E_\infty \oplus N$ by [3, Theorem 3]. Now, let $R = \boldsymbol{Z}/4\boldsymbol{Z}$. Then R satisfies the hypothesis in Corollary 2, but E does not form an ideal.

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